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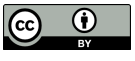

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Two-field reconstructions and polynomial-degree-robust efficiency for the transient acoustic wave equation

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Abstract

We consider the transient acoustic wave equation discretised using the spectral finite element method in space and the explicit leapfrog scheme in time. We reconstruct two $\mathcal{C}^0$-in-time approximations from the leapfrog snapshots: one of the solution and one of its first time derivative. This reconstruction pair verifies a pointwise-in-time Galerkin orthogonality. This allows us to reconstruct patchwise optimal fluxes with pointwise-in-time equilibration. Therefrom, we develop polynomial degree-robust *a posteriori* error estimates. We prove the local space-time efficiency using the robust stability bound developed in [SIAM J. Numer. Anal. 55 (2017), 2811-2834] for simplicial meshes in up to 3 space dimensions. These results are based on the inf-sup stable weak formulation from [J. Math. Anal. Appl. 505 (2022), 125457]. We then relax the pointwise-in-time Galerkin orthogonality into an average-in-time one. This allows for more general estimates with less restrictive hypotheses, which are, however, not necessarily p-robust anymore. These include residual- and flux-based (averaged and equilibrated) estimators. There, the proof of the local space-time efficiency uses appropriately scaled space-time bubble functions and norms on possibly anisotropic space-time meshes following [SIAM J. Numer. Anal. 51 (2013), 773-793].

Keywords: acoustic wave equation, a posteriori estimates, polynomial degree robustness, two-field reconstruction, spectral finite element method

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1 Introduction

We consider the acoustic wave equation with zero initial and boundary conditions which consists in finding $u : \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that

$$\partial_{tt} u - \Delta u = f \qquad \text{in } (0,T)\times\Omega, \tag{1.1a}$$
$$u = 0 \qquad \text{on } (0,T)\times\partial\Omega, \tag{1.1b}$$
$$u = 0 \qquad \text{on } \{0\}\times\Omega, \tag{1.1c}$$
$$\partial_t u = 0 \qquad \text{on } \{0\}\times\Omega. \tag{1.1d}$$

Here Ω is an open, bounded, connected, and polytopal domain of $\mathbb{R}^d$, $d \geq 1$, with a Lipschitz boundary $\partial\Omega$, and $T > 0$ is the final simulation time. We denote as Q the space-time domain $\Omega \times (0,T)$ and suppose that $f \in L^2(Q)$.

In [23], we introduce some *a posteriori* estimates based on the inf−sup stable weak formulation of the acoustic wave equation from [26]. For the estimates in [23, Theorem 6.3] to hold, the discrete approximation $u_{h\tau}$ (obtained by a suitable temporal reconstruction from the given discretisation) is required to have an additional regularity of $H^2(0,T;L^2(\Omega))$ and to verify the Galerkin orthogonality [23, Equation (6.7)]

$$\int_{I_n} (f_{h\tau}(t) - \partial_{tt} u_{h\tau}(t), \psi_a)_\Omega - (\nabla u_{h\tau}(t), \nabla\psi_a)_\Omega \mathrm{d}t = 0 \quad \forall a \in \mathcal{S}_h^{\mathrm{int}}(\Omega),\ \forall I_n \in \mathcal{T}_\tau(0,T), \tag{1.2}$$

where $f_{h\tau}$ is a suitable discrete reconstruction of the source term f and ψ_a is the finite element hat function associated with a vertex a.

In this follow-up work, we investigate a way to reduce the constraints on $u_{h\tau}$. Specifically, in addition to $u_{h\tau}$, we introduce a second reconstruction $s_{h\tau}$ that takes the role previously taken by $\partial_t u_{h\tau}$. Therefore, we now reconstruct $u_{h\tau}$ in $H^1(Q)$ such that $u_{h\tau}(0) = 0$ and, for all $t \in (0,T)$, $u_{h\tau}(t)|_{\partial\Omega} = 0$ together with $s_{h\tau}$ in $H^1(0,T;L^2(\Omega))$ such that $s_{h\tau}(0) = 0$. We use this two-field reconstruction in Theorem 4.5 to extend the *a posteriori* estimates from [23, Theorem 6.3]. The previous Galerkin orthogonality (1.2) is here relaxed to

$$\int_{I_n} (f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_\Omega - (\nabla u_{h\tau}(t), \nabla\psi_a)_\Omega \mathrm{d}t = 0 \quad \forall a \in \mathcal{S}_h^{\mathrm{int}}(\Omega),\ \forall I_n \in \mathcal{T}_\tau(0,T), \tag{1.3}$$

and this naturally adds the following additional term to the estimators:

$$\frac{1}{\tau_n} ||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n}. \tag{1.4}$$

In Theorem 4.5, the efficiency of this new term is dealt with by bounding it from above by residual terms. We specifically bound it by the jumps in time of $\partial_t u_{h\tau}$ (just like $\frac{1}{h_K}||\nabla u_{h\tau} + \boldsymbol{p}_{h\tau}||_{K\times I_n}$ is bounded by $||[\![\nabla u_{h\tau}\cdot\boldsymbol{n}_F]\!]||_{\partial K\times I_n}$, where $\boldsymbol{p}_{h\tau}$ is some reconstructed spatial flux).

This new setting also allows us to derive polynomial-degree-robust *a posteriori* estimates in Theorem 3.3. There, the multiplicative constants in the reliability and efficiency results do not depend on the polynomial degree of the space discretisation. To achieve this, we reproduce the procedure devised for the heat equation in [17] in the case of the wave equation. In particular, to derive the local efficiency, we augment the error by adding (1.4). Then, we assume global equivalence between the native and augmented errors to recover a global efficiency result that uses the native error. We show that this holds, in space dimensions $d \in \{1,2,3\}$ for simplicial meshes if the discrete approximation verifies the now pointwise-in-time Galerkin orthogonality (in comparison with the average-in-time one in (1.3))

$$(f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_\Omega - (\nabla u_{h\tau}(t), \nabla\psi_a)_\Omega = 0 \quad \forall a \in \mathcal{S}_h^{\mathrm{int}}(\Omega),\ \forall I_n \in \mathcal{T}_\tau(0,T),\ \forall t \in I_n. \tag{1.5}$$

Time stepping methods are common to discretise time dependent problems. In this case, the *a posteriori* analysis can either be directly based on the iterates, or on a reconstructed discrete approximation. In [2, 3, 14, 17], the authors study parabolic problems discretised with time stepping schemes (Crank-Nicolson, Runge-Kutta, and backward Euler). They reconstruct a space-time discrete approximation from the time stepping snapshots. In [19] the authors analyse the wave equation discretised in time by an implicit time stepping scheme before dealing with the leapfrog and cosine schemes in [20]. In [5], the acoustic wave equation is discretised in space by low order piecewise polynomials and by the backward Euler scheme in time. Both reliability and efficiency are shown, though in different norms. In [10] the potential is reconstructed for the leapfrog scheme with a higher regularity (not required by the wave equation, but by the *a posteriori* estimates). In [20, 21, 22], the authors also use two-field reconstructions to base their *a posteriori* estimates on. It is similar to the current work, where $s_{h\tau}$ is reconstructed independently of the pressure $u_{h\tau}$. It has its own regularity but both are still linked by a consistency requirement. Direct two-field schemes also exist for the wave equation. Mixed schemes exist for the acoustic wave equation in [15] and for the elastic one in [4, 12] (velocity–stress).

Polynomial-degree robustness of the equilibrated flux was first established by [9] using a patchwise equilibration procedure. It was then extended to other space-only problems [8] and space-time ones [17]. In the parabolic case, [17] augments the native norm of the error with a jump-in-time term to recover local space-time efficiency. We follow the same methodology to obtain our local space-time polynomial-degree-robust efficiency result. The current work extends the p-robust methodology of [17] from parabolic to hyperbolic problems using the pointwise-in-time Galerkin orthogonality (1.5).

This work is organised as follows. Section 2 sets useful notation. In Section 3, we adapt the procedure from [17] to reconstruct fluxes with the pointwise-in-time equilibration (1.5) and use them to derive polynomial-degree-robust *a posteriori* estimates. There, we present this more restrictive but sharper result. Then, we relax some of the restrictions in Section 4, where we adapt both the fluxes and the *a posteriori* estimates from [23] to the new setting with two-field reconstruction. This gives a general framework only average-in-time but accommodating a larger class of estimators, still reliable and locally space-time efficient, though not p-robust. In Sections 5 and 6, we respectively prove the main results, Theorems 3.3 and 4.5. Section 7 finally shows how to apply the results from Sections 3 and 4 to the case of a discretisation using a conforming finite element method in space with an explicit leapfrog scheme in time.

2 Setting

This section sets the notation.

2.1 Basic notation

For any time subinterval $I \subset (0,T)$ and space subdomain $\omega \subset \Omega$, we identify $L^2(\omega \times I)$ with the Bochner space $L^2(I;L^2(\omega))$. We use the following shorthand notation for the L^2 inner products and norms:

$$(u,v)_\omega := \int_\omega u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad ||u||_\omega := \sqrt{(u,u)_\omega}, \qquad u,v \in L^2(\omega),$$
$$(u,v)_{\omega\times I} := \int_I (u(t),v(t))_\omega \mathrm{d}t, \qquad ||u||_{\omega\times I} := \sqrt{(u,u)_{\omega\times I}}, \qquad u,v \in L^2(\omega\times I).$$

We denote by $\mathcal{P}_k(I)$ the space containing all real-valued polynomials on I of total degree less than or equal to $k \in \mathbb{N}$. For V any vector space, we also denote by $\mathcal{P}_k(I;V)$ the set containing all V-valued polynomials on I of degree less than or equal to k. For the reference simplex $\hat{K}$, we define

$$\mathcal{P}_k(\hat{K}) := \text{span}\left\{ \boldsymbol{x} \mapsto x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d},\ \text{s.t.}\ \sum_{i=1}^{d} \alpha_i \leq k \right\}$$

the space of polynomials of total degree at most k over $\hat{K}$. We finally define

$$\mathcal{Q}_k([0,1]^d) := \text{span}\left\{ \boldsymbol{x} \mapsto x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d},\ \text{s.t.}\ \max_{i \in [\![1,d]\!]} \alpha_i \leq k \right\}$$

the space of polynomials of componentwise degrees at most k over the reference cube $[0,1]^d$.

2.2 Space, time, and space-time meshes

Let $\mathcal{T}_h(\Omega)$ be a mesh of Ω composed of either simplices or of affine-mapped hypercubes such that any face of an element is either the whole face of another one, or a part of the boundary $\partial\Omega$. Let $\mathcal{F}_h(\Omega)$ and $\mathcal{S}_h(\Omega)$ be the sets containing the faces and vertices of $\mathcal{T}_h(\Omega)$, respectively. We specifically denote by $\mathcal{F}_h^{\text{int}}(\Omega)$ and $\mathcal{F}_h^{\text{ext}}(\Omega)$ the sets of interior and boundary faces of $\mathcal{T}_h(\Omega)$ and $\mathcal{S}_h^{\text{int}}(\Omega)$ and $\mathcal{S}_h^{\text{ext}}(\Omega)$ the sets of interior and boundary vertices. We use the notation $\hat{K}$ for the reference element: either the reference simplex $\{\boldsymbol{x} \in [0,1]^d \text{ s.t. } \sum_{k=1}^{d} x_k \leq 1\}$ or the unit hypercube $[0,1]^d$. For any element $K \in \mathcal{T}_h(\Omega)$, we assume that K is an affine map of $\hat{K}$, and we denote as:

- $h_K > 0$ its diameter;
- $\boldsymbol{T}_K : \hat{K} \longrightarrow K$ the affine mapping between the reference element $\hat{K}$ and K;
- $\mathbf{J}_K : \hat{K} \longrightarrow \mathbb{R}^{d\times d}$ the constant Jacobian matrix of $\boldsymbol{T}_K$ (we assume that $\det \mathbf{J}_K > 0$);
- $\mathcal{F}_h(K)$ and $\mathcal{S}_h(K)$ the sets containing respectively the faces and vertices of K.

Let $N \in \mathbb{N} \setminus \{0\}$ and $\mathcal{S}_\tau(0,T) := \{0 = t^0 < t^1 < \ldots < t^{N-1} < t^N = T\}$ be a partition of the time interval $(0,T)$ by the discrete times t^n. For each $n \in [\![1,N]\!]$, we denote

- $I_n := (t^{n-1}, t^n)$ be the n-th time interval;
- $\tau_n := t^n - t^{n-1}$ the length of I_n;
- $\mathcal{S}_\tau(I_n) := \{t^{n-1}, t^n\}$ the set containing the boundary times of I_n.

We denote the time mesh as $\mathcal{T}_\tau(0,T) := \{I_n,\ n \in [\![1,N]\!]\}$. Then $\mathcal{T}_{h\tau}(Q) := \mathcal{T}_h(\Omega) \times \mathcal{T}_\tau(0,T)$ stands for the space-time mesh.

2.3 Scalar-valued piecewise polynomial spaces

Let $p > 0$ be the polynomial degree of the space discretisation. Then, we define two piecewise polynomial, finite-dimensional, conforming approximation spaces respectively for $L^2(\Omega)$ and $H_0^1(\Omega)$:

$$U_h^p := \left\{ v_h \in L^2(\Omega) \text{ s.t. } \forall K \in \mathcal{T}_h(\Omega),\ v_h|_K \circ \boldsymbol{T}_K \in \begin{vmatrix} \mathcal{P}_p(\hat{K}) \text{ if } \hat{K} \text{ is a simplex} \\ \mathcal{Q}_p(\hat{K}) \text{ if } \hat{K} \text{ is a hypercube} \end{vmatrix} \right\}$$

and

$$V_h^p := U_h^p \cap H_0^1(\Omega).$$

Here, V_h^p imposes the piecewise polynomials to be continuous on Ω and zero on $\partial\Omega$.

Let $k > 0$ be the polynomial order of the time-discretisation. We similarly introduce $U_{h\tau}^{pk}$ as a space-time piecewise polynomial space to discretise $L^2(Q)$,

$$U_{h\tau}^{pk} := \left\{ u_{h\tau} \in L^2(Q) \text{ s.t. } \forall I_n \in \mathcal{T}_\tau(0,T),\ u_{h\tau}|_{I_n} \in \mathcal{P}_k(I_n; U_h^p) \right\}. \tag{2.1a}$$

Like above, we then impose global time or space-time continuity onto $U_{h\tau}^{pk}$ to build the following discrete spaces:

$$V_{h\tau}^{pk} := U_{h\tau}^{pk} \cap \{ v \in H^1(Q) \text{ s.t. } v = 0 \text{ on } \partial Q \setminus (\Omega \times \{T\}) \}, \tag{2.1b}$$

$$R_{h\tau}^{pk} := U_{h\tau}^{p,k+1} \cap \{ v \in H^1(0,T; L^2(\Omega)) \text{ s.t. } v(0) = 0 \}. \tag{2.1c}$$

In what follows, $V_{h\tau}^{pk}$ and $R_{h\tau}^{pk}$ are used to reconstruct the respective approximations $u_{h\tau}$ and $s_{h\tau}$ of u and $\partial_t u$.

The equilibration compares $f_{h\tau} \in U_{h\tau}$ to $\partial_t s_{h\tau}$ and the divergence of a flux. We build $R_{h\tau}^{pk}$ with an additional polynomial degree in time compared to $U_{h\tau}^{pk}$ to be able to obtain a space-time strong equilibration.

2.4 Vector-valued piecewise polynomial spaces

In this section, we define vector-valued piecewise polynomial spaces. Later, in Sections 3 and 4, we construct our fluxes in these spaces.

Recall that $p > 0$ is the polynomial order of space discretisation and $k > 0$ is the polynomial order of time discretisation. On the unit hypercube $\hat{K} = [0,1]^d$, we introduce, for $i \in [\![1, d]\!]$,

$$\mathcal{Q}_{p,i}^-(\hat{K}) := \operatorname{span}\left\{ \boldsymbol{x} \mapsto \prod_{j=1}^d x_j^{\alpha_j},\ \text{s.t. } \forall j \in [\![1,d]\!],\ \alpha_j \in \begin{cases} [\![0, p-1]\!] & \text{if } i = j \\ [\![0, p]\!] & \text{else} \end{cases} \right\}, \tag{2.2}$$

$$\mathcal{Q}_{p,i}^+(\hat{K}) := \operatorname{span}\left\{ \boldsymbol{x} \mapsto \prod_{j=1}^d x_j^{\alpha_j},\ \text{s.t. } \forall j \in [\![1,d]\!],\ \alpha_j \in \begin{cases} [\![0, p+1]\!] & \text{if } i = j \\ [\![0, p]\!] & \text{else} \end{cases} \right\}. \tag{2.3}$$

Then, the Raviart–Thomas space on the reference cell $\hat{K}$ is defined by [7, Remark 2.3.1, Equation (2.4.1)] as

$$\boldsymbol{RT}_p(\hat{K}) := \begin{cases} \mathcal{P}_p(\hat{K})^d + \boldsymbol{x}\mathcal{P}_p(\hat{K}) & \text{if } \hat{K} \text{ is a simplex,} \\ \mathcal{Q}_{p,1}^+(\hat{K}) \times \mathcal{Q}_{p,2}^+(\hat{K}) \times \cdots \times \mathcal{Q}_{p,d}^+(\hat{K}) & \text{if } \hat{K} \text{ is a hypercube.} \end{cases}$$

It is defined on any cell $K \in \mathcal{T}_h(\Omega)$ by [7, Remark 2.3.1] as

$$\boldsymbol{RT}_p(K) := \frac{1}{\det \mathbf{J}_K} \mathbf{J}_K \boldsymbol{RT}_p(\hat{K}) \circ \boldsymbol{T}_K^{-1}.$$

The global $\boldsymbol{H}(\operatorname{div}, \Omega)$-conforming space and space-time approximation spaces are then

$$\boldsymbol{W}_h^p := \left\{ \boldsymbol{w}_h \in \boldsymbol{H}(\operatorname{div}, \Omega) \text{ s.t. } \forall K \in \mathcal{T}_h(\Omega),\ \boldsymbol{w}_h|_K \in \boldsymbol{RT}_p(K) \right\},$$

$$\boldsymbol{W}_{h\tau}^{pk} := \left\{ \boldsymbol{w}_{h\tau} \in L^2(0,T; \boldsymbol{H}(\operatorname{div}, \Omega)) \text{ s.t. } \forall I_n \in \mathcal{T}_\tau(0,T),\ \boldsymbol{w}_{h\tau}|_{I_n} \in \mathcal{P}_k(I_n; \boldsymbol{W}_h^p) \right\}.$$

2.5 Partition of unity and local subdomains

Let $a \in \mathcal{S}_h(\Omega)$ be a mesh vertex. We introduce $\psi_a \in \mathcal{C}^0(\overline{\Omega})$, the hat function around a, such that for $K \in \mathcal{T}_h(\Omega)$,

$$\psi_a|_K \circ \boldsymbol{T}_K \in \begin{cases} \mathcal{P}_1(\hat{K}) & \text{if } \hat{K} \text{ is a simplex,} \\ \mathcal{Q}_1(\hat{K}) & \text{if } \hat{K} \text{ is a hypercube,} \end{cases}$$

and that for $a' \in \mathcal{S}_h(\Omega)$,

$$\psi_a(a') := \begin{cases} 1 & \text{if } a = a', \\ 0 & \text{else.} \end{cases}$$

Remarks:

- For $a \in \mathcal{S}_h(\Omega)$, ψ_a is in $U_h^1 \cap \mathcal{C}^0(\overline{\Omega})$, and if $a \in \mathcal{S}_h^{\text{int}}(\Omega)$, ψ_a is in V_h^1.
- The family of the hat functions forms a partition of unity in the sense that

$$\sum_{a \in \mathcal{S}_h(\Omega)} \psi_a = 1.$$

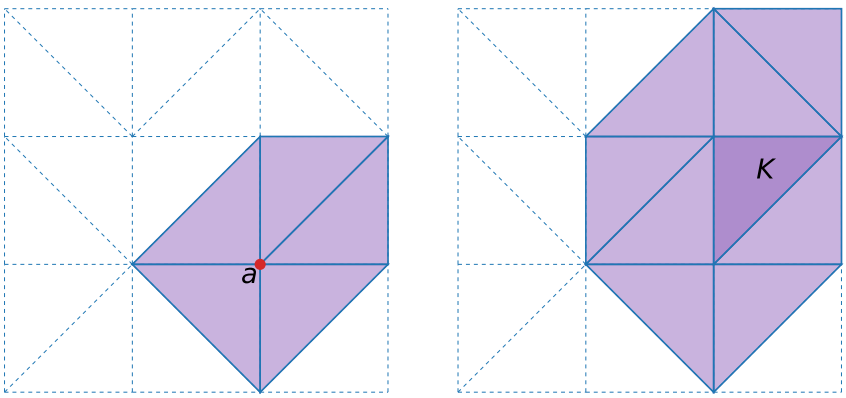
We define the local patch around a vertex $a \in \mathcal{S}_h(\Omega)$ as

$$\mathcal{T}_h(\omega_a) := \{K \in \mathcal{T}_h(\Omega) \text{ s.t. } a \in \mathcal{S}_h(K)\},$$

with ω_a the associated open subdomain. We denote as h_{ω_a} the diameter of ω_a. We similarly define the local patch around an element $K \in \mathcal{T}_h(\Omega)$ as

$$\mathcal{T}_h(\omega_K) := \bigcup_{a \in \mathcal{S}_h(K)} \mathcal{T}_h(\omega_a),$$

with $\omega_K := \bigcup_{a \in \mathcal{S}_h(K)} \omega_a$ the associated open subdomain. Examples of such patches are given in Figure 1.



