Evanescent Plane Wave Approximation of Helmholtz Solutions in Spherical Domains

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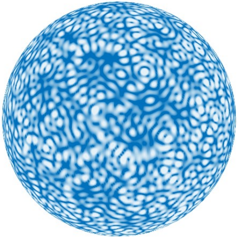
27 Aprile 2023

Helmholtz equation and Trefftz methods

Let u be a solution of the homogeneous Helmholtz equation $(n = 2, 3)$:



$$\Delta u + \kappa^2 u = 0, \quad \text{in } \Omega \subset \mathbb{R}^n.$$

The wavenumber is $\kappa = \omega/c > 0$ and the wavelength is $\lambda = \frac{2\pi}{\kappa}$.

$u(\mathbf{x})$ represents the space dependence of time-harmonic solutions $U(\mathbf{x}, t) = \Re\{e^{-i\omega t} u(\mathbf{x})\}$ of the wave equation $\frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

- 'Easy' PDE for small κ: perturbation of Laplace,
- 'Difficult' PDE for large κ: high-frequency problem.

Helmholtz equation and Trefftz methods

Trefftz method: computing the approximation $\tilde{u}$ of the form:

$$\tilde{u} := \sum_{p=1}^{P} \xi_p \, \phi_p,$$

where each element of the Trefftz space $\operatorname{span}\{\phi_p\}_{p=1}^{P}$ satisfies

$$\Delta \phi_p + \kappa^2 \phi_p = 0.$$

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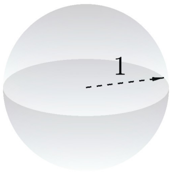
The setting of this presentation:

- Single-cell mesh, i.e. no h-refinement, and

$$\Omega \equiv B_1 \subset \mathbb{R}^n \text{ is the unit ball,}$$

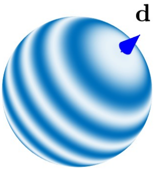
- ϕ_p are plane waves.



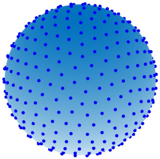
Propagative plane waves (PPWs)

Propagative plane waves have the form:

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}}, \quad \text{where } \mathbf{d} \in \mathbb{R}^n \text{ and } \mathbf{d}\cdot\mathbf{d} = 1.$$



PPWs are complex exponentials, thus easy and cheap to evaluate, differentiate, integrate...



For isotropic approximations, one can use (almost) evenly-spaced propagation direction $\{\mathbf{d}_p\}_p$:

$$\phi_p = e^{i\kappa \mathbf{d}_p\cdot\mathbf{x}}.$$

If $n = 3$, e.g. extremal point systems [Sloan, Womersley 2004].

Instability of PPWs

Can we construct accurate approximations $u(\mathbf{x}) \approx \sum_{p=1}^{P} \xi_p \, e^{i\kappa \mathbf{d}_p \cdot \mathbf{x}}$?

In theory, yes: better rates w.r.t #DOFs than polynomial spaces:

- [Cessenat, Després 1998],
- [Melenk 1995], [Moiola, Hiptmair, Perugia 2011].

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In pratice (finite-precision arithmetic), not always:

The issue is 'instability'.

Increasing #PPWs, at some point convergence stagnates.

- Numerical phenomenon due to finite-precision arithmetic and cancellation,
- PPW instability already observed in all PPW-based Trefftz methods and usually described as ill-conditioning issue.

Adcock–Huybrechs theory

Regardless of the reconstruction strategy, the linear system matrix $A \in \mathbb{C}^{S \times P}$ is ill-conditioned [Moiola, Hiptmair, Perugia 2011].

Oversampling ($S \gg P$) & SVD ϵ-regularization:

$$A = U\Sigma V^* \quad \rightarrow \quad A \approx U\Sigma_\epsilon V^* \quad \rightarrow \quad \boldsymbol{\xi}_{S,\epsilon} := V\Sigma_\epsilon^\dagger U^*\mathbf{b},$$

where the singular values below ϵ have been trimmed in Σ_ϵ.