(a) $\mathcal{T}_h(\omega_a)$ around a vertex $a \in \mathcal{S}_h(\Omega)$ (b) $\mathcal{T}_h(\omega_K)$ around an element $K \in \mathcal{T}_h(\Omega)$

Figure 1: Examples of patches in a triangular mesh, $d = 2$

For $t^n \in \mathcal{S}_\tau(0,T)$ we also define the time hat function $\psi^n \in \mathcal{C}^0(0,T)$ such that

$$\begin{cases} \psi^n|_{I_m} \in \mathcal{P}_1(I_m;\mathbb{R}) & \forall I_m \in \mathcal{T}_\tau(0,T), \\ \psi^n(t^m) = 0 & \forall t^m \in \mathcal{S}_\tau(0,T) \setminus \{t^n\}, \\ \psi^n(t^n) = 1. & \end{cases} \tag{2.4}$$

We denote as $J_{t^n} := \mathrm{supp}(\psi^n)$ the time patch subdomain around t^n, formed by the one or two time intervals sharing t^n.

2.6 Space-time setting of Steinbach and Zank

In order to derive space-time *a posteriori* estimates, we use well-posedness results for the wave equation from [25] and the inf−sup formulation developed in [26]. We introduce the trial space Y_0 by imposing zero values on $\partial\Omega \times (0,T)$ and on $\Omega \times \{0\}$ for functions of $H^1(Q)$:

$$Y_0 := \left\{ v \in H^1(Q) \text{ s.t. } v = 0 \text{ on } \partial Q \setminus (\{T\} \times \Omega) \right\}. \tag{2.5}$$

Then, we introduce the test space Y_T similarly by imposing zero values on $\partial\Omega \times (0,T)$ and on $\Omega \times \{T\}$ for functions of $H^1(Q)$:

$$Y_T := \left\{ v \in H^1(Q) \text{ s.t. } v = 0 \text{ on } \partial Q \setminus (\{0\} \times \Omega) \right\}. \tag{2.6}$$

By default, Y_T is equipped with the norm,

$$||\cdot||_{Y_T}^2 := ||\partial_t \cdot||_\Omega^2 + ||\nabla \cdot||_\Omega^2 \tag{2.7}$$

and its dual space Y_T' is equipped with the dual norm

$$||\cdot||_{Y_T'} := \sup_{v \in Y_T \setminus \{0\}} \frac{\langle \cdot, v \rangle_{Y_T}}{||v||_{Y_T}},$$

where $\langle \cdot, \cdot \rangle_{Y_T}$ is the duality pairing between Y_T' and Y_T. Let $Q_- := \Omega \times (-T,T)$ be the extended space-time domain and let $\Box : L^2(Q) \longrightarrow \mathcal{C}_0^\infty(Q_-)'$ be the extended d'Alembert operator defined such that for $u \in L^2(Q)$ and $\phi \in \mathcal{C}_0^\infty(Q_-)$,

$$\langle \Box u, \phi \rangle := (u, \partial_{tt}\phi - \Delta\phi)_Q.$$

In [26], Steinbach and Zank introduce the Banach space

$$\mathcal{H}(Q) := \left\{ v \in L^2(Q), \text{ s.t. } \Box v \in H_0^1(Q_-)' \right\} \quad \text{with} \quad ||\cdot||_{\mathcal{H}(Q)}^2 := ||\cdot||_Q^2 + ||\Box \cdot||_{H_0^1(Q_-)'}^2.$$

Then, by completion, they define

$$\mathcal{Y}_0 := \left\{ v \in \mathcal{H}(Q) \text{ s.t. } \exists (v_i)_{i\in\mathbb{N}} \subset Y_0, \ ||v - v_i||_{\mathcal{H}(Q)} \underset{i\to\infty}{\longrightarrow} 0 \right\}. \tag{2.8}$$

For $f \in Y_T'$, [26, Theorem 3.9] shows the well-posedness of the weak formulation

$$\text{find } u \in \mathcal{Y}_0 \text{ such that } \langle \Box u, \mathcal{E}v \rangle_{H_0^1(Q_-)} = \langle f, v \rangle_{Y_T} \quad \forall v \in Y_T, \tag{2.9}$$

where $\mathcal{E} : Y_T \longrightarrow H_0^1(Q_-)$ is any given continuous and injective extension operator. Additionally, the inf−sup stability of $\langle \Box \cdot, \mathcal{E} \cdot \rangle_{H_0^1(Q_-)}$ from [26, Equation (3.10)] ensures that,

$$||\cdot||_{\mathcal{Y}_0} := \sup_{v \in Y_T \setminus \{0\}} \frac{\langle \Box \cdot, \mathcal{E}v \rangle_{H_0^1(Q_-)}}{||v||_{Y_T}}$$

is a norm on $\mathcal{Y}_0$. This in turn ensures that for $u \in \mathcal{Y}_0$ the weak solution of (2.9) and any $u_{h\tau} \in \mathcal{Y}_0$, there holds

$$||u - u_{h\tau}||_{\mathcal{Y}_0} = ||r(u_{h\tau})||_{Y_T'}, \tag{2.10}$$

where $r : \mathcal{Y}_0 \to Y_T'$ is the residual of (2.9):

$$\langle r(u_{h\tau}), v\rangle_{Y_T} := \langle f, v\rangle_{Y_T} - \langle \Box u_{h\tau}, \mathcal{E}v\rangle_{H_0^1(Q_-)}, \qquad v \in Y_T. \tag{2.11}$$

Notably, $r(u) = 0$ if and only if u is the solution of (2.9). This equivalence between the norm of the error and the dual norm of the residual is the basis of the *a posteriori* estimates derived in what follows. Here, (2.10) is the counterpart of [17, Equation (2.7)] for the wave equation instead of the heat equation.

In what follows, we assume that $f \in L^2(Q)$. In this case, [25, Theorem 5.1] ensures the well-posedness of the following weak formulation,

$$\text{find } u \in Y_0 \text{ such that } -(\partial_t u, \partial_t v)_Q + (\nabla u, \nabla v)_Q = (f, v)_Q \quad \forall v \in Y_T, \tag{2.12}$$

and [26, Equation (3.11)] ensures that (2.9) and (2.12) are equivalent. Furthermore, if $f \in L^2(Q)$ and $u_{h\tau} \in Y_0$, the residual can be equivalently simplified to become

$$\langle r(u_{h\tau}), v\rangle_{Y_T} = (f, v)_Q + (\partial_t u_{h\tau}, \partial_t v)_Q - (\nabla u_{h\tau}, \nabla v)_Q, \qquad v \in Y_T. \tag{2.13}$$

2.7 Space-time anisotropic meshes and norms

Now that the variational setting is set, we still need to address an issue with the use of $||\cdot||_{Y_T}$ as a norm over Y_T. Indeed, $||\cdot||_{Y_T}$ does not have a coherent scaling of the time and space derivatives (they are, in particular, in different physical units). This lack of appropriate scaling would also cause problems later when showing the efficiency (constants that depend on the anisotropy of the space-time mesh would arise). We follow the methodology of [14] and introduce the following equivalent norm on Y_T:

$$|||\cdot|||_{Y_T}^2 := \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} |||\cdot|||_{Y_T,K\times I_n}^2 \tag{2.14a}$$

with

$$|||\cdot|||_{Y_T,K\times I_n}^2 := \tau_n^2||\partial_t \cdot||_{K\times I_n}^2 + h_K^2||\nabla \cdot||_{K\times I_n}^2 \quad (K, I_n) \in \mathcal{T}_{h\tau}(Q). \tag{2.14b}$$

We also update the dual norms for Y_T' and $\mathcal{Y}_0$ (in both local and global forms) such that for any subinterval $I \subset (0, T)$ and subdomain $\omega \subset \Omega$ (either a mesh cell or a patch of mesh cells),

$$|||\cdot|||_{Y_T',\omega\times I} := \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega\times I}} \frac{\langle\cdot, v\rangle_{Y_T}}{|||v|||_{Y_T}} \qquad \text{and} \qquad |||\cdot|||_{Y_T'} := |||\cdot|||_{Y_T',Q}. \tag{2.14c}$$

Then, the local and global norms of the error become, for $u_{h\tau} \in \mathcal{Y}_0$,

$$|||u - u_{h\tau}|||_{\mathcal{Y}_0,\omega\times I} := |||r(u_{h\tau})|||_{Y_T',\omega\times I} \qquad \text{and} \qquad |||u - u_{h\tau}|||_{\mathcal{Y}_0} := |||r(u_{h\tau})|||_{Y_T'}. \tag{2.14d}$$

3 Polynomial-degree-robust reliable and locally space-time efficient *a posteriori* estimates (pointwise-in-time setting)

In this section, we present our *a posteriori* estimates that are robust with regards to the space discretisation polynomial-degree. We first define the patchwise optimal pointwise-in-time equilibrated flux and the associated estimator in Section 3.1. Then, we identify a suitable augmented error and state the *a posteriori* estimates in Section 3.2. The proofs are postponed until Section 6.

3.1 Patchwise optimal pointwise-in-time equilibrated flux

Recall the spaces $U_{h\tau}^{pk}$, $V_{h\tau}^{pk}$, and $R_{h\tau}^{pk}$ from (2.1).

Definition 3.1 (Patchwise optimal pointwise-in-time equilibrated flux)**.** *Let*

$$u_{h\tau} \in V_{h\tau}^{pk}, \quad s_{h\tau} \in R_{h\tau}^{pk}, \text{ and } \quad f_{h\tau} \in U_{h\tau}^{pk} \tag{3.1}$$

be such that:

$$(f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_{\omega_a} - (\nabla u_{h\tau}(t), \nabla \psi_a)_{\omega_a} = 0 \quad \forall (a, I_n) \in \mathcal{S}_h^{\mathrm{int}}(\Omega) \times \mathcal{T}_\tau(0,T),\ \forall t \in I_n. \tag{3.2}$$

For $I_n \in \mathcal{T}_\tau(0,T)$ and $a \in \mathcal{S}_h(\Omega)$, we define the patchwise optimal equilibrated local flux as:

$$\begin{aligned} \boldsymbol{p}_{h\tau,a}^{\mathrm{rob}}|_{I_n} := \operatorname{argmin} \Bigg\{ & \sum_{K \in \mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_{h\tau} + \psi_a \nabla u_{h\tau}||_{K \times I_n}^2,\ \boldsymbol{p}_{h\tau} \in \boldsymbol{W}_{h\tau}^{p+1,k}|_{\omega_a \times I_n} \\ & \text{with } \boldsymbol{p}_{h\tau} \cdot \boldsymbol{n} = 0 \text{ on } (\partial\omega_a \cap \{\psi_a = 0\}) \times I_n \text{ and} \\ & \operatorname{div} \boldsymbol{p}_{h\tau} = \psi_a (f_{h\tau} - \partial_t s_{h\tau}) - \nabla \psi_a \cdot \nabla u_{h\tau} \text{ on } \omega_a \times I_n \Bigg\}. \end{aligned} \tag{3.3a}$$

The global flux is defined for each $I_n \in \mathcal{T}_\tau(0,T)$ by

$$\boldsymbol{p}_{h\tau}^{\mathrm{rob}}|_{I_n} := \sum_{a \in \mathcal{S}_h(\Omega)} \boldsymbol{p}_{h\tau,a}^{\mathrm{rob}}|_{I_n}. \tag{3.3b}$$

In Definition 3.1, the equilibration of the flux is pointwise both in space and time. In practice, the stationary procedure known from [9, 13, 14] and depicted in Figure 2 is applied to all time-orthogonal components of the flux, as described in Section 5.2. In comparison with [23] and Section 4 below, Definition 3.1 comes with a stronger requirement (pointwise-in-time Galerkin orthogonality with respect to the hat function ψ_a) on the reconstructions in (3.2), but is rewarded by a stronger equilibration property which follows from (3.3):

$$\operatorname{div} \boldsymbol{p}_{h\tau}^{\mathrm{rob}} = f_{h\tau} - \partial_t s_{h\tau}. \tag{3.4}$$

This procedure is the counterpart of [17, Definition 4.1] but in our setting for the wave equation.

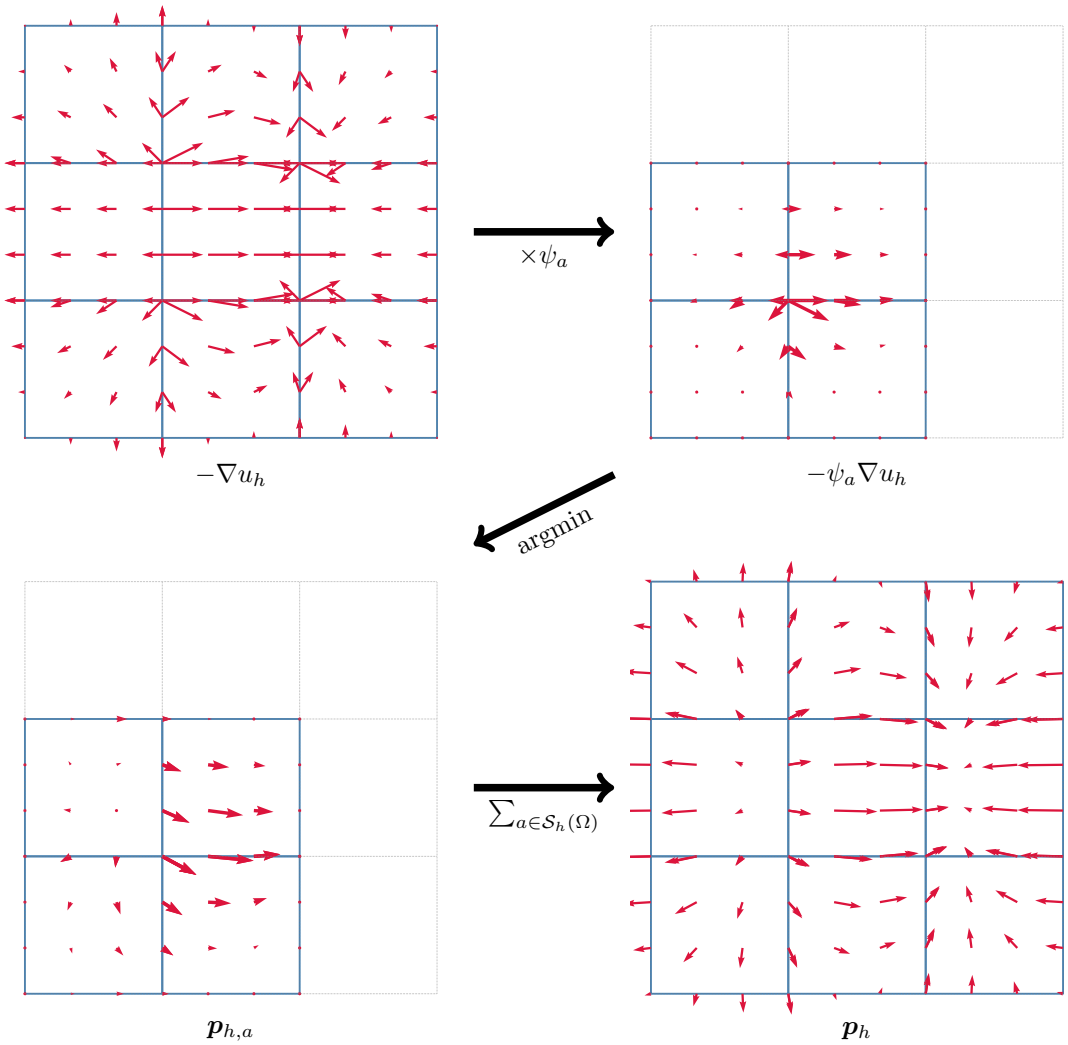


Figure 2: Construction of the locally optimal equilibrated flux in the stationary case

For the flux $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ from (3.3), the term evaluating the quality of the equilibration is zero and the estimator takes the following form:

Definition 3.2 (*A posteriori* estimator based on $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$)**.** *Let the assumptions of Definition 3.1 hold. For $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$, we define the local estimator based on $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ from Definition 3.1 as:*

$$(\eta_{K\times I_n}^{\mathrm{rob}})^2 := \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n}^2 + \frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n}^2. \tag{3.5}$$

The global estimator is defined by

$$(\eta_Q^{\mathrm{rob}})^2 := \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} (\eta_{K\times I_n}^{\mathrm{rob}})^2. \tag{3.6}$$

3.2 Polynomial-degree-robust reliability and efficiency in an augmented norm

To derive the polynomial-degree-robust efficiency result, we follow the methodology from [17]. In [17, Lemma 6.1], the authors devise an augmented error and show that the global error and the augmented global error are equivalent in the setting of the heat equation. This allows them to state the global reliability and efficiency results in the native global error norm.

Here, we proceed similarly. We consider the native error norm from (2.14d)

$$|||u - u_{h\tau}|||_{\mathcal{Y}_0}, \tag{3.7}$$

and its augmentation by

$$\sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n}}.$$

We assume that for $u_{h\tau}$ and $s_{h\tau}$, there holds

$$\sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n}} \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0} + |||f - f_{h\tau}|||_{Y_T'}. \tag{3.8}$$

Then, the native and augmented errors are equivalent, up to the data oscillation term $|||f - f_{h\tau}|||_{Y_T'}$:

$$\begin{aligned}|||u - u_{h\tau}|||_{\mathcal{Y}_0} \le |||u - u_{h\tau}|||_{\mathcal{Y}_0} + \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n}}\\ \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0} + |||f - f_{h\tau}|||_{Y_T'}\end{aligned} \tag{3.9}$$

This allows us to state the global reliability and efficiency results directly using the native error (3.7) together with global reliability and local space-time efficiency in the augmented error. Hypothesis (3.8) is carefully verified in the applications below.

Theorem 3.3 (Polynomial-degree-robust reliable and locally space-time efficient a posteriori estimates)**.** *Let $f \in L^2(Q)$ and $u \in Y_0$ be the weak solution from (2.12). Let $d \in \{1,2,3\}$ with $\mathcal{T}_h(\Omega)$ a simplicial mesh.*

If the hypotheses (3.1) and (3.2) of Definition 3.1 and (3.8) are verified, then the following properties hold for the a posteriori *estimator η^{rob} from Definition 3.2:*

- ***Global space-time reliability**:*

$$|||u - u_{h\tau}|||_{\mathcal{Y}_0} \le \eta_Q^{\mathrm{rob}} + |||f - f_{h\tau}|||_{Y_T'}. \tag{3.10a}$$

- ***Global space-time efficiency**:*

$$\eta_Q^{\mathrm{rob}} \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0} + |||f - f_{h\tau}|||_{Y_T'}. \tag{3.10b}$$

- ***Global space-time reliability** in the augmented error:*

$$|||u - u_{h\tau}|||_{\mathcal{Y}_0} + \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n}} \le 2\eta_Q^{\mathrm{rob}} + |||f - f_{h\tau}|||_{Y_T'}. \tag{3.10c}$$

- ***Local space-time efficiency*** *in the augmented error: For all* $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$,

$$\eta^{\mathrm{rob}}_{K\times I_n} \lesssim |||u - u_{h\tau}|||_{y_0,\omega_K\times I_n} + \frac{1}{\tau_n}||s_{h\tau} - \partial_t u_{h\tau}||_{\omega_K\times I_n} + |||f - f_{h\tau}|||_{Y'_T,\omega_K\times I_n}. \quad (3.10\mathrm{d})$$

Here $\lesssim_p$ *means up to a constant that can only depend on the space dimension* d*, the polynomial orders* p *and* k*, and the shape-regularity of* $\mathcal{T}_h(\Omega)$*. Here,* $\lesssim$ *means* $\lesssim_p$*, but* p*-independent.*

Specifically, the proof of Theorem 3.3 still uses finite-dimensional arguments for inequalities in time. Therefore, the results of Theorem 3.3 are not robust with regard to the polynomial order k of the time discretisation.

Recall that $s_{h\tau}$ and the flux $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}$ are not only mathematical tools, but also have important physical meaning. The goal of $s_{h\tau}$ is to be an approximation of the time derivative of the solution $\partial_t u$. We compare it to $\partial_t u_{h\tau}$ to evaluate how consistent the two discrete approximations of $\partial_t u$ are with each other. Later, we prove that the distance between the two is equivalent (reliability and efficiency) to the actual discretisation error as expressed by (3.9) (that follows immediately from (3.8)). The same applies to the flux $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}$ that, when compared to $-\nabla u_{h\tau}$ evaluates how close it is to the physical flux $-\nabla u$.

4 General reliable and locally space-time efficient *a posteriori* estimates (average-in-time setting)

In this section we extend the estimates from [23, Section 6.4] to the case with a two-field reconstruction $u_{h\tau}$, $s_{h\tau}$. In Sections 4.1 and 4.2, we recall and adapt the fluxes introduced in [23, Section 3.3] to the current setting. Then, we define our new space-time local and global estimators in Section 4.3, before stating the reliability and local space-time efficiency results in Section 4.4.

4.1 Average-in-space flux reconstruction

The reconstruction procedure for the averaged flux does not change from [23, Section 3.3]. Here, we recall it as is.

Definition 4.1 (Averaged flux)**.** *Let* $u_{h\tau} \in V^{pk}_{h\tau}$*. We define the averaged flux* $\boldsymbol{p}^{\mathrm{avg}}_{h\tau} \in \boldsymbol{W}^{pk}_{h\tau}$ *such that, for any* $t \in (0,T)$ *and* $K \in \mathcal{T}_h(\Omega)$*, there holds:*

$$\begin{cases} (\boldsymbol{p}^{\mathrm{avg}}_{h\tau}(t), \boldsymbol{z}_h)_K := -(\nabla u_{h\tau}(t), \boldsymbol{z}_h)_K & \forall \boldsymbol{z}_h \in \boldsymbol{Z}^p_h(K), \\ \boldsymbol{p}^{\mathrm{avg}}_{h\tau}(t)|_K\cdot\boldsymbol{n}_F := -\langle\!\langle \nabla u_{h\tau}(t)\cdot\boldsymbol{n}_F \rangle\!\rangle_F & \forall F \in \mathcal{F}_h(K), \end{cases}$$

with $\boldsymbol{Z}^p_h(K)$ *a space that matches the local interior degrees of freedom of the Raviart–Thomas space:*

$$\boldsymbol{Z}^p_h(K) := \mathbf{J}_K^{-T} \begin{cases} \mathcal{P}_{p-1}(\hat{K})^d & \text{if } \hat{K} \text{ is a simplex,} \\ \mathcal{Q}^-_{p,1}(\hat{K}) \times \mathcal{Q}^-_{p,2}(\hat{K}) \times \cdots \times \mathcal{Q}^-_{p,d}(\hat{K}) & \text{if } \hat{K} \text{ is a hypercube.} \end{cases} \circ \boldsymbol{T}_K^{-1},$$

and with the averaging operator on any face F*:*

$$\langle\!\langle g_h \rangle\!\rangle_F := \begin{cases} g_h|_F & \text{if } F \in \mathcal{F}^{\mathrm{ext}}_h(\Omega), \\ \frac{1}{2}\left(g_h|_{K_1}|_F + g_h|_{K_2}|_F\right) & \text{if } F \in \mathcal{F}^{\mathrm{int}}_h(\Omega) \text{ if shared by } K_1, K_2 \in \mathcal{T}_h(\Omega). \end{cases}$$

The flux reconstruction procedure in Definition 4.1 is illustrated in Figure 3. It is inspired by the one in [16, Equations (5) and (6)] and [27, Equation (4.1)] for simplices.

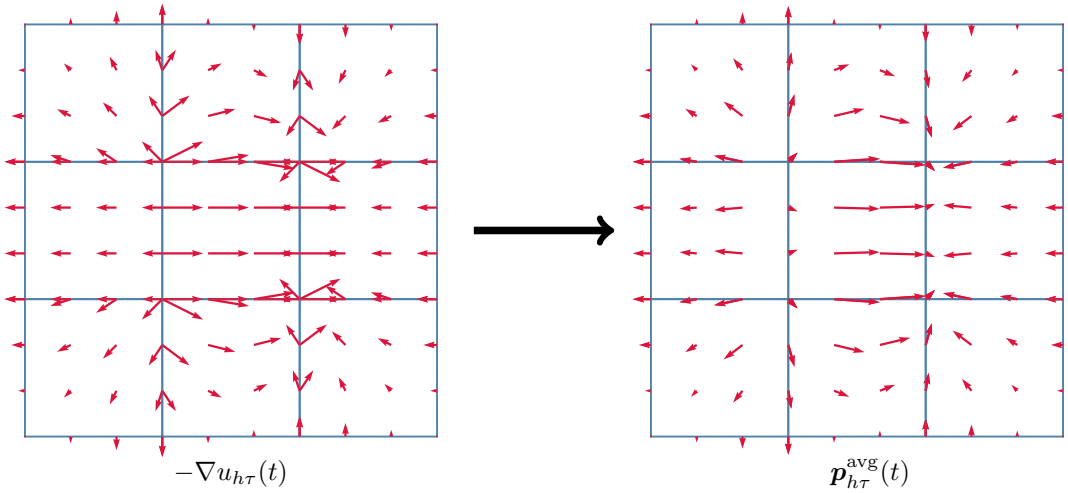


Figure 3: Construction of the averaged flux $\boldsymbol{p}_{h\tau}^{\mathrm{avg}}(t)$ from Definition 4.1

4.2 Patchwise optimal average-in-time equilibrated flux reconstruction

In this section, we present a procedure to reconstruct a patchwise optimal flux with average-in-time equilibration adapted from [23, Definition 3.5]. The only difference is that $\partial_{tt} u_{h\tau}$ is replaced by $\partial_t s_{h\tau}$ in the assumed Galerkin orthogonality and in the equilibration constraint. Both these equalities are strong in space but average-in-time.

Definition 4.2 (Patchwise optimal average-in-time equilibrated flux)**.** *Let $u_{h\tau} \in V_{h\tau}^{pk}$, $s_{h\tau} \in R_{h\tau}^{pk}$, and $f_{h\tau} \in U_{h\tau}^{pk}$ be such that:*

$$\int_{I_n} (f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_{\omega_a} - (\nabla u_{h\tau}(t), \nabla \psi_a)_{\omega_a} \mathrm{d}t = 0 \quad \forall (a, I_n) \in \mathcal{S}_h^{\mathrm{int}}(\Omega) \times \mathcal{T}_\tau(0,T). \tag{4.1a}$$

For $I_n \in \mathcal{T}_\tau(0,T)$ and $a \in \mathcal{S}_h(\Omega)$, we define the patchwise optimal equilibrated local flux as:

$$\begin{aligned} \boldsymbol{p}_{h\tau,a}^{\mathrm{equ}}|_{I_n} := \operatorname{argmin} \Bigg\{ & \sum_{K \in \mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_{h\tau} + \psi_a \nabla u_{h\tau}||_{K \times I_n}^2, \ \boldsymbol{p}_{h\tau} \in \boldsymbol{W}_{h\tau}^{p+1,k}|_{\omega_a \times I_n} \\ & \text{with } \boldsymbol{p}_{h\tau} \cdot \boldsymbol{n} = 0 \text{ on } (\partial\omega_a \cap \{\psi_a = 0\}) \times I_n \text{ and} \\ & \int_{I_n} \operatorname{div} \boldsymbol{p}_{h\tau}(\cdot,t) - \psi_a(\cdot)(f_{h\tau}(\cdot,t) - \partial_t s_{h\tau}(\cdot,t)) + \nabla \psi_a(\cdot) \cdot \nabla u_{h\tau}(\cdot,t) \mathrm{d}t = 0 \text{ on } \omega_a \Bigg\}. \end{aligned} \tag{4.1b}$$

The global flux is defined for all $I_n \in \mathcal{T}_\tau(0,T)$ by:

$$\boldsymbol{p}_{h\tau}^{\mathrm{equ}}|_{I_n} := \sum_{a \in \mathcal{S}_h(\Omega)} \boldsymbol{p}_{h\tau,a}^{\mathrm{equ}}|_{I_n}. \tag{4.1c}$$

Here, the stationary flux equilibration procedure depicted in Figure 2 only applies to the piecewise average-in-time of the flux from Definition 4.2. It consists in approximating the localised discrete gradient $-\psi_a \nabla u_{h\tau}$ in the best possible way in $\boldsymbol{H}(\mathrm{div}, \omega_a)$ while keeping the piecewise average-in-time divergence constrained to mimic (1.1a). This is reflected by the equilibration property which follows from (4.1b):

$$\int_{I_n} f_{h\tau}(\cdot,t) - \partial_t s_{h\tau}(\cdot,t) - \mathrm{div}\, \boldsymbol{p}_{h\tau}^{\mathrm{equ}}(\cdot,t)\mathrm{d}t = 0 \text{ on } \Omega \qquad \forall I_n \in \mathcal{T}_\tau(0,T). \tag{4.2}$$

4.3 Residual and flux estimators

We introduce here both the residual- and flux-based estimators. Note that their scaling comes from the choices of norms described in Section 2.7.

Definition 4.3 (Residual-based *a posteriori* estimator)**.** *Let $u_{h\tau} \in V_{h\tau}^{pk}$, $f_{h\tau} \in U_{h\tau}^{pk}$, and $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$. We define the local residual-based estimators as*

$$\eta_{K\times I_n}^{\mathrm{R,res}} := c^{\mathrm{res}} ||f_{h\tau} - \partial_{tt} u_{h\tau} + \Delta u_{h\tau}||_{K\times I_n}, \tag{4.3a}$$

$$(\eta_{K\times I_n}^{\mathrm{JS,res}})^2 := \frac{(c^{\mathrm{res}})^2}{h_K} \sum_{F\in\mathcal{F}_h(K)\cap\mathcal{F}_h^{\mathrm{int}}(\Omega)} ||[\![\nabla u_{h\tau}\cdot \boldsymbol{n}_F]\!]||_{F\times I_n}^2, \tag{4.3b}$$

$$(\eta_{K\times I_n}^{\mathrm{JT,res}})^2 := \frac{(c^{\mathrm{res}})^2}{\tau_n} \sum_{t^k\in\mathcal{S}_\tau(I_n)\setminus\{T\}} ||[\![\partial_t u_{h\tau}]\!]_{t^k}||_K^2, \tag{4.3c}$$

where c^{res} is a positive constant that can only depend on the space dimension d, the polynomial orders p and k, and the shape-regularities of $\mathcal{T}_h(\Omega)$ and $\mathcal{T}_\tau(0,T)$.

Definition 4.4 (Flux-based *a posteriori* estimators)**.** *Let the assumptions of Definition 4.2 be satisfied. Let $\star \in \{\mathrm{avg}, \mathrm{equ}\}$. For $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$, we define the local estimator based on $\boldsymbol{p}_{h\tau}^\star$, respectively from Definitions 4.1 and 4.2 as:*

$$\eta_{K\times I_n}^{\mathrm{R},\star} := c^\star ||f_{h\tau} - \partial_t s_{h\tau} - \mathrm{div}\, \boldsymbol{p}_{h\tau}^\star||_{K\times I_n}, \tag{4.4a}$$

$$(\eta_{K\times I_n}^{\mathrm{JS},\star})^2 := \frac{1}{h_K^2} ||\boldsymbol{p}_{h\tau}^\star + \nabla u_{h\tau}||_{K\times I_n}^2, \tag{4.4b}$$

$$(\eta_{K\times I_n}^{\mathrm{JT},\star})^2 := \frac{1}{\tau_n^2} ||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n}^2. \tag{4.4c}$$

Here, the constants c^{avg} and c^{equ} are

$$c^{\mathrm{avg}} := \sqrt{\max_{K\in\mathcal{T}_h(\Omega)} \sum_{a\in\mathcal{S}_h(K)} \frac{c_{\mathrm{P},a}^2 h_{\omega_a}^2}{h_K^2}} \quad \text{and} \quad c^{\mathrm{equ}} := \frac{1}{\pi}, \tag{4.5}$$

where $c_{\mathrm{P},a}$ denotes the Poincaré constant on ω_a such that for all $v \in H_0^1(\Omega)$,

$$||v - \bar{v}_a||_{\omega_a} \le c_{\mathrm{P},a} h_{\omega_a} ||\nabla v||_{\omega_a} \quad \text{with } \bar{v}_a = \begin{cases} \frac{1}{|\omega_a|}(v,1)_{\omega_a} & \text{if } a \in \mathcal{S}^{\mathrm{int}}(\Omega), \\ 0 & \text{if } a \in \mathcal{S}^{\mathrm{ext}}(\Omega). \end{cases} \tag{4.6}$$

For $\star \in \{\mathrm{res}, \mathrm{avg}, \mathrm{equ}\}$, we also denote the global estimators as:

$$\begin{aligned}(\eta_Q^{\mathrm{R},\star})^2 &:= \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} (\eta_{K\times I_n}^{\mathrm{R},\star})^2,\\ (\eta_Q^{\mathrm{JS},\star})^2 &:= \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} (\eta_{K\times I_n}^{\mathrm{JS},\star})^2,\\ (\eta_Q^{\mathrm{JT},\star})^2 &:= \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} (\eta_{K\times I_n}^{\mathrm{JT},\star})^2,\end{aligned} \tag{4.7}$$

and the local and global aggregate estimators as:

$$\eta_{K\times I_n}^{\star} := \eta_{K\times I_n}^{\mathrm{R},\star} + \sqrt{(\eta_{K\times I_n}^{\mathrm{JS},\star})^2 + (\eta_{K\times I_n}^{\mathrm{JT},\star})^2} \quad \text{and} \quad \eta_Q^{\star} := \eta_Q^{\mathrm{R},\star} + \sqrt{(\eta_Q^{\mathrm{JS},\star})^2 + (\eta_Q^{\mathrm{JT},\star})^2}. \tag{4.8}$$

4.4 Reliability and efficiency

In this section, we extend the results from [23, Theorem 6.3] to the case with a pair of reconstructions $u_{h\tau}$ and $s_{h\tau}$ (to respectively approximate u and $\partial_t u$). Since we replace $\partial_t u_{h\tau}$ by $s_{h\tau}$, we adapt the Galerkin orthogonality from [23, Theorem 6.3] by replacing $\partial_{tt} u_{h\tau}$ by $\partial_t s_{h\tau}$.

We introduce the term $s_{h\tau}$ to correct the H^1-in-time non-conformity of $\partial_t u_{h\tau}$. Therefore, we expect the difference $s_{h\tau} - \partial_t u_{h\tau}$ to be bounded by the residual terms linked to this non-conformity. This is stated in (4.9c). We ensure that $s_{h\tau}$ fits the wave equation using the updated Galerkin orthogonality (4.9b).

In what follows, we denote the space-time patch around $K\times I_n$ by $\mathcal{Q}_{K\times I_n} := \omega_K \times I_n \cup K \times J_{I_n}$, where $J_{I_n} := J_{t^n} \cup J_{t^{n-1}}$.

Theorem 4.5 (Reliable and locally space-time efficient a posteriori estimates)**.** *Let $f \in L^2(Q)$ and $u \in Y_0$ be the weak solution of (2.12). Let*

$$f_{h\tau} \in U_{h\tau}^{pk}, \quad u_{h\tau} \in V_{h\tau}^{pk}, \text{ and } \quad s_{h\tau} \in R_{h\tau}^{pk} \tag{4.9a}$$

be such that for all $I_n \in \mathcal{T}_\tau(0,T)$,

$$\int_{I_n} (f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_\Omega - (\nabla u_{h\tau}(t), \nabla\psi_a)_\Omega \mathrm{d}t = 0 \qquad \forall a \in \mathcal{S}_h^{\mathrm{int}}(\Omega), \tag{4.9b}$$

$$\frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n}^2 \lesssim_p \sum_{t^k\in\mathcal{S}_\tau(I_n)\setminus\{T\}} \frac{1}{\tau_n}||[\![\partial_t u_{h\tau}]\!]_{t^k}||_K^2 \qquad \forall K \in \mathcal{T}_h(\Omega). \tag{4.9c}$$

For $\star \in \{\mathrm{res}, \mathrm{avg}, \mathrm{equ}\}$ (and also $\star = \mathrm{rob}$ under the additional hypotheses from Definition 3.1), the following properties hold for $\eta^\star$:

- ***Global space-time reliability**:*

$$|||u - u_{h\tau}|||_{\mathcal{Y}_0} \le \eta_Q^\star + |||f - f_{h\tau}|||_{Y_T'}; \tag{4.10a}$$

- ***Global space-time efficiency**:*

$$\eta_Q^\star \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0} + |||f - f_{h\tau}|||_{Y_T'}. \tag{4.10b}$$

- ***Local space-time efficiency**: For $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$,*

$$\eta_{K\times I_n}^\star \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0, \mathcal{Q}_{K\times I_n}} + |||f - f_{h\tau}|||_{Y_T', \mathcal{Q}_{K\times I_n}}; \tag{4.10c}$$

Here, $\lesssim_p$ means up to a constant that can only depend on the space dimension d, the polynomial orders p and k, and the shape-regularities of $\mathcal{T}_h(\Omega)$ and $\mathcal{T}_\tau(0,T)$.

Notice that Theorem 4.5 also holds for η^{rob}, independently of Theorem 3.3. This, of course, is only true if $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ exists, which is not ensured by only the hypotheses of Theorem 4.5. Specifically, the pointwise-in-time Galerkin orthogonality from Definition 3.1 needs to be verified, in addition to the hypotheses of Theorem 4.5, for $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ to exist. On the other hand, (3.8) does not need to hold, being replaced by (4.9c).

5 Proof of Theorem 3.3

In this section, we prove the guaranteed upper bound and polynomial-degree robust efficiency from Theorem 3.3.

5.1 Guaranteed upper bound

We first show the guaranteed upper bound of the estimator from Definition 3.2. To achieve this, we recall, in Lemma 5.1, a general reliability result similar to [23, Lemma 7.2]. Then, we show that it applies to the flux $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ from Definition 3.1.

Lemma 5.1 (Guaranteed upper bound for elementwise equilibration in space and time)**.** *Let $f \in L^2(Q)$, $f_{h\tau} \in L^2(Q)$, $u_{h\tau} \in Y_0$, a space flux $\boldsymbol{p}_{h\tau} \in L^2(0,T;\boldsymbol{H}(\mathrm{div},\Omega))$, and a time flux $s_{h\tau} \in H^1(0,T,L^2(\Omega))$ such that $s_{h\tau}(0) = 0$ be given. If, for all $(K,I_n) \in \mathcal{T}_{h\tau}(Q)$,*

$$(f_{h\tau} - \partial_t s_{h\tau} - \mathrm{div}\,\boldsymbol{p}_{h\tau}, 1)_{K\times I_n} = 0, \tag{5.1}$$

then

$$\begin{aligned}|||u-u_{h\tau}|||_{\mathcal{Y}_0} \le |||f-f_{h\tau}|||_{Y_T'} + \frac{1}{\pi}\sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} ||f_{h\tau} - \partial_t s_{h\tau} - \mathrm{div}\,\boldsymbol{p}_{h\tau}||^2_{K\times I_n}}\\ + \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n^2}||s_{h\tau}-\partial_t u_{h\tau}||^2_{K\times I_n} + \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau}+\nabla u_{h\tau}||^2_{K\times I_n}}.\end{aligned}$$

Proof. Let $v \in Y_T$. Since $s_{h\tau} \in H^1(0,T;L^2(\Omega))$ with $s_{h\tau}(0) = 0$ and $\boldsymbol{p}_{h\tau} \in L^2(0,T;\boldsymbol{H}(\mathrm{div},\Omega))$, we obtain by integration by parts

$$\begin{aligned}\langle r(u_{h\tau}), v\rangle &\overset{(2.13)}{=} (f,v)_Q + (\partial_t u_{h\tau}, \partial_t v)_Q - (\nabla u_{h\tau}, \nabla v)_Q\\ &= (f-f_{h\tau}, v)_Q + (\underbrace{f_{h\tau} - \partial_t s_{h\tau} - \mathrm{div}\,\boldsymbol{p}_{h\tau}}_{\xi_{\mathrm{R}}\in L^2(Q)}, v)_Q\\ &\qquad - (\underbrace{\boldsymbol{p}_{h\tau}+\nabla u_{h\tau}}_{\boldsymbol{\xi}_{\mathrm{JS}}\in L^2(Q)^d}, \nabla v)_Q - (\underbrace{s_{h\tau}-\partial_t u_{h\tau}}_{\xi_{\mathrm{JT}}\in L^2(Q)}, \partial_t v)_Q.\end{aligned}$$

Next, $(f-f_{h\tau},v)_Q \le |||f-f_{h\tau}|||_{Y_T}|||v|||_{Y_T}$ by definition, whereas $(\xi_{\mathrm{R}},v)_Q$ is bounded using (5.1) and a space-time Poincaré inequality, just like in the proof of [23, Lemma 7.2]. As for $(\boldsymbol{\xi}_{\mathrm{JS}},\nabla v)_Q$ and $(\xi_{\mathrm{JT}},\partial_t v)_Q$, they are first separately bounded similarly to [23, Lemma 7.2] by

the Cauchy-Schwarz inequality to obtain

$$-(\boldsymbol{\xi}_{\mathrm{JS}}, \nabla v)_Q \leq \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{h_K} ||\boldsymbol{\xi}_{\mathrm{JS}}||_{K\times I_n} \times h_K ||\nabla v||_{K\times I_n},$$
$$-(\xi_{\mathrm{JT}}, \partial_t v)_Q \leq \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{\tau_n} ||\xi_{\mathrm{JT}}||_{K\times I_n} \times \tau_n ||\partial_t v||_{K\times I_n}.$$

Then, we add both terms and conclude by applying the Cauchy–Schwarz inequality twice:

$$\begin{aligned}
&-(\boldsymbol{\xi}_{\mathrm{JS}}, \nabla v)_Q - (\xi_{\mathrm{JT}}, \partial_t v)_Q \\
&\quad\leq \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \left(\frac{1}{h_K} ||\boldsymbol{\xi}_{\mathrm{JS}}||_{K\times I_n} \times h_K ||\nabla v||_{K\times I_n} + \frac{1}{\tau_n} ||\xi_{\mathrm{JT}}||_{K\times I_n} \times \tau_n ||\partial_t v||_{K\times I_n} \right) \\
&\quad\leq \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \sqrt{\frac{1}{h_K^2} ||\boldsymbol{\xi}_{\mathrm{JS}}||^2_{K\times I_n} + \frac{1}{\tau_n^2} ||\xi_{\mathrm{JT}}||^2_{K\times I_n}} \times \sqrt{h_K^2 ||\nabla v||^2_{K\times I_n} + \tau_n^2 ||\partial_t v||^2_{K\times I_n}} \\
&\quad\leq \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \left(\frac{1}{h_K^2} ||\boldsymbol{\xi}_{\mathrm{JS}}||^2_{K\times I_n} + \frac{1}{\tau_n^2} ||\xi_{\mathrm{JT}}||^2_{K\times I_n} \right)} \times |||v|||_{Y_T},
\end{aligned}$$

where we crucially employ the scaling from (2.14). □

Now, to use Lemma 5.1, we only need to verify its hypotheses. The correct spaces containing the functions f, $f_{h\tau}$, $u_{h\tau}$, $s_{h\tau}$, and $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}$ are assumptions of Definition 3.1. The definition of the local contributions of $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}$ in (3.3a) ensures that for all $I_n \in \mathcal{T}_\tau(0,T)$ and all $a \in \mathcal{S}_h(\Omega)$, there holds

$$\big(\operatorname{div} \boldsymbol{p}^{\mathrm{rob}}_{h\tau,a}(t) - \psi_a(f_{h\tau}(t) - \partial_t s_{h\tau}(t)) + \nabla\psi_a \cdot \nabla u_{h\tau}(t), v_h\big)_{\omega_a} = 0 \quad \forall v_h \in U_h^{p+1}|_{\omega_a},\ \forall t \in I_n.$$

Therefore, by summing over $\mathcal{S}_h(\Omega)$, we obtain

$$\big(\operatorname{div} \boldsymbol{p}^{\mathrm{rob}}_{h\tau}(t) - f_{h\tau}(t) + \partial_t s_{h\tau}(t), v_h\big)_{\Omega} = 0 \quad \forall v_h \in U_h^{p+1},\ \forall t \in I_n,$$

which is stronger than (5.1). This establishes the global reliabilities (3.10a) and (3.10c).

5.2 Decoupling time and space components of the flux equilibration

In this section, we use the methodology and pre-existing results from [17] to set up a crucial component of the proof of the efficiency of Theorem 3.3. This is a space-time decoupling of the flux. In the next section, we use it to recall and adapt existing results to prove the polynomial-degree-robust efficiency.

Let $\Phi^n := (\phi^{n,l})_{l\in\mathbb{N}}$ be the orthogonal basis of $\mathcal{P}(I_n, \mathbb{R})$ based on the Legendre polynomials as described in [17, Lemma 4.3]. Let $I_n \in \mathcal{T}_\tau(0,T)$ and $a \in \mathcal{S}_h(\Omega)$. In Sections 5.2 and 5.3, we use the following notation:

$$u_{h\tau}|_{I_n} = (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k} u_h^{n,l}(\boldsymbol{x})\phi^{n,l}(t), \qquad \partial_t s_{h\tau}|_{I_n} = (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k} \dot{s}_h^{n,l}(\boldsymbol{x})\phi^{n,l}(t),$$
$$f_{h\tau}|_{I_n} = (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k} f_h^{n,l}(\boldsymbol{x})\phi^{n,l}(t), \qquad \boldsymbol{p}^{\mathrm{rob}}_{h\tau,a}|_{I_n} = (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k} \boldsymbol{p}_{h,a}^{n,l}(\boldsymbol{x})\phi^{n,l}(t),$$

with

$$u_h^{n,l} \subset V_h^p, \quad \dot{s}_h^{n,l} \subset U_h^p, \quad f_h^{n,l} \subset U_h^p, \quad \boldsymbol{p}_{h,a}^{n,l} \subset \boldsymbol{W}_h^{p+1}|_{\omega_a}, \quad \text{for } l \in [\![0,k]\!].$$

Recall that we define $\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}}|_{I_n}$ in (3.3a) as

$$\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}}|_{I_n} := \operatorname{argmin} \left\{ \sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_{h\tau} + \psi_a \nabla u_{h\tau}||_{K\times I_n}^2,\ \boldsymbol{p}_{h\tau} \in \boldsymbol{W}_{h\tau}^{p+1,k}|_{\omega_a\times I_n} \right.$$
$$\text{with } \boldsymbol{p}_{h\tau}\cdot\boldsymbol{n} = 0 \text{ on } (\partial\omega_a \cap \{\psi_a = 0\}) \times I_n \text{ and}$$
$$\left. \operatorname{div} \boldsymbol{p}_{h\tau} = \psi_a(f_{h\tau} - \partial_t s_{h\tau}) - \nabla\psi_a \cdot \nabla u_{h\tau} \text{ on } \omega_a \times I_n \right\}.$$

Similarly to [17] and [23, Section 7.5], for any $\boldsymbol{p}_{h\tau}|_{I_n} = (\boldsymbol{x},t) \longmapsto \sum_{l=0}^k \boldsymbol{p}_h^{n,l}(\boldsymbol{x})\phi^{n,l}(t) \in \boldsymbol{W}_{h\tau}^{p+1,k}|_{\omega_a\times I_n}$, the minimised functional can be easily decoupled into a sum of independent terms:

$$\sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_{h\tau} + \psi_a \nabla u_{h\tau}||_{K\times I_n}^2 = \sum_{l=0}^k \left(||\phi^{n,l}||_{I_n}^2 \sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_h^{n,l} + \psi_a \nabla u_h^{n,l}||_K^2 \right). \quad (5.2)$$

Using the uniqueness of decomposition in an orthogonal basis, the boundary condition becomes:

$$\boldsymbol{p}_h^{n,l}\cdot\boldsymbol{n} = 0 \text{ on } \partial\omega_a \cap \{\psi_a = 0\} \qquad \forall l \in [\![0,k]\!].$$

Again, using the uniqueness of decomposition in an orthogonal basis, the equilibration constraint becomes:

$$\operatorname{div} \boldsymbol{p}_h^{n,l} = \psi_a(f_h^{n,l} - \dot{s}_h^{n,l}) - \nabla\psi_a \cdot \nabla u_h^{n,l} \text{ on } \omega_a \qquad \forall l \in [\![0,k]\!].$$

Consequently, when decomposed in the basis Φ^n, there is no coupling between components of $\boldsymbol{p}_{h\tau}$ in both the minimised functional and constraints. Therefore we can reformulate (3.3a) equivalently as patchwise space-only problems, the solutions of which are the components of $\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}}|_{I_n}$. For $l \in [\![0,k]\!]$, this gives

$$\boldsymbol{p}_{h,a}^{n,l} := \operatorname{argmin} \left\{ \sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2} ||\boldsymbol{p}_h + \psi_a \nabla u_h^{n,l}||_K^2,\ \boldsymbol{p}_h \in \boldsymbol{W}_h^{p+1}|_{\omega_a} \right.$$
$$\text{with } \boldsymbol{p}_h\cdot\boldsymbol{n} = 0 \text{ on } \partial\omega_a \cap \{\psi_a = 0\} \text{ and}$$
$$\left. \operatorname{div} \boldsymbol{p}_h^{n,l} = \psi_a(f_h^{n,l} - \dot{s}_h^{n,l}) - \nabla\psi_a \cdot \nabla u_h^{n,l} \text{ on } \omega_a \right\}. \quad (5.3)$$

For $I_n \in \mathcal{T}_\tau(0,T)$, we introduce the components of the global flux $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}|_{I_n}$ in Φ^n:

$$\boldsymbol{p}_h^{n,l} := \sum_{a\in\mathcal{S}_h(\Omega)} \boldsymbol{p}_{h,a}^{n,l}, \qquad l \in [\![0,k]\!]. \quad (5.4)$$

5.3 Polynomial-degree-robust stability results

To prove the polynomial-degree-robust efficiency of the patchwise optimal pointwise-in-time equilibrated flux, we first recall the stationary polynomial-degree-robust stability bound found in [17, Lemma 8.1], relying on [8, 18].

Lemma 5.2 (Stationary polynomial-degree-robust stability bound)**.** *Let $d \in \{1,2,3\}$ and let $\mathcal{T}_h(\Omega)$ be a simplicial mesh. Let $a \in \mathcal{S}_h(\Omega)$, $f_h \in U_h^p$, and $\boldsymbol{\xi}_h$ be such that $\boldsymbol{\xi}_h|_K \in \boldsymbol{W}_h^p|_K$ for all $K \in \mathcal{T}_h(\Omega)$. Assume that if $a \in \mathcal{S}_h^{\mathrm{int}}(\Omega)$, there holds*

$$(\boldsymbol{\xi}_h, \nabla\psi_a)_\Omega = (f_h, \psi_a)_\Omega. \tag{5.5}$$

Then, for

$$\boldsymbol{p}_{h,a} := \operatorname{argmin}\Bigg\{ ||\boldsymbol{p}_h + \psi_a\boldsymbol{\xi}_h||_{\omega_a}^2,\ \boldsymbol{p}_h \in \boldsymbol{W}_h^{p+1}|_{\omega_a} \\ \text{with } \boldsymbol{p}_h\cdot\boldsymbol{n} = 0 \text{ on } \partial\omega_a \cap \{\psi_a = 0\} \text{ and} \\ (\operatorname{div}\boldsymbol{p}_h - \psi_a f_h + \nabla\psi_a\cdot\boldsymbol{\xi}_h, v_h)_{\omega_a} = 0 \quad \forall v_h \in U_h^{p+1}|_{\omega_a}\Bigg\}, \tag{5.6}$$

there holds that

$$||\boldsymbol{p}_{h,a} + \psi_a\boldsymbol{\xi}_h||_{\omega_a} \lesssim \sup_{v\in H_0^1(\omega_a)\setminus\{0\}} \frac{(f_h, v)_{\omega_a} - (\boldsymbol{\xi}_h, \nabla v)_{\omega_a}}{||\nabla v||_{\omega_a}}. \tag{5.7}$$

Here, $\lesssim$ means up to a constant that can only depend on the space dimension d and the shape-regularity of $\mathcal{T}_h(\Omega)$ ***but does not depend on the polynomial degree*** *p.*

We now use Lemma 5.2 to prove the following anisotropic space-time stability result.

Lemma 5.3 (Anisotropic space-time polynomial-degree-robust stability bound)**.** *Let $d \in \{1,2,3\}$ and let $\mathcal{T}_h(\Omega)$ be a simplicial mesh. Let $(a, I_n) \in \mathcal{S}_h(\Omega)\times\mathcal{T}_\tau(0,T)$. Let $f_{h\tau} \in U_{h\tau}^{pk}$, $s_{h\tau} \in R_{h\tau}^{pk}$, and $u_{h\tau} \in V_{h\tau}^{pk}$ be such that for all $a \in \mathcal{S}_h^{\mathrm{int}}(\Omega)$, there holds*

$$(f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_{\omega_a} - (\nabla u_{h\tau}(t), \nabla\psi_a)_{\omega_a} = 0 \qquad \forall t \in I_n. \tag{5.8}$$

Then, there holds:

$$\sqrt{\sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a\nabla u_{h\tau}||_{K\times I_n}^2} \lesssim \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega_a\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}}. \tag{5.9}$$

Here, $\lesssim$ means up to a constant that can only depend on the space dimension d, the polynomial order k, and the shape-regularity of $\mathcal{T}_h(\Omega)$ ***but does not depend on the polynomial degree*** *p. This result specifically does not depend on the shape regularity of the time mesh $\mathcal{T}_\tau(0,T)$.*

Proof. Let $I_n \in \mathcal{T}_\tau(0,T)$, $l \in [\![0,k]\!]$, and $a \in \mathcal{S}_h(\Omega)$. Denote $\xi^{n,l} \in H_0^1(\omega_a)$ such that,

$$(\nabla\xi^{n,l}, \nabla v)_{\omega_a} := (f_h^{n,l} - \dot{s}_h^{n,l}, v)_{\omega_a} - (\nabla u_h^{n,l}, \nabla v)_\Omega \qquad \forall v \in H_0^1(\omega_a). \tag{5.10}$$

We denote as $\mathfrak{p}_{h,a}^{n,l} \in \boldsymbol{W}_h^{p+1}|_{\omega_a}$, the solution of (5.6) for $f_h = f_h^{n,l} - \dot{s}_h^{n,l}$ and $\boldsymbol{\xi}_h = \nabla u_h^{n,l}$. Then, there holds:

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\boldsymbol{p}_{h,a}^{n,l} + \psi_a\nabla u_h^{n,l}||_K^2 &\le \sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\mathfrak{p}_{h,a}^{n,l} + \psi_a\nabla u_h^{n,l}||_K^2 \\
&\le \left(\frac{1}{\min_{K\in\mathcal{T}_h(\omega_a)} h_K}\right)^2 ||\mathfrak{p}_{h,a}^{n,l} + \psi_a\nabla u_h^{n,l}||_{\omega_a}^2 \\
&\overset{\text{Lemma 5.2 and (5.10)}}{\lesssim} \left(\frac{1}{\min_{K\in\mathcal{T}_h(\omega_a)} h_K}\right)^2 ||\nabla\xi^{n,l}||_{\omega_a}^2.
\end{aligned} \tag{5.11}$$

We denote

$$\xi_\tau^n := (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k} \phi^{n,l}(t)\xi^{n,l}(\boldsymbol{x}) \in \mathcal{P}_k(I_n, H_0^1(\omega_a)). \tag{5.12}$$

Then, by (5.11), there holds

$$\begin{aligned}\sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a\nabla u_{h\tau}||_{K\times I_n}^2 &\overset{(5.2)}{=} \sum_{l=0}^{k}\sum_{K\in\mathcal{T}_h(\omega_a)} ||\phi^{n,l}||_{I_n}^2 \frac{1}{h_K^2}||\boldsymbol{p}_{h,a}^{n,l} + \psi_a\nabla u_h^{n,l}||_K^2 \\ &\overset{(5.11)}{\lesssim} \left(\frac{1}{\min_{K\in\mathcal{T}_h(\omega_a)} h_K}\right)^2 \sum_{l=0}^{k} ||\phi^{n,l}||_{I_n}^2 ||\nabla\xi^{n,l}||_{\omega_a}^2 \\ &\overset{(5.12)}{\lesssim} \left(\frac{1}{\min_{K\in\mathcal{T}_h(\omega_a)} h_K}\right)^2 ||\nabla\xi_\tau^n||_{\omega_a\times I_n}^2.\end{aligned} \tag{5.13}$$

We now want to introduce another function $\xi_{\tau,\mathrm{c}}^n$ that is

- equivalent to ξ_τ^n in the $||\nabla\cdot||_{\omega_a\times I_n}$ norm,
- supported in I_n, vanishes at t^{n-1} and t^n.

To achieve this, we introduce, for $t \in I_n$,

$$\xi_{\tau,\mathrm{c}}^n(t) := \xi_\tau^n(t) - \frac{(-1)^k}{2}[\phi^{n,k+2}(t) - \phi^{n,k+1}(t)]\xi_\tau^n(t^{n-1}) - \frac{1}{2}[\phi^{n,k+2}(t) + \phi^{n,k+1}(t)]\xi_\tau^n(t^n). \tag{5.14}$$

For simplicity of notation later, we denote

$$\begin{aligned}\xi^{n,k+1} &:= \frac{1}{2}\left((-1)^k\xi_\tau^n(t^{n-1}) - \xi_\tau^n(t^n)\right), \\ \xi^{n,k+2} &:= -\frac{1}{2}\left((-1)^k\xi_\tau^n(t^{n-1}) + \xi_\tau^n(t^n)\right),\end{aligned}$$

so that $\xi_{\tau,\mathrm{c}}^n$ can be written as follows

$$\xi_{\tau,\mathrm{c}}^n := (\boldsymbol{x}, t) \longmapsto \sum_{l=0}^{k+2} \phi^{n,l}(t)\xi^{n,l}(\boldsymbol{x}) \in \mathcal{P}_{k+2}(I_n, H_0^1(\omega_a)). \tag{5.15}$$

Since the $(\phi^{n,l})_{l\in[\![0,k+2]\!]}$ are an $L^2(I_n)$-orthogonal basis of $\mathcal{P}_{k+2}(I_n,\mathbb{R})$, there holds

$$(\nabla\xi_{\tau,\mathrm{c}}^n - \nabla\xi_\tau^n, \nabla\xi_\tau^n)_{\omega_a\times I_n} = \sum_{l=k+1}^{k+2}\sum_{m=0}^{k} \left(\nabla\xi_\tau^{n,l}, \nabla\xi^{n,m}\right)_{\omega_a} \underbrace{(\phi^{n,l}, \phi^{n,m})_{I_n}}_{=0 \text{ if } l\neq m} = 0.$$

Therefore, there holds

$$(\nabla\xi_\tau^n, \nabla\xi_\tau^n)_{\omega_a\times I_n} = (\nabla\xi_\tau^n, \nabla\xi_{\tau,\mathrm{c}}^n)_{\omega_a\times I_n} = (f_{h\tau} - \partial_t s_{h\tau}, \xi_{\tau,\mathrm{c}}^n)_{\omega_a\times I_n} - (\nabla u_{h\tau}, \nabla\xi_{\tau,\mathrm{c}}^n)_{\omega_a\times I_n},$$

and $\xi_{\tau,\mathrm{c}}^n$ vanishes at t^{n-1} and t^n. Therefore $\xi_{\tau,\mathrm{c}}^n$ is in $Y_T\backslash\{0\}$ with $\mathrm{supp}(\xi_{\tau,\mathrm{c}}^n) \subset \omega_a \times I_n$ and

$$(\nabla\xi_\tau^n, \nabla\xi_\tau^n)_{\omega_a\times I_n} \leq \sup_{\substack{v\in Y_T\backslash\{0\} \\ \mathrm{supp}(v)\subset\omega_a\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}} |||\xi_{\tau,\mathrm{c}}^n|||_{Y_T,\omega_a\times I_n}. \tag{5.16}$$

We apply an inverse inequality in time and the Poincaré inequality to get that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h(\omega_a)} \tau_n^2||\partial_t\xi^n_{\tau,\mathrm{c}}||^2_{K\times I_n} &\lesssim ||\xi^n_{\tau,\mathrm{c}}||^2_{\omega_a\times I_n} \lesssim h^2_{\omega_a}||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{\omega_a\times I_n}\\ &\lesssim c_a^2 \sum_{K\in\mathcal{T}_h(\omega_a)} h_K^2||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{K\times I_n},\end{aligned} \tag{5.17}$$

with $c_a^2 := \max_{K\in\mathcal{T}_h(\omega_a)} \left(\frac{h_{\omega_a}}{h_K}\right)^2$ which also does not depend on p. Then

$$\begin{aligned}|||\xi^n_{\tau,\mathrm{c}}|||^2_{Y_T,\omega_a\times I_n} &\overset{(2.14b)}{=} \sum_{K\in\mathcal{T}_h(\omega_a)} \tau_n^2||\partial_t\xi^n_{\tau,\mathrm{c}}||^2_{K\times I_n} + h_K^2||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{K\times I_n}\\ &\overset{(5.17)}{\lesssim} \sum_{K\in\mathcal{T}_h(\omega_a)} h_K^2||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{K\times I_n}\\ &\lesssim \left(\max_{K\in\mathcal{T}_h(\omega_a)} h_K\right)^2 ||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{\omega_a\times I_n}.\end{aligned} \tag{5.18}$$

We use the reformulation of $\xi^n_{\tau,\mathrm{c}}$ in (5.15) and the orthogonality of $(\phi^{n,l})_{l\in[\![0,k+2]\!]}$, to get

$$||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{\omega_a\times I_n} = \sum_{l=0}^{k+2} ||\nabla\xi^{n,l}||^2_{\omega_a}||\phi^{n,l}||^2_{I_n} = ||\nabla\xi^n_\tau||^2_{\omega_a\times I_n} + \sum_{l=k+1}^{k+2} ||\nabla\xi^{n,l}||^2_{\omega_a}||\phi^{n,l}||^2_{I_n}. \tag{5.19}$$

Then, using Young's inequality, we have

$$\begin{aligned}||\nabla\xi^{n,k+1}||^2_{\omega_a} &= \frac{1}{4}\left|\left|(-1)^k\nabla\xi^n_\tau(t^{n-1}) - \nabla\xi^n_\tau(t^n)\right|\right|^2_{\omega_a} \le \frac{1}{2}\left(||\nabla\xi^n_\tau(t^{n-1})||^2_{\omega_a} + ||\nabla\xi^n_\tau(t^n)||^2_{\omega_a}\right),\\ ||\nabla\xi^{n,k+2}||^2_{\omega_a} &= \frac{1}{4}\left|\left|(-1)^k\nabla\xi^n_\tau(t^{n-1}) + \nabla\xi^n_\tau(t^n)\right|\right|^2_{\omega_a} \le \frac{1}{2}\left(||\nabla\xi^n_\tau(t^{n-1})||^2_{\omega_a} + ||\nabla\xi^n_\tau(t^n)||^2_{\omega_a}\right),\end{aligned}$$

that we reinject in (5.19) to obtain

$$||\nabla\xi^n_{\tau,\mathrm{c}}||^2_{\omega_a\times I_n} \le ||\nabla\xi^n_\tau||^2_{\omega_a\times I_n} + \frac{1}{2}\left(||\phi^{n,k+1}||^2_{I_n} + ||\phi^{n,k+2}||^2_{I_n}\right)\left(||\nabla\xi^n_\tau(t^{n-1})||^2_{\omega_a} + ||\nabla\xi^n_\tau(t^n)||^2_{\omega_a}\right).$$

We apply a trace inequality in time followed by an inverse inequality in time to obtain:

$$||\nabla\xi^n_\tau(t^{n-1})||^2_{\omega_a} + ||\nabla\xi^n_\tau(t^n)||^2_{\omega_a} \lesssim \frac{1}{\tau_n}||\nabla\xi^n_\tau||^2_{\omega_a\times I_n} + \tau_n||\partial_t\nabla\xi^n_\tau||^2_{\omega_a\times I_n} \lesssim \frac{1}{\tau_n}||\nabla\xi^n_\tau||^2_{\omega_a\times I_n}.$$

Since we also have $\left(||\phi^{n,k+1}||^2_{I_n} + ||\phi^{n,k+2}||^2_{I_n}\right) = \tau_n\left(\frac{1}{2k+3} + \frac{1}{2k+5}\right)$ (see [1, 22.1.2 and 22.2.10]), there holds

$$||\nabla\xi^n_{\tau,\mathrm{c}}||_{\omega_a\times I_n} \lesssim ||\nabla\xi^n_\tau||_{\omega_a\times I_n}. \tag{5.20}$$

We conclude, using (5.13), (5.16), (5.18), and (5.20), that

$$\sqrt{\sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\boldsymbol{p}^{\mathrm{rob}}_{h\tau,a} + \psi_a\nabla u_{h\tau}||^2_{K\times I_n}} \lesssim \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega_K\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}},$$

also using that c_a and $\frac{\max_{K\in\mathcal{T}_h(\omega_a)} h_K}{\min_{K\in\mathcal{T}_h(\omega_a)} h_K}$ only depend on the shape regularity of $\mathcal{T}_h(\Omega)$. □

Finally, we use Lemma 5.3 to prove the space-time efficiency of the flux $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ from Definition 3.1.

Lemma 5.4 (Space-time polynomial-degree-robust efficiency)**.** *Let $d \in \{1,2,3\}$ and let $\mathcal{T}_h(\Omega)$ be a simplicial mesh. Let $f_{h\tau} \in U_{h\tau}^{pk}$, $s_{h\tau} \in R_{h\tau}^{pk}$, and $u_{h\tau} \in V_{h\tau}^{pk}$ be such that*

$$(f_{h\tau}(t) - \partial_t s_{h\tau}(t), \psi_a)_{\omega_a} - (\nabla u_{h\tau}(t), \nabla \psi_a)_{\omega_a} = 0 \qquad \forall I_n \in \mathcal{T}_\tau(0,T),\ \forall t \in I_n,\ \forall a \in \mathcal{S}_h^{\mathrm{int}}(\Omega). \tag{5.21}$$

Then, there holds:

- ***Local efficiency**: For $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$,*

$$\frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n} \lesssim \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega_a\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}}.$$

- ***Global efficiency**:*

$$\sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n}^2} \lesssim \sup_{v\in Y_T\setminus\{0\}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}}.$$

*Here, $\lesssim$ means up to a constant that can only depend on the space dimension d, the polynomial order k, and the shape-regularity of $\mathcal{T}_h(\Omega)$ **but does not depend on the polynomial degree p and the shape regularity of the time mesh** $\mathcal{T}_\tau(0,T)$.*

Proof. Let $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$. Then, there holds

$$\begin{aligned}
\frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n} &\le \sum_{a\in\mathcal{S}_h(K)} \frac{1}{h_K}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a \nabla u_{h\tau}||_{K\times I_n} \\
&\le \sum_{a\in\mathcal{S}_h(K)} \sqrt{\sum_{K'\in\mathcal{T}_h(\omega_a)} \frac{1}{h_{K'}^2}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a \nabla u_{h\tau}||_{K'\times I_n}^2} \\
&\overset{(5.9)}{\lesssim} \sum_{a\in\mathcal{S}_h(K)} \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega_a\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}} \\
&\lesssim (d+1) \sup_{\substack{v\in Y_T\setminus\{0\}\\ \mathrm{supp}(v)\subset\omega_K\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}}.
\end{aligned}$$

This gives the local efficiency.

For the global efficiency, let $\xi \in Y_T$ be the Riesz representer of the functional $v \mapsto (f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q$, in the scalar product over Y_T corresponding to (2.14a) and (2.14b). This means that for all $v \in Y_T$, there holds

$$\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \tau_n^2(\partial_t\xi, \partial_t v)_{K\times I_n} + h_K^2(\nabla\xi, \nabla v)_{K\times I_n} = (f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q.$$

Recall that this is merely a weighted $H^1(Q)$ scalar product, so that existence and uniqueness of ξ are immediate. By applying the Cauchy–Schwarz inequality, we obtain that for any $a \in \mathcal{S}_h(\Omega)$,

$I_n \in \mathcal{T}_\tau(0,T)$, and $v \in Y_T$ such that $\mathrm{supp}(v) \subset \omega_a \times I_n$,

$$\begin{aligned}(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q &= \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \tau_n^2(\partial_t\xi, \partial_t v)_{K\times I_n} + h_K^2(\nabla\xi, \nabla v)_{K\times I_n} \\ &\overset{\mathrm{supp}(v)}{=} \sum_{K\in\mathcal{T}_h(\omega_a)} \tau_n^2(\partial_t\xi, \partial_t v)_{K\times I_n} + h_K^2(\nabla\xi, \nabla v)_{K\times I_n} \\ &\leq |||\xi|||_{Y_T,\omega_a\times I_n}|||v|||_{Y_T,\omega_a\times I_n} \\ &\overset{\mathrm{supp}(v)}{=} |||\xi|||_{Y_T,\omega_a\times I_n}|||v|||_{Y_T}.\end{aligned} \tag{5.22}$$

Then, we conclude using

$$\begin{aligned}&\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n}^2 \\ &\quad\lesssim_p \sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \sum_{a\in\mathcal{S}_h(K)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a\nabla u_{h\tau}||_{K\times I_n}^2 \\ &\quad= \sum_{(a,I_n)\in\mathcal{S}_h(\Omega)\times\mathcal{T}_\tau(0,T)} \sum_{K\in\mathcal{T}_h(\omega_a)} \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau,a}^{\mathrm{rob}} + \psi_a\nabla u_{h\tau}||_{K\times I_n}^2 \\ &\quad\overset{(5.9)}{\lesssim} \sum_{(a,I_n)\in\mathcal{S}_h(\Omega)\times\mathcal{T}_\tau(0,T)} \left(\sup_{\substack{v\in Y_T\setminus\{0\} \\ \mathrm{supp}(v)\subset\omega_a\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}} \right)^2 \\ &\quad\overset{(5.22)}{\leq} \sum_{(a,I_n)\in\mathcal{S}_h(\Omega)\times\mathcal{T}_\tau(0,T)} |||\xi|||_{Y_T,\omega_a\times I_n}^2 \\ &\quad= (d+1)|||\xi|||_{Y_T}^2 \\ &\quad= (d+1)\left(\sup_{v\in Y_T\setminus\{0\}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}} \right)^2.\end{aligned}$$

□

Here, we obtain (5.22) using a unique global Riesz representation. Another way is to define local Riesz representations and sum their contributions as in [6, 11].

5.4 Polynomial-degree-robust efficiency

We now prove the efficiency results (3.10b) and (3.10d). Lemma 5.4 shows that for $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$,

$$\frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{rob}} + \nabla u_{h\tau}||_{K\times I_n} \lesssim \sup_{\substack{v\in Y_T\setminus\{0\} \\ \mathrm{supp}(v)\subset\omega_K\times I_n}} \frac{(f_{h\tau} - \partial_t s_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q}{|||v|||_{Y_T}}.$$

By applying the triangle inequality, integrating by parts the term with $\partial_t s_{h\tau}$, and using the definition (2.14d) of the error norms, we get

$$\begin{aligned}
&\sup_{\substack{v\in Y_T\setminus\{0\}\\ \operatorname{supp}(v)\subset\omega_K\times I_n}} \frac{(f_{h\tau}-\partial_t s_{h\tau},v)_Q-(\nabla u_{h\tau},\nabla v)_Q}{|||v|||_{Y_T}}\\
&\qquad\le |||r(u_{h\tau})|||_{Y_T',\omega_K\times I_n}+|||f-f_{h\tau}|||_{Y_T',\omega_K\times I_n}+\sup_{\substack{v\in Y_T\setminus\{0\}\\ \operatorname{supp}(v)\subset\omega_K\times I_n}}\frac{(s_{h\tau}-\partial_t u_{h\tau},\partial_t v)_Q}{|||v|||_{Y_T}},\\
&\qquad\le |||u-u_{h\tau}|||_{\mathcal{Y}_0,\omega_K\times I_n}+|||f-f_{h\tau}|||_{Y_T',\omega_K\times I_n}+\frac{1}{\tau_n}||s_{h\tau}-\partial_t u_{h\tau}||_{\omega_K\times I_n}.
\end{aligned}$$

We add $\frac{1}{\tau_n}||s_{h\tau}-\partial_t u_{h\tau}||_{K\times I_n}$ on both sides to obtain (3.10d).

The global efficiency is obtained by (5.9), by proceeding as above.

6 Proof of Theorem 4.5

In this section, we prove the reliability and efficiency from Theorem 4.5.

6.1 Reliability of the equilibrated fluxes

First, recall that the reliability of η^{rob} is shown in Section 5.1.

Next, we apply the general reliability result for equilibrated fluxes from Lemma 5.1 to the flux $\boldsymbol{p}_{h\tau,a}^{\mathrm{equ}}|_{I_n}$ from Definition 4.2. The definition of the local contributions of $\boldsymbol{p}_{h\tau}^{\mathrm{equ}}$ in (4.1b) ensures that for all $I_n\in\mathcal{T}_\tau(0,T)$ and all $a\in\mathcal{S}_h(\Omega)$, there holds

$$\int_{I_n}\Big(\operatorname{div}\boldsymbol{p}_{h\tau,a}^{\mathrm{equ}}(t)-\psi_a(f_{h\tau}(t)-\partial_t s_{h\tau}(t))+\nabla\psi_a\cdot\nabla u_{h\tau}(t),v_h\Big)_{\omega_a}\,\mathrm{d}t=0\quad\forall v_h\in U_h^{p+1}|_{\omega_a}.$$

Therefore, by summing over $\mathcal{S}_h(\Omega)$, since $\sum_{a\in\mathcal{S}_h(\Omega)}\psi_a=1$, we obtain

$$\int_{I_n}(\operatorname{div}\boldsymbol{p}_{h\tau}(t)-f_{h\tau}(t)+\partial_t s_{h\tau}(t),v_h)_\Omega\,\mathrm{d}t=0\quad\forall v_h\in U_h^{p+1},$$

which verifies the equilibration hypothesis (5.1) of Lemma 5.1. Therefore, (4.10a) holds for both $\boldsymbol{p}_{h\tau}^{\mathrm{equ}}$ and $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$.

6.2 Reliability of the averaged-in-space flux

Similarly to the reliability of the equilibrated fluxes, we introduce a general reliability result in Lemma 6.1, adapted from [23, Lemma 7.3]. This is used to show the reliability of the averaged-in-space flux.

Lemma 6.1 (Guaranteed upper bound for patchwise equilibration in space)**.** *Let $f\in L^2(Q)$, $f_{h\tau}\in L^2(Q)$, $u_{h\tau}\in Y_0$, a space flux $\boldsymbol{p}_{h\tau}\in L^2(0,T;\boldsymbol{H}(\operatorname{div},\Omega))$, and a time flux $s_{h\tau}\in H^1(0,T,L^2(\Omega))$ such that $s_{h\tau}(0)=0$ be given. If*

$$\int_{I_n}(f_{h\tau}(t)-\partial_t s_{h\tau}(t)-\operatorname{div}\boldsymbol{p}_{h\tau}(t),\psi_a)_{\omega_a}\mathrm{d}t=0\quad\forall(a,I_n)\in\mathcal{S}_h^{\mathrm{int}}(\Omega)\times\mathcal{T}_\tau(0,T),\tag{6.1}$$

holds, then

$$|||u-u_{h\tau}|||_{\mathcal{Y}_0} \le |||f-f_{h\tau}|||_{Y_T'} + c^{\mathrm{avg}} \times \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} ||f_{h\tau}-\partial_t s_{h\tau}-\operatorname{div}\boldsymbol{p}_{h\tau}||^2_{K\times I_n}}$$
$$+ \sqrt{\sum_{(K,I_n)\in\mathcal{T}_{h\tau}(Q)} \left(\frac{1}{\tau_n^2}||s_{h\tau}-\partial_t u_{h\tau}||^2_{K\times I_n} + \frac{1}{h_K^2}||\boldsymbol{p}_{h\tau}+\nabla u_{h\tau}||^2_{K\times I_n}\right)}.$$

where c^{avg} is given by (4.5).

Proof. This proof is the same as for Lemma 5.1, except for the term $(f_{h\tau}-\partial_t s_{h\tau}-\operatorname{div}\boldsymbol{p}_{h\tau}, v)_Q$, which is bounded in the same way as $(f_{h\tau}-\partial_{tt}u_{h\tau}-\operatorname{div}\boldsymbol{p}_{h\tau}, v)_Q$ in the proof of [23, Lemma 7.3]. □

We only need to ensure that the hypothesis (6.1) of Lemma 6.1 holds to prove the reliability of $\boldsymbol{p}_{h\tau}^{\mathrm{avg}}$. Let $a\in\mathcal{S}_h^{\mathrm{int}}(\Omega)$. Since ψ_a is in V_h^p, $\nabla\psi_a$ itself is in $\boldsymbol{Z}_h^p$ elementwise. Then, by Definition 4.1, we know that

$$\int_{I_n} (\nabla u_{h\tau}(t)+\boldsymbol{p}_{h\tau}^{\mathrm{avg}}(t), \nabla\psi_a)_{\omega_a}\mathrm{d}t = 0.$$

Using the assumption (4.9b), it follows that

$$\int_{I_n} (f_{h\tau}(t)-\partial_t s_{h\tau}(t)-\operatorname{div}\boldsymbol{p}_{h\tau}^{\mathrm{avg}}(t), \psi_a)_{\omega_a}\mathrm{d}t = 0.$$

Therefore, Lemma 6.1 shows the reliability (4.10a) of $\boldsymbol{p}_{h\tau}^{\mathrm{avg}}$.

6.3 Efficiency of the residual estimator

In [23, Sections 7.2 and 7.3], we already prove the efficiencies (4.10b) and (4.10c) of (4.3a) and (4.3b). To complete the proof of the efficiency of the residual-based estimator in Definition 4.3, we only need to prove it for the time jump term in (4.3c). Again, we use a discrete bubble function defined on the patch around each internal space-time face of the form $K\times\{t^n\}$ for $K\in\mathcal{T}_h(\Omega)$ and $t^n\in\mathcal{S}_\tau(0,T)\setminus\{T\}$.

Lemma 6.2 (Efficiency of the time jump residual)**.** *Let $f\in L^2(Q)$, $f_{h\tau}\in U_{h\tau}^{pk}$, and $u_{h\tau}\in V_{h\tau}^{pk}$. Then there holds:*

- ***Local efficiency**: For $(K,t^n)\in\mathcal{T}_h(\Omega)\times\mathcal{S}_\tau(0,T)\setminus\{0,T\}$,*

$$\frac{1}{\sqrt{\tau_n+\tau_{n+1}}}||[\![\partial_t u_{h\tau}]\!]_{t^n}||_K \lesssim_p |||r(u_{h\tau})|||_{Y_T',K\times(t^{n-1},t^{n+1})} + |||f-f_{h\tau}|||_{Y_T',K\times(t^{n-1},t^{n+1})}, \quad (6.2)$$

and for $[\![\partial_t u_{h\tau}]\!]_{t^0} := \partial_t u_{h\tau}|_{I_1}(t^0) - 0$, there holds

$$\frac{1}{\sqrt{\tau_1}}||[\![\partial_t u_{h\tau}]\!]_{t^0}||_K \lesssim_p |||r(u_{h\tau})|||_{Y_T',K\times I_1} + |||f-f_{h\tau}|||_{Y_T',K\times I_1}. \quad (6.3)$$

- ***Global efficiency**: For simplicity, denote $\tau_0 := 0$. Then*

$$\sqrt{\sum_{(K,t^n)\in\mathcal{T}_h(\Omega)\times\mathcal{S}_\tau(0,T)\setminus\{T\}} \frac{1}{\tau_n+\tau_{n+1}}||[\![\partial_t u_{h\tau}]\!]_{t^n}||_K^2} \lesssim_p |||r(u_{h\tau})|||_{Y_T'} + |||f-f_{h\tau}|||_{Y_T'}. \quad (6.4)$$

Here, $\lesssim_p$ means up to a constant that can only depend on the space dimension d, the polynomial orders p and k, and the shape-regularity of $\mathcal{T}_h(\Omega)$.

Proof. To prove the bound around t^0, we introduce the finite-dimensional local function

$$\psi_K^0 := (\boldsymbol{x}, t) \longmapsto \psi^0(t) \prod_{a \in \mathcal{S}_h(K)} \psi_a(\boldsymbol{x}).$$

Then we introduce v_K^0, the extension by constants of $[\![\partial_t u_{h\tau}]\!]_{t^0}$ from K to $K \times I_1$. By equivalence of norms in finite dimension and integration by parts, there holds:

$$||[\![\partial_t u_{h\tau}]\!]_{t^0}||_K^2 \lesssim_p (\partial_t u_{h\tau}, \psi_K^0 v_K^0)_{K\times\{t^0\}} = -(\partial_t u_{h\tau}, \partial_t(\psi_K^0 v_K^0))_{K\times I_1} - (\partial_{tt} u_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1}.$$

There, just like in the proof of [23, Lemma 7.8], we use

$$\begin{aligned}
0 &= (\nabla u_{h\tau}, \nabla(\psi_K^0 v_K^0))_{K\times I_1} + (\Delta u_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1}, \\
0 &= (f, \psi_K^0 v_K^0)_{K\times I_1} - (f, \psi_K^0 v_K^0)_{K\times I_1}, \\
0 &= (f_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1} - (f_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1}, \\
r_{h\tau}|_{K\times I_1} &:= (f_{h\tau} - \partial_{tt} u_{h\tau} + \Delta u_{h\tau})\,|_{K\times I_1}
\end{aligned}$$

to get

$$||[\![\partial_t u_{h\tau}]\!]_{t^0}||_K^2 \lesssim_p -\langle r(u_{h\tau}), \psi_K^0 v_K^0\rangle_{Y_T, K\times I_1} + (f - f_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1} + (r_{h\tau}, \psi_K^0 v_K^0)_{K\times I_1}.$$

Recall that $\psi_K^0 v_K^0 \in H^1(K \times I_1)$ with $\psi_K^0 v_K^0 = 0$ on $\partial(K \times I_1) \setminus K \times \{t^0\}$. Therefore, it can be extended by zero to obtain $\psi_K^0 v_K^0 \in Y_T$ with $\mathrm{supp}(\psi_K^0 v_K^0) \subset K \times I_1$, so that the following holds too

$$||[\![\partial_t u_{h\tau}]\!]_{t^0}||_K^2 \lesssim_p -\langle r(u_{h\tau}), \psi_K^0 v_K^0\rangle_{Y_T} + (f - f_{h\tau}, \psi_K^0 v_K^0)_Q + (r_{h\tau}, \psi_K^0 v_K^0)_Q. \tag{6.5}$$

We similarly show that for each $t^n \in \mathcal{S}_\tau(0,T) \setminus \{0, T\}$, $\psi_K^n v_K^n \in Y_T$ with $\mathrm{supp}(\psi_K^0 v_K^0) \subset K \times (t^{n-1}, t^{n+1})$, and there holds

$$||[\![\partial_t u_{h\tau}]\!]_{t^n}||_K^2 \lesssim_p -\langle r(u_{h\tau}), \psi_K^n v_K^n\rangle_{Y_T} + (f - f_{h\tau}, \psi_K^n v_K^n)_Q + (r_{h\tau}, \psi_K^n v_K^n)_Q. \tag{6.6}$$

We then get (6.2) and (6.3) using

- the Cauchy–Schwarz inequality,
- [23, Lemma 7.6] to bound $r_{h\tau}$,
- local inverse inequalities on $K \times I_n$ and $K \times I_{n+1}$,
- and the definition of the extension by constants.

To get (6.4), the procedure is similar to the proof of [23, Lemma 7.8] for the global bound. The idea is to:

- sum all local contributions from (6.5) and (6.6),
- use the Cauchy–Schwarz inequality and [23, Lemma 7.6] to bound $r_{h\tau}$,
- use (2.14) to bound the other terms,
- and use the definition of the extension by constants to conclude.

□

6.4 Efficiency of the averaged flux estimator

In this section, we recall and generalise the results found in [23, Section 7.4]. This allows to prove the efficiencies (4.10b) and (4.10c) of the averaged-in-space flux estimators from Definitions 4.1 and 4.4 in Theorem 4.5. Let $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$. Just like in [23, Section 7.4], we use the triangle inequality to show that

$$\begin{aligned}||f_{h\tau} - \partial_t s_{h\tau} - \operatorname{div} \boldsymbol{p}_{h\tau}^{\mathrm{avg}}||_{K\times I_n} \leq\ & ||f_{h\tau} - \partial_{tt} u_{h\tau} + \Delta u_{h\tau}||_{K\times I_n} \\ & + ||\partial_t(s_{h\tau} - \partial_t u_{h\tau})||_{K\times I_n} + ||\operatorname{div}(\boldsymbol{p}_{h\tau}^{\mathrm{avg}} + \nabla u_{h\tau})||_{K\times I_n}.\end{aligned}$$

Then, by applying elementwise inverse inequalities, we get that

$$\begin{aligned}||f_{h\tau} - \partial_t s_{h\tau} - \operatorname{div} \boldsymbol{p}_{h\tau}^{\mathrm{avg}}||_{K\times I_n} \leq\ & ||f_{h\tau} - \partial_{tt} u_{h\tau} + \Delta u_{h\tau}||_{K\times I_n} \\ & + \frac{1}{\tau_n}||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n} + \frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{avg}} + \nabla u_{h\tau}||_{K\times I_n}.\end{aligned}$$

Under the assumption (4.9c), namely that

$$\frac{1}{\tau_n^2}||s_{h\tau} - \partial_t u_{h\tau}||_{K\times I_n}^2 \lesssim_p \sum_{t^k \in \mathcal{S}_\tau(I_n)\setminus\{T\}} \frac{1}{\tau_n}|||[\![\partial_t u_{h\tau}]\!]|||_{K\times\{t^k\}}^2 \qquad \forall (K, I_n) \in \mathcal{T}_{h\tau}(Q), \tag{6.7}$$

and recalling the notation of Definition 4.3 and (4.8), the previous inequality becomes:

$$||f_{h\tau} - \partial_t s_{h\tau} - \operatorname{div} \boldsymbol{p}_{h\tau}^{\mathrm{avg}}||_{K\times I_n} \lesssim_p \eta_{K\times I_n}^{\mathrm{res}} + \frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{avg}} + \nabla u_{h\tau}||_{K\times I_n}. \tag{6.8}$$

Therefore, the local and global efficiencies of the estimator η^{avg}, again, follow from $\frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{avg}} + \nabla u_{h\tau}||_{K\times I_n}$ being bounded from above by $\eta_{K\times I_n}^{\mathrm{res}}$.

Lemma 6.3 (Local boundedness of the averaged flux estimator)**.** *Let $f_{h\tau} \in U_{h\tau}^{pk}$, $u_{h\tau} \in V_{h\tau}^{pk}$, $s_{h\tau} \in R_{h\tau}^{pk}$ and let assumption (4.9c) hold. Then, for all $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$, there holds*

$$\eta_{K\times I_n}^{\mathrm{avg}} \lesssim_p \eta_{K\times I_n}^{\mathrm{res}},$$

where $\lesssim_p$ means up to a constant that can only depend on the space dimension d, the polynomial orders p and k, and the shape-regularity of $\mathcal{T}_h(\Omega)$.

Proof. Given (4.4b), (4.4c), and (6.8), we only need to bound $\frac{1}{h_K}||\boldsymbol{p}_{h\tau}^{\mathrm{avg}} + \nabla u_{h\tau}||_{K\times I_n}$. This is achieved by [23, Lemma 7.10]. □

Given Lemma 6.3, the efficiency of η^{avg} follows from the results of [23, Sections 7.2 and 7.3] and Section 6.3.

6.5 Efficiency of the equilibrated fluxes

Similarly to Section 6.4, we derive the efficiency of the equilibrated fluxes by comparing them locally to the residual-based estimator.

Lemma 6.4 (Local boundedness of the patchwise optimal equilibrated flux estimators)**.** *Let $f_{h\tau} \in U_{h\tau}^{pk}$, $u_{h\tau} \in V_{h\tau}^{pk}$, $s_{h\tau} \in R_{h\tau}^{pk}$ and let assumption (4.9c) hold. Then, for all $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$*

and $\star \in \{\mathrm{equ}, \mathrm{rob}\}$*, there holds*

$$(\eta^{\star}_{K\times I_n})^2 \lesssim_p (\eta^{\mathrm{res}}_{K\times I_n})^2 + \sum_{K'\in\mathcal{T}_h(\omega_K)} ||f_{h\tau} - \partial_{tt}u_{h\tau} + \Delta u_{h\tau}||^2_{K'\times I_n} + \frac{1}{h_K}\sum_{\substack{F\in\mathcal{F}_h(\omega_K)\\ F\cap\partial\omega_K=\emptyset}} |||[\![\nabla u_{h\tau}\cdot\boldsymbol{n}_F]\!]||^2_{F\times I_n}, \quad (6.9)$$

where $\lesssim_p$ *means up to a constant that can only depend on the space dimension* d*, the polynomial orders* p *and* k*, and the shape-regularity of* $\mathcal{T}_h(\Omega)$*.*

This result is preferred to the less sharp

$$(\eta^{\star}_{K\times I_n})^2 \lesssim_p \sum_{K'\in\mathcal{T}_h(\omega_K)} (\eta^{\mathrm{res}}_{K'\times I_n})^2. \quad (6.10)$$

The difference between (6.9) and (6.10) is that with (6.10), the efficiency bound becomes less sharp:

$$\eta^{\star}_{K\times I_n} \lesssim_p |||u-u_{h\tau}|||_{\mathcal{Y}_0,\tilde{\omega}_K} + |||f-f_{h\tau}|||_{Y'_T,\tilde{\omega}_K}$$

with $\tilde{\omega}_K := \omega_K \cup \left(\bigcup_{\substack{F\in\mathcal{F}_h(K')\\ K'\in\mathcal{T}_h(\omega_K)}} \omega_F\right) \supsetneq \omega_K$.

Proof. Trivially, (6.8) also holds for $\boldsymbol{p}^{\mathrm{equ}}_{h\tau}$ and $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}$ in place of $\boldsymbol{p}^{\mathrm{avg}}_{h\tau}$. Therefore, we only need to bound $\frac{1}{h_K}||\boldsymbol{p}^{\star}_{h\tau} + \nabla u_{h\tau}||_{K\times I_n}$.

For $\star = \mathrm{equ}$, the proof is similar to the one of [23, Lemma 7.13], only substituting $\partial_{tt}u_{h\tau}$ with $\partial_t s_{h\tau}$.

For $\star = \mathrm{rob}$, the proof is similar to the one of [23, Lemma 7.13]. We substitute $\partial_{tt}u_{h\tau}$ with $\partial_t s_{h\tau}$ and we use [23, Lemma 7.11] for all the decoupled components of $\boldsymbol{p}^{\mathrm{rob}}_{h\tau}|_{I_n}$ (instead of [23, Lemma 7.12] for $l \in [\![1,k]\!]$). □

Given Lemma 6.4, both local and global efficiencies of η^{equ} and η^{rob} follow from the results of [23, Sections 7.2 and 7.3] and Section 6.3.

6.6 Reliability of the residual estimator

The reliability (4.10a) is shown for $\star = \mathrm{rel}$ using Lemma 6.3 and the reliability of η^{avg}_Q.

7 Example of application: finite element space discretisation with an explicit leapfrog time-stepping scheme

In this section, we discretise the acoustic wave equation (1.1) using a conforming high-order finite element method in space and the explicit leapfrog time-stepping scheme in time. We cannot apply the previous results directly as, at first, we only dispose of snapshots of the discrete approximation u^n_h for $n \in [\![-1, N+1]\!]$. We introduce concrete space-time reconstructed discrete approximations $u_{h\tau}$, $s_{h\tau}$ and show how the *a posteriori* estimates in Theorems 3.3 and 4.5 can be applied to these reconstructions.

7.1 Finite element space-discretisation with explicit leapfrog time-stepping

The leapfrog time-discretisation can be derived from the second-order weak formulation of the wave equation [24]. Here, only uniform time-stepping is considered with the time step size $\tau > 0$. We approximate the solution u by a sequence of pointwise-in-time snapshots

$$(u_h^n)_{n\in[\![-1,N+1]\!]} \subset V_h^p$$

representing the discrete approximation at each discrete time (also including the additional values at $t^{-1} := -\tau$ and $t^{N+1} := T+\tau$). We also approximate the first and second time derivatives in (1.1) at the same times for $n \in [\![0, N]\!]$ by the following second order finite-difference approximations

$$\dot{u}_h^n := \frac{1}{2\tau}\left(u_h^{n+1} - u_h^{n-1}\right) \quad \text{and} \quad \ddot{u}_h^n := \frac{1}{\tau^2}\left(u_h^{n+1} - 2u_h^n + u_h^{n-1}\right). \tag{7.1}$$

The initial conditions (1.1c) and (1.1d) are approximated by

$$u_h^0 = 0, \tag{7.2a}$$
$$\dot{u}_h^0 = 0. \tag{7.2b}$$

Then, the fully discrete version of (1.1a) consists, for all $n \in [\![0, N]\!]$, in finding $u_h^{n+1} \in V_h^p$ such that

$$(\ddot{u}_h^n, v_h)_\Omega + (\nabla u_h^n, \nabla v_h)_\Omega = (f(t^n), v_h)_\Omega \qquad \forall v_h \in V_h^p. \tag{7.2c}$$

For this to work, we assume that $f \in \mathcal{C}^0(0,T;L^2(\Omega))$. We also define for each $t^n \in \mathcal{S}_\tau(0,T)$, the orthogonal projection f_h^n of $f(t^n)$ onto U_h^p, that is to say

$$f_h^n \in U_h^p \text{ such that} \quad (f_h^n, v_h)_\Omega = (f(t^n), v_h)_\Omega \quad \forall v_h \in U_h^p. \tag{7.3}$$

7.2 Potential reconstruction with average-in-time Galerkin orthogonality

We now present a reconstruction pair that fits Theorem 4.5.

Definition 7.1 (Reconstructions in time of the leapfrog iterates)**.** *Let $(u_h^n)_{t^n\in\mathcal{S}_\tau(0,T)}$ be the snapshots of the discrete approximation given by (7.2) and let $(f_h^n)_{t^n\in\mathcal{S}_\tau(0,T)}$ be the projected snapshots of the source term defined by (7.3). We define $f_{h\tau} \in U_{h\tau}^{p1} \cap \mathcal{C}^0(0,T;U_h^p)$ such that for $I_n \in \mathcal{T}_\tau(0,T)$,*

$$f_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} f_h^{n-1} + \frac{t - t^{n-1}}{\tau} f_h^n. \tag{7.4}$$

We next define $u_{h\tau} \in V_{h\tau}^{p4}$ and $s_{h\tau} \in R_{h\tau}^{p4} \cap V_{h\tau}^{p3}$ such that, for $I_n \in \mathcal{T}_\tau(0,T) \setminus \{I_N\}$,

$$u_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} u_h^{n-1} + \frac{t - t^{n-1}}{\tau} u_h^n, \tag{7.5}$$
$$s_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} \dot{u}_h^{n-1} + \frac{t - t^{n-1}}{\tau} \dot{u}_h^n. \tag{7.6}$$

On I_N, we define

$$s_{h\tau}|_{I_N} := \dot{u}_h^{N-1} + (\dot{u}_h^N - \dot{u}_h^{N-1}) \times \varphi_1\left(\frac{t - t^{N-1}}{\tau}\right) + \frac{\tau}{2}\ddot{u}_h^{N-1} \times \varphi_2\left(\frac{t - t^{N-1}}{\tau}\right), \tag{7.7}$$

with

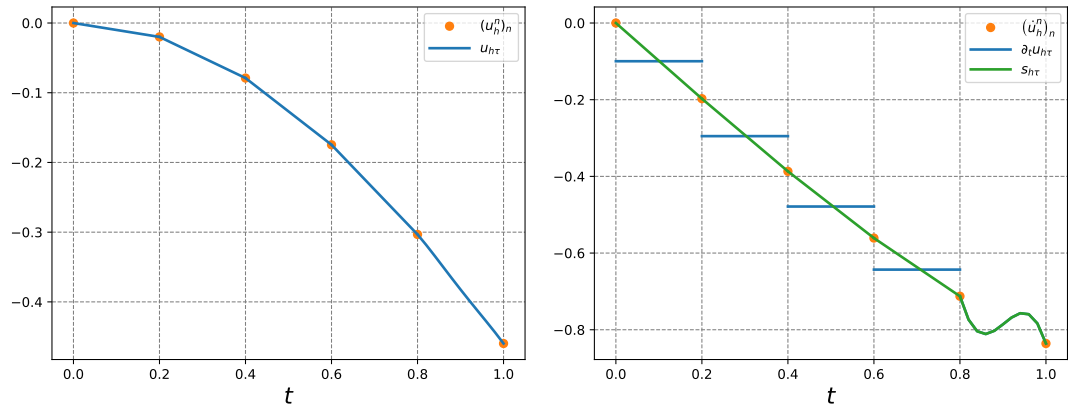
$$\varphi_1 := \theta \in [0,1] \longmapsto 3\theta - 12\theta^2 + 10\theta^3 \quad \textit{and} \quad \varphi_2 := \theta \in [0,1] \longmapsto 6\theta - 6\theta^2. \tag{7.8}$$

Then, on I_N*, we define*

$$u_{h\tau}|_{I_N} := t \longmapsto u_h^{N-1} + \int_{t^{N-1}}^{t} s_{h\tau}(\tilde{t})\mathrm{d}\tilde{t}. \tag{7.9}$$

On $(0, t^{N-1})$, both $u_{h\tau}$ and $s_{h\tau}$ are piecewise affine in time, interpolating $(u_h^n)_{t^n \in \mathcal{S}_\tau(0,T)}$ and $(\dot{u}_h^n)_{t^n \in \mathcal{S}_\tau(0,T)}$. To verify the hypotheses (4.9) of Theorem 4.5, an adjustment is necessary on the last time interval I_N. This means that $u_{h\tau}$ and $s_{h\tau}$ are globally fourth-order polynomials in time but with $u_{h\tau}|_{\Omega\times(0,t^{N-1})} \in V_{h\tau}^{p1}|_{\Omega\times(0,t^{N-1})}$ and $s_{h\tau}|_{\Omega\times(0,t^{N-1})} \in R_{h\tau}^{p1}|_{\Omega\times(0,t^{N-1})}$. All of the advantages of the lowest orders (sharper constants when locally depending on k, less computing costs for numerical evaluation...) can be recovered on nearly the whole space-time domain. Specifically for our setting with time stepping, this means that the estimation will use the more favourable low-order behaviour for all except the last step.

Figure 4 shows an example of the reconstruction pair from Definition 7.1. Notice that on the last time step I_N, of Figure 4a, the reconstruction is close, at least visually, to the affine interpolation used on the other time steps. Figure 4b shows $\partial_t u_{h\tau}$ and $s_{h\tau}$ for the same example. There, we recognise the piecewise constant $\partial_t u_{h\tau}$ and the piecewise affine $s_{h\tau}$ on all but the last time step. On I_N, we also find (as known from (7.9) in Definition 7.1), that $\partial_t u_{h\tau}$ overlaps with $s_{h\tau}$.



(a) $u_{h\tau}$ (b) $\partial_t u_{h\tau}$ and $s_{h\tau}$

Figure 4: Example of the reconstruction pair $(u_{h\tau}, s_{h\tau})$ from Definition 7.1 for pointwise values of $t \mapsto \cos(t) - 1$

Now, we want to verify that the reconstructions introduced in Definition 7.1 verify the hypotheses of Theorem 4.5. Lemmas 7.2, 7.3 and 7.5 respectively verify the regularity and initial conditions, the average-in-time Galerkin orthogonality, and the bound on $s_{h\tau} - \partial_t u_{h\tau}$. Lemma 7.4 is a helper for the proof of Lemma 7.5 giving the values of time jumps of $\partial_t u_{h\tau}$.

Lemma 7.2 (Regularity and initial conditions)**.** *Consider* $u_{h\tau}$ *and* $s_{h\tau}$ *from Definition 7.1. Then,*

- $u_{h\tau}$ *is in* $V_{h\tau}^{p4}$ *(*i.e. *in* $U_{h\tau}^{p4} \cap Y_0$*);*
- $s_{h\tau}$ *is in* $R_{h\tau}^{p4}$ *(*i.e. *in* $U_{h\tau}^{p5} \cap H^1(0,T;L^2(\Omega))$ *with* $s_{h\tau}(0) = 0$*).*

Actually, $s_{h\tau}$ is built in $V_{h\tau}^{p3}$ which is a strict subset of $R_{h\tau}^{p4}$, but Theorem 4.5 only requires the regularity from Lemma 7.2.

Proof. $u_{h\tau}(0) = 0$ comes from (7.2a) and (7.5) and $s_{h\tau}(0) = 0$ comes from (7.2b) and (7.6).

On $(0, t^{N-1})$, $u_{h\tau}$ and $s_{h\tau}$ are respectively the affine interpolators of the sequences

$$(u_h^n)_{n\in[\![0,N-1]\!]} \subset V_h^p \quad \text{and} \quad (\dot{u}_h^n)_{n\in[\![0,N-1]\!]} \subset U_h^p,$$

ensuring continuity in time. We only need to verify that

$$u_{h\tau}|_{I_N}(t^{N-1}) = u_h^{N-1} \quad \text{and} \quad s_{h\tau}|_{I_N}(t^{N-1}) = \dot{u}_h^{N-1}$$

to complete the proof. First, $u_{h\tau}|_{I_N}(t^{N-1}) = u_h^{N-1}$ comes from (7.9). Second,

$$\begin{aligned} s_{h\tau}|_{I_N}(t^{N-1}) &\overset{(7.7)}{=} \dot{u}_h^{N-1} + (\dot{u}_h^N - \dot{u}_h^{N-1})\times\varphi_1(0) + \frac{\tau}{2}\ddot{u}_h^{N-1}\times\varphi_2(0) \\ &\overset{(7.8)}{=} \dot{u}_h^{N-1}. \end{aligned}$$

□

Lemma 7.3 (Average-in-time Galerkin orthogonality)**.** *Consider $u_{h\tau}$, $s_{h\tau}$, and $f_{h\tau}$ from Definition 7.1. Then, for $I_n \in \mathcal{T}_\tau(0,T)$,*

$$\int_{I_n} (f_{h\tau}(t) - \partial_t s_{h\tau}(t), v_h)_\Omega - (\nabla u_{h\tau}(t), \nabla v_h)_\Omega \mathrm{d}t = 0 \qquad \forall v_h \in V_h^p.$$

Proof. For $I_n \in \mathcal{T}_\tau(0,T)$,

$$\int_{I_n} f_{h\tau}(t)\mathrm{d}t \overset{(7.4)}{=} \int_{I_n} \frac{t^n - t}{\tau} f_h^{n-1} + \frac{t - t^{n-1}}{\tau} f_h^n \mathrm{d}t = \frac{\tau}{2}(f_h^n + f_h^{n-1}). \tag{7.10}$$

Similarly, for $I_n \in \mathcal{T}_\tau(0,T) \setminus \{I_N\}$,

$$\int_{I_n} u_{h\tau}(t)\mathrm{d}t = \frac{\tau}{2}(u_h^n + u_h^{n-1}). \tag{7.11}$$

Additionally, denoting respectively $\Phi_1 := \frac{3}{2}X^2 - 4X^3 + \frac{5}{2}X^4$ and $\Phi_2 := 3X^2 - 2X^3$ the antiderivatives of φ_1 and φ_2 from (7.8), there holds

$$\begin{aligned} u_{h\tau}|_{I_N}(t) &\overset{(7.9)}{=} u_h^{N-1} + \int_{t^{N-1}}^t s_{h\tau}(\tilde{t})\mathrm{d}\tilde{t}, \\ &\overset{(7.7)}{=} u_h^{N-1} + \tau\left[\dot{u}_h^{N-1}\times\theta + (\dot{u}_h^N - \dot{u}_h^{N-1})\times\Phi_1(\theta) + \frac{\tau}{2}\ddot{u}_h^{N-1}\times\Phi_2(\theta)\right], \end{aligned}$$

where $\theta := \frac{t-t^{N-1}}{\tau}$. Therefore, since $\int_0^1 \Phi_1(\theta)\mathrm{d}\theta = 0$ and $\int_0^1 \Phi_2(\theta)\mathrm{d}\theta = \frac{1}{2}$, there holds

$$\begin{aligned} \int_{I_N} u_{h\tau}(t)\mathrm{d}t &= \tau u_h^{N-1} + \tau^2\left[\frac{1}{2}\dot{u}_h^{N-1} + \frac{\tau}{4}\ddot{u}_h^{N-1}\right] \\ &= \frac{\tau}{2}\left[2u_h^{N-1} + \frac{1}{2}\left(u_h^N - u_h^{N-2}\right) + \frac{1}{2}\left(u_h^N - 2u_h^{N-1} + u_h^{N-2}\right)\right] \\ &= \frac{\tau}{2}\left(u_h^N + u_h^{N-1}\right). \end{aligned} \tag{7.12}$$

Now, by (7.6), $s_{h\tau}(t^n) = \dot{u}_h^n$ for $t^n \in \mathcal{S}_\tau(0,T) \setminus \{T\}$. By (7.7),

$$\begin{aligned} s_{h\tau}(t^N) &= \dot{u}_h^{N-1} + (\dot{u}_h^N - \dot{u}_h^{N-1}) \times \varphi_1(1) + \frac{\tau}{2} \ddot{u}_h^{N-1} \times \varphi_2(1) \\ &\overset{(7.8)}{=} \dot{u}_h^{N-1} + (\dot{u}_h^N - \dot{u}_h^{N-1}) = \dot{u}_h^N. \end{aligned}$$

Therefore, for $I_n \in \mathcal{T}_\tau(0,T)$,

$$\int_{I_n} \partial_t s_{h\tau}(t) \mathrm{d}t = s_{h\tau}(t^n) - s_{h\tau}(t^{n-1}) = \dot{u}_h^n - \dot{u}_h^{n-1} \overset{(7.1)}{=} \frac{\tau}{2} \left(\ddot{u}_h^n + \ddot{u}_h^{n-1} \right). \tag{7.13}$$

Using (7.10) to (7.13) we prove that for $v_h \in V_h^p$,

$$\begin{aligned} &\frac{1}{\tau} \int_{I_n} (f_{h\tau}(t) - \partial_t s_{h\tau}(t), v_h)_\Omega - (\nabla u_{h\tau}(t), \nabla v_h)_\Omega \mathrm{d}t = \\ &\qquad \frac{1}{2}(f_h^n + f_h^{n-1}, v_h)_\Omega - \frac{1}{2}(\ddot{u}_h^n + \ddot{u}_h^{n-1}, v_h) - \frac{1}{2}(\nabla(u_h^n + u_h^{n-1}), \nabla v_h)_\Omega \overset{(7.2c)}{=} 0, \end{aligned}$$

where the conclusion uses the scheme (7.2c). □

Lemma 7.4 (Time jump values of $\partial_t u_{h\tau}$)**.** *Consider $u_{h\tau}$ from Definition 7.1. Then, for $t^n \in \mathcal{S}_\tau(0,T) \setminus \{T\}$,*

$$[\![\partial_t u_{h\tau}]\!]_{t^n} = \begin{cases} \frac{\tau}{2} \ddot{u}_h^0 & \text{if } n = 0, \\ \tau \ddot{u}_h^n & \text{if } n \in [\![1, N-2]\!], \\ \frac{\tau}{2} \ddot{u}_h^{N-1} & \text{if } n = N-1. \end{cases}$$

Proof. For $I_n \in \mathcal{T}_\tau(0,T) \setminus \{I_N\}$,

$$\partial_t u_{h\tau}|_{I_n} \overset{(7.5)}{=} \frac{1}{\tau}(u_h^n - u_h^{n-1}). \tag{7.14}$$

Together with (7.1), this yields the following jump at t^n for $n \in [\![1, N-2]\!]$:

$$\begin{aligned} [\![\partial_t u_{h\tau}]\!]_{t^n} &\overset{(7.14)}{=} \frac{1}{\tau}(u_h^{n+1} - u_h^n) - \frac{1}{\tau}(u_h^n - u_h^{n-1}) \\ &= \frac{1}{\tau}(u_h^{n+1} - 2u_h^n + u_h^{n-1}) \\ &\overset{(7.1)}{=} \tau \ddot{u}_h^n. \end{aligned}$$

At t^0:

$$\begin{aligned} [\![\partial_t u_{h\tau}]\!]_{t^0} &\overset{(1.1d) \text{ and } (7.14)}{=} \frac{1}{\tau}(u_h^1 - u_h^0) - 0 \\ &\overset{(7.1) \text{ and } (7.2b)}{=} \frac{1}{2\tau}(u_h^1 - 2u_h^0 + u_h^{-1}) \\ &\overset{(7.1)}{=} \frac{\tau}{2} \ddot{u}_h^0. \end{aligned}$$

There also holds that,

$$\begin{aligned} \partial_t u_{h\tau}|_{I_N}(t^{N-1}) &\overset{(7.9)}{=} s_{h\tau}|_{I_N}(t^{N-1}) \\ &\overset{(7.7)}{=} \dot{u}_h^{N-1} + (\dot{u}_h^N - \dot{u}_h^{N-1}) \times \varphi_1(0) - \frac{\tau}{2} \ddot{u}_h^{N-1} \varphi_2(0) \\ &\overset{(7.8)}{=} \dot{u}_h^{N-1}. \end{aligned} \tag{7.15}$$

Therefore, at t^{N-1}:

$$\begin{aligned}
[\![\partial_t u_{h\tau}]\!]_{t^{N-1}} &\overset{(7.1),\ (7.14),\text{ and }(7.15)}{=} \frac{1}{2\tau}(u_h^N - u_h^{N-2}) - \frac{1}{\tau}(u_h^{N-1} - u_h^{N-2}) \\
&= \frac{1}{2\tau}(u_h^N - u_h^{N-2} - 2u_h^{N-1} + 2u_h^{N-2}) \\
&= \frac{1}{2\tau}(u_h^N - 2u_h^{N-1} + u_h^{N-2}) \\
&\overset{(7.1)}{=} \frac{\tau}{2}\ddot{u}_h^{N-1}.
\end{aligned}$$

□

Lemma 7.5 (Bound on $s_{h\tau} - \partial_t u_{h\tau}$)**.** *Consider $u_{h\tau}$, $s_{h\tau}$, and $f_{h\tau}$ from Definition 7.1. Then, for $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$,*

$$\frac{1}{\tau^2}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n} \le \frac{c_n}{\tau} \sum_{t^k \in \mathcal{S}_\tau(I_n)\setminus\{T\}} ||[\![\partial_t u_{h\tau}]\!]_{t^k}||^2_K,$$

with

$$c_n := \begin{cases} \frac{1}{2} & \text{if } n = 1, \\ \frac{1}{8} & \text{if } n \in [\![2, N-2]\!], \\ \frac{1}{2} & \text{if } n = N-1, \\ 0 & \text{if } n = N. \end{cases}$$

Proof. The result is easily proven for I_N given (7.9), where $s_{h\tau} = \partial_t u_{h\tau}$.

Let $I_n \in \mathcal{T}_\tau(0,T) \setminus \{I_N\}$, $t^{n-1/2} := \frac{1}{2}(t^n + t^{n-1})$, and $t \in I_n$. Then, there holds that

$$\begin{aligned}
s_{h\tau}(t) - \partial_t u_{h\tau}(t) &\overset{(7.17)\text{ and }(7.18)}{=} \frac{t^n - t}{\tau}\dot{u}_h^{n-1} + \frac{t - t^{n-1}}{\tau}\dot{u}_h^n - \frac{1}{\tau}(u_h^n - u_h^{n-1}) \\
&= 2\frac{t - t^{n-1/2}}{\tau} \times \frac{1}{2}(\dot{u}_h^n - \dot{u}_h^{n-1}) + \frac{1}{2}(\dot{u}_h^n + \dot{u}_h^{n-1}) - \frac{1}{\tau}(u_h^n - u_h^{n-1}).
\end{aligned}$$

Using that

$$\begin{aligned}
\frac{1}{2}(\dot{u}_h^n - \dot{u}_h^{n-1}) &\overset{(7.1)}{=} \frac{\tau}{4}\left(\ddot{u}_h^n + \ddot{u}_h^{n-1}\right), \\
\frac{1}{2}(\dot{u}_h^n + \dot{u}_h^{n-1}) - \frac{1}{\tau}(u_h^n - u_h^{n-1}) &\overset{(7.1)}{=} \frac{\tau}{4}\left(\ddot{u}_h^n - \ddot{u}_h^{n-1}\right),
\end{aligned}$$

there holds, for $K \in \mathcal{T}_h(\Omega)$,

$$\frac{1}{\tau}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n} = \frac{1}{3}\left(\frac{\tau}{4}\right)^2 ||\ddot{u}_h^n + \ddot{u}_h^{n-1}||^2_K + \left(\frac{\tau}{4}\right)^2 ||\ddot{u}_h^n - \ddot{u}_h^{n-1}||^2_K,$$

that is simplified using Young's inequality to obtain

$$\frac{1}{\tau}||s_{h\tau} - \partial_t u_{h\tau}||^2_{K\times I_n} \le \frac{\tau^2}{8}\left(||\ddot{u}_h^{n-1}||^2_K + ||\ddot{u}_h^n||^2_K\right).$$

The proof is concluded using Lemma 7.4. □

7.3 Application of Theorem 4.5

We now prove reliability and efficiency for this setting using the general results from Theorem 4.5.

Theorem 7.6 (Hypotheses of Theorem 4.5)**.** *Let $f \in L^2(Q)$ and $u \in Y_0$ be the weak solution of (2.12). Consider $f_{h\tau}$, $u_{h\tau}$, and $s_{h\tau}$ from Definition 7.1. Then, the hypotheses (4.9) of Theorem 4.5 hold.*

Proof. We verify the hypotheses of Theorem 4.5:

- The regularities of $f_{h\tau}$, $u_{h\tau}$, and $s_{h\tau}$ required in (4.9a) are verified by Lemma 7.2, with in particular $k = 4$.
- The Galerkin orthogonality (4.9b) is verified by Lemma 7.3.
- Hypothesis (4.9c) is verified by Lemma 7.5.

□

Notice that here η^{rob} is not included. The Galerkin orthogonality from Lemma 7.3 is only average-in-time. Therefore, the main hypothesis of Definition 3.1 is not verified and the existence of $\boldsymbol{p}_{h\tau}^{\mathrm{rob}}$ is not proven.

7.4 Potential reconstruction with pointwise-in-time Galerkin orthogonality

We now present a reconstruction pair fitting Theorem 3.3.

Definition 7.7 (Reconstructions in time of the leapfrog iterates)**.** *Let $(u_h^n)_{n\in[\![-1,N+1]\!]}$ be the iterates of the discrete approximation given by (7.2) and $(f_h^n)_{t^n\in\mathcal{S}_\tau(0,T)}$ be the iterates of the discrete source term defined by (7.3). We define $u_{h\tau} \in V_{h\tau}^{p1}$, $s_{h\tau} \in R_{h\tau}^{p1}$, and $f_{h\tau} \in U_{h\tau}^{p1} \cap \mathcal{C}^0(0,T;U_h^p)$ such that, for all $I_n \in \mathcal{T}_\tau(0,T)$,*

$$f_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} f_h^{n-1} + \frac{t - t^{n-1}}{\tau} f_h^n, \tag{7.16}$$

$$u_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} u_h^{n-1} + \frac{t - t^{n-1}}{\tau} u_h^n, \tag{7.17}$$

$$s_{h\tau}|_{I_n} := t \longmapsto \frac{1}{\tau}(u_h^n - u_h^{n-1}) + \frac{\tau}{2}\left(\frac{t - t^{n-1}}{\tau}\right)^2 \ddot{u}_h^n - \frac{\tau}{2}\left(\frac{t^n - t}{\tau}\right)^2 \ddot{u}_h^{n-1}. \tag{7.18}$$

Figure 5 shows an example of the reconstruction pair $(u_{h\tau}, s_{h\tau})$ from Definition 7.7. Here, $s_{h\tau}$ is actually built taking $\partial_t u_{h\tau}$ and correcting its time jumps while keeping

$$\partial_t s_{h\tau}|_{I_n} := t \longmapsto \frac{t^n - t}{\tau} \ddot{u}_h^{n-1} + \frac{t - t^{n-1}}{\tau} \ddot{u}_h^n.$$

Note that $s_{h\tau}$ is piecewise quadratic but Figure 5 shows it close to the piecewise affine reconstruction from Definition 7.1.

Lemma 7.8 (Regularity and initial conditions)**.** *Consider $u_{h\tau}$ and $s_{h\tau}$ from Definition 7.7. Then*

- *$u_{h\tau}$ is in $V_{h\tau}^{p1}$ (*i.e. *in $U_{h\tau}^{p1} \cap Y_0$),*

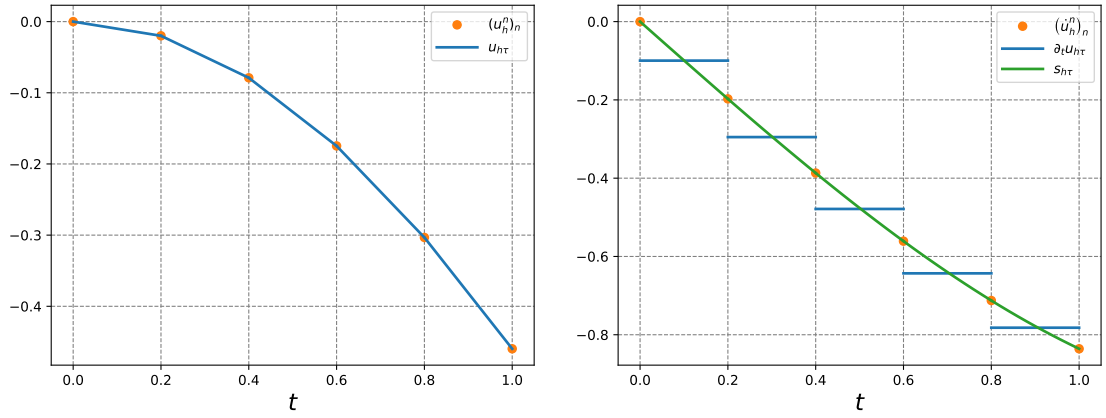


Figure 5: Example of the reconstruction pair $(u_{h\tau}, s_{h\tau})$ with pointwise-in-time Galerkin orthogonality from Definition 7.7 for pointwise values of $t \mapsto \cos(t) - 1$

- $s_{h\tau}$ *is in* $R_{h\tau}^{p1}$ *(*i.e. *in* $U_{h\tau}^{p2} \cap H^1(0,T;L^2(\Omega))$ *with* $s_{h\tau}(0) = 0$*).*

Proof. Let $I_n \in \mathcal{T}_\tau(0,T)$, then

$$\begin{aligned} s_{h\tau}|_{I_n}(t^{n-1}) &\overset{(7.18)}{=} \frac{1}{\tau}(u_h^n - u_h^{n-1}) - \frac{\tau}{2}\ddot{u}_h^{n-1} \overset{(7.1)}{=} \dot{u}_h^{n-1}, \\ s_{h\tau}|_{I_n}(t^{n}) &\overset{(7.18)}{=} \frac{1}{\tau}(u_h^n - u_h^{n-1}) + \frac{\tau}{2}\ddot{u}_h^{n} \overset{(7.1)}{=} \dot{u}_h^{n}. \end{aligned} \tag{7.19}$$

Therefore, $s_{h\tau}$ is piecewise quadratic in time and continuous. Next, $u_{h\tau}(0) = 0$ comes from (7.2a) and (7.17) and $s_{h\tau}(0) = 0$ comes from (7.2b) and (7.19). $u_{h\tau}$ is continuous because it is the affine interpolation of $(u_h^n)_{n\in[\![0,N-1]\!]} \subset V_h^p$. □

Lemma 7.9 (Pointwise-in-time Galerkin orthogonality)**.** *Consider* $u_{h\tau}$*,* $s_{h\tau}$*, and* $f_{h\tau}$ *from Definition 7.7. Then, for* $I_n \in \mathcal{T}_\tau(0,T)$*,*

$$(f_{h\tau}(t) - \partial_t s_{h\tau}(t), v_h)_\Omega - (\nabla u_{h\tau}(t), \nabla v_h)_\Omega = 0 \qquad \forall v_h \in V_h^p,\ \forall t \in I_n.$$

Proof. For $I_n \in \mathcal{T}_\tau(0,T)$ and $t \in I_n$, there holds

$$\partial_t s_{h\tau}(t) \overset{(7.18)}{=} \ddot{u}_h^{n-1}\left(\frac{t^n - t}{\tau}\right) + \ddot{u}_h^n\left(\frac{t - t^{n-1}}{\tau}\right).$$

Therefore, for $I_n \in \mathcal{T}_\tau(0,T)$, $t \in I_n$, and $v_h \in V_h^p$,

$$\begin{aligned} &(f_{h\tau}(t) - \partial_t s_{h\tau}(t), v_h)_\Omega - (\nabla u_{h\tau}(t), \nabla v_h)_\Omega = \\ &\underbrace{\left[(f_h^{n-1} - \ddot{u}_h^{n-1}, v_h)_\Omega - (\nabla u_h^{n-1}, \nabla v_h)_\Omega\right]}_{=0 \text{ by } (7.2c)}\left(\frac{t^n - t}{\tau}\right) + \underbrace{\left[(f_h^{n} - \ddot{u}_h^{n}, v_h)_\Omega - (\nabla u_h^{n}, \nabla v_h)_\Omega\right]}_{=0 \text{ by } (7.2c)}\left(\frac{t - t^{n-1}}{\tau}\right). \end{aligned}$$

□

Finally, we prove the equivalence between the native error and the augmented error.

Lemma 7.10 (Equivalence between the error and the augmented error)**.** *Consider $f_{h\tau}$, $u_{h\tau}$, and $s_{h\tau}$ from Definition 7.7. Then, there holds*

$$\sqrt{\sum_{I_n\in\mathcal{T}_\tau(0,T)}\frac{1}{\tau^2}||s_{h\tau}-\partial_t u_{h\tau}||^2_{\Omega\times I_n}}\lesssim_p |||u-u_{h\tau}|||_{\mathcal{Y}_0}+|||f-f_{h\tau}|||_{Y_T'}. \tag{7.20}$$

Proof. Let $(K,I_n)\in\mathcal{T}_{h\tau}(Q)$ and $t\in I_n$. Using the fact that $\partial_t u_{h\tau}(t)=\frac{1}{\tau}(u_h^n-u_h^{n-1})$, we obtain

$$s_{h\tau}(t)-\partial_t u_{h\tau}(t)=\frac{\tau}{2}\left(\frac{t-t^{n-1}}{\tau}\right)^2\ddot{u}_h^n-\frac{\tau}{2}\left(\frac{t^n-t}{\tau}\right)^2\ddot{u}_h^{n-1}.$$

Therefore, there holds that

$$\frac{1}{\tau}||s_{h\tau}-\partial_t u_{h\tau}||^2_{K\times I_n}=\frac{\tau^2}{20}\left(||\ddot{u}_h^n||^2_K+||\ddot{u}_h^{n-1}||^2_K\right)-\frac{\tau^2}{60}(\ddot{u}_h^n,\ddot{u}_h^{n-1})_K.$$

Using Young's inequality, we obtain

$$\frac{1}{\tau^2}||s_{h\tau}-\partial_t u_{h\tau}||^2_{K\times I_n}\leq\frac{7\tau}{120}\left(||\ddot{u}_h^n||^2_K+||\ddot{u}_h^{n-1}||^2_K\right). \tag{7.21}$$

Let $v\in H^1(0,T,V_h^p)\cap Y_T$, defined for $I_n\in\mathcal{T}_\tau(0,T)$ by

$$v|_{I_n}:=t\longmapsto\frac{\theta(1-\theta)}{2}\left(\frac{1}{10}\left[\ddot{u}_h^n+\ddot{u}_h^{n-1}\right]+\left(\theta-\frac{1}{2}\right)\left[\ddot{u}_h^n-\ddot{u}_h^{n-1}\right]\right), \tag{7.22}$$

with $\theta:=\frac{t-t^{n-1}}{\tau}$. This specifically ensures that for all $I_n\in\mathcal{T}_\tau(0,T)$, $v|_{I_n}(t^{n-1})=v|_{I_n}(t^n)=0$ so that $v\in H^1_0(0,T;V_h^p)\cap Y_T$. Then,

$$\begin{aligned}(\partial_t u_{h\tau},\partial_t v)_Q&\overset{(7.17)}{=}\sum_{I_n\in\mathcal{T}_\tau(0,T)}\int_{I_n}\left(\frac{1}{\tau}\left(u_h^n-u_h^{n-1}\right),\partial_t v(t)\right)_\Omega\mathrm{d}t\\&=\sum_{I_n\in\mathcal{T}_\tau(0,T)}\left(\frac{1}{\tau}\left(u_h^n-u_h^{n-1}\right),v(t^n)-v(t^{n-1})\right)_\Omega\\&=0.\end{aligned} \tag{7.23}$$

Note the values of the following integrals:

$$\begin{aligned}\int_0^1\theta^2(1-\theta)\mathrm{d}\theta&=\frac{1}{12},&\int_0^1\theta(1-\theta)^2\mathrm{d}\theta&=\frac{1}{12},\\\int_0^1\theta^2(1-\theta)(\theta-1/2)\mathrm{d}\theta&=\frac{1}{120},&\int_0^1\theta(1-\theta)^2(\theta-1/2)\mathrm{d}\theta&=\frac{-1}{120}.\end{aligned} \tag{7.24}$$

Let $I_n\in\mathcal{T}_\tau(0,T)$. Using (7.18), we rewrite

$$(\partial_t s_{h\tau},v)_{\Omega\times I_n}=\tau\int_0^1\left(\theta\,\ddot{u}_h^n+(1-\theta)\,\ddot{u}_h^{n-1},\frac{\theta(1-\theta)}{2}\left(\frac{1}{10}\left[\ddot{u}_h^n+\ddot{u}_h^{n-1}\right]+\left(\theta-\frac{1}{2}\right)\left[\ddot{u}_h^n-\ddot{u}_h^{n-1}\right]\right)\right)_\Omega\mathrm{d}\theta.$$

This and the values in (7.24) allow us to rewrite

$$(\partial_t s_{h\tau}, v)_{\Omega\times I_n} = \frac{\tau}{120}\Big((\ddot{u}_h^n, \ddot{u}_h^n + \ddot{u}_h^{n-1})_\Omega + (\ddot{u}_h^{n-1}, \ddot{u}_h^n + \ddot{u}_h^{n-1})_\Omega \\ + (\ddot{u}_h^n, \ddot{u}_h^n - \ddot{u}_h^{n-1})_\Omega - (\ddot{u}_h^{n-1}, \ddot{u}_h^n - \ddot{u}_h^{n-1})_\Omega\Big),$$

that we simplify to get

$$(\partial_t s_{h\tau}, v)_{\Omega\times I_n} = \frac{\tau}{60}\left(||\ddot{u}_h^n||_\Omega^2 + ||\ddot{u}_h^{n-1}||_\Omega^2\right). \tag{7.25}$$

There holds

$$\begin{aligned}(\partial_t s_{h\tau}, v)_Q &\overset{\text{Lemma 7.9}}{=} (f_{h\tau}, v)_Q - (\nabla u_{h\tau}, \nabla v)_Q \\ &\overset{(7.23)}{=} (f_{h\tau} - f, v)_Q + (f, v)_Q + (\partial_t u_{h\tau}, \partial_t v)_Q - (\nabla u_{h\tau}, \nabla v)_Q \\ &\overset{(2.13)}{=} (f_{h\tau} - f, v)_Q + \langle r(u_{h\tau}), v\rangle_{Y_T} \\ &\overset{(2.14)}{\leq} \left(|||f - f_{h\tau}|||_{Y_T'} + |||u - u_{h\tau}|||_{\mathcal{Y}_0}\right)|||v|||_{Y_T}.\end{aligned} \tag{7.26}$$

Let $(K, I_n) \in \mathcal{T}_{h\tau}(Q)$. By applying a space-time inverse inequality to $|||v|||_{Y_T, K\times I_n}$, we obtain

$$|||v|||_{Y_T,K\times I_n}^2 = \tau^2||\partial_t v||_{K\times I_n}^2 + h_K^2||\nabla v||_{K\times I_n}^2 \lesssim_p ||v||_{K\times I_n}^2.$$

Note the values of the following integrals:

$$\int_0^1 \theta^2(1-\theta)^2 \mathrm{d}\theta = \frac{1}{30}, \quad \int_0^1 \theta^2(1-\theta)^2(\theta - 1/2)\mathrm{d}\theta = 0, \text{ and } \int_0^1 \theta^2(1-\theta)^2(\theta-1/2)^2\mathrm{d}\theta = \frac{1}{840}.$$

Using this and Young's inequality, we bound $||v||_{K\times I_n}^2$ as follows:

$$\begin{aligned}||v||_{K\times I_n}^2 &= \frac{\tau}{3000}||\ddot{u}_h^n + \ddot{u}_h^{n-1}||_\Omega^2 + \frac{\tau}{840}||\ddot{u}_h^n - \ddot{u}_h^{n-1}||_\Omega^2 \\ &= \frac{4\tau}{2625}\left(||\ddot{u}_h^n||_\Omega^2 + ||\ddot{u}_h^{n-1}||_\Omega^2\right) - \frac{3\tau}{1750}(\ddot{u}_h^n, \ddot{u}_h^{n-1})_\Omega \\ &\leq \frac{\tau}{420}\left(||\ddot{u}_h^n||_\Omega^2 + ||\ddot{u}_h^{n-1}||_\Omega^2\right)\end{aligned}$$

This means that

$$|||v|||_{Y_T,K\times I_n}^2 \lesssim_p \tau\left(||\ddot{u}_h^n||_{K\times I_n}^2 + ||\ddot{u}_h^{n-1}||_{K\times I_n}^2\right). \tag{7.27}$$

We conclude the proof of (7.20) using (7.21) and (7.25) to (7.27). □

Notice that (7.27) uses a local space-time inverse inequality. Because of this, (7.20) is not polynomial degree robust. The implications of this are discussed in the next section.

7.5 Application of Theorem 3.3

In this section, we apply Theorem 3.3 to prove reliability and polynomial-degree robust efficiency for our reconstruction pair with pointwise-in-time Galerkin orthogonality. Here, the equivalence between the error and the augmented error (7.20) is not polynomial-degree-robust (we use a local space-time inequality to prove it). This means that the global efficiency result

$$\eta_Q^{\text{rob}} \lesssim_p |||u - u_{h\tau}|||_{\mathcal{Y}_0} + |||f - f_{h\tau}|||_{Y_T'}$$

still holds in Theorem 7.11, but is not polynomial-degree robust.

Theorem 7.11 (Hypotheses of Theorem 3.3)**.** *Let $f \in L^2(Q)$ and $u \in Y_0$ be the weak solution from (2.12). Let $d \in \{1,2,3\}$ with $\mathcal{T}_h(\Omega)$ a simplicial mesh. Consider $f_{h\tau}$, $u_{h\tau}$, and $s_{h\tau}$ from Definition 7.7. Then, the hypotheses (3.1), (3.2), and (3.8) of Theorem 3.3 hold.*

Proof. To verify the hypotheses of Theorem 3.3, we use:

- Definition 7.7 and Lemma 7.8 for the regularity and initial conditions of $f_{h\tau}$, $u_{h\tau}$ and $s_{h\tau}$, required by (3.1);
- Lemma 7.9 for the Galerkin orthogonality (3.2);
- Lemma 7.10 for the control of $s_{h\tau} - \partial_t u_{h\tau}$ by the native error (3.8).

□

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