Consider approximations $\tilde{u}[\boldsymbol{\mu}](\mathbf{x}) := \sum_{p=1}^{P} \mu_p \, e^{i\kappa \mathbf{d}_p \cdot \mathbf{x}}$, with $\boldsymbol{\mu} := (\mu_p)_p$

Theorem: [Parolin, Huybrechs, Moiola 2022]

Given $\epsilon \in (0,1]$, $\forall \boldsymbol{\mu} \in \mathbb{C}^P$ we have that, if S is large enough,

$$\left\| u - \tilde{u}\left[\boldsymbol{\xi}_{S,\epsilon}\right]\right\| \lesssim \left\| u - \tilde{u}\left[\boldsymbol{\mu}\right]\right\| + \epsilon \|\boldsymbol{\mu}\|.$$

Outline

Outline of the presentation:

- Instability of propagative plane waves (PPWs)
- Stability of evanescent plane waves (EPWs)
- Recipe for choosing the EPWs
- Numerical results

2D setting	3D setting
[Parolin, Huybrechs, Moiola 2022]	This work

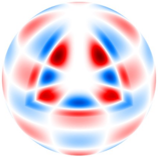
Instability of propagative plane waves (PPWs)

Spherical waves — Fourier–Bessel functions

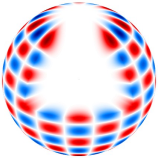
Spherical waves are separable solutions in spherical coordinates:

$$b_\ell^m(\mathbf{x}) := \beta_\ell \, j_\ell(\kappa|\mathbf{x}|) Y_\ell^m(\mathbf{x}/|\mathbf{x}|), \quad 0 \le |m| \le \ell, \ \forall \mathbf{x} \in B_1,$$

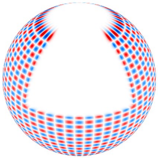
where $\beta_\ell \overset{\ell\to\infty}{\sim} 2\sqrt{2}\kappa\left(\frac{2}{e\kappa}\right)^\ell \ell^{\ell+\frac{1}{2}}$ is a H^1-normalization constant.



Propagative mode
$\ell = 2m = \kappa/2 = 8$



Grazing mode
$\ell = 2m = \kappa = 16$



Evanescent mode
$\ell = 2m = 3\kappa = 48$

Orthonormal basis for $\mathcal{B} := \{u \in H^1(B_1) : \Delta u + \kappa^2 u = 0\}$.

Modal analysis — PPW instability

The Jacobi–Anger identity relates PPWs to spherical waves b_ℓ^m:

$$e^{i\kappa\mathbf{d}\cdot\mathbf{x}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(4\pi i^\ell \overline{Y_\ell^m(\mathbf{d})}\beta_\ell^{-1}\right) b_\ell^m(\mathbf{x}).$$

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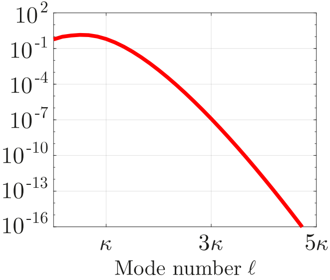

Asymptotics of Fourier coefficients:

$$\left|4\pi i^\ell \overline{Y_\ell^m(\mathbf{d})}\beta_\ell^{-1}\right| \overset{\ell\to\infty}{\sim} \mathcal{O}\left(\ell^{-\ell}\right)$$

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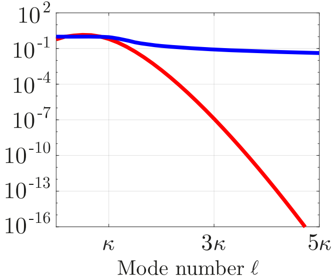

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Approximating $u = \sum_\ell \sum_m \hat{u}_\ell^m b_\ell^m \in \mathcal{B}$ needs exponentially large coefficients:

$$u \in H^s(B_1),\ s \geq 1 \iff |\hat{u}_\ell^m| \overset{\ell\to\infty}{\sim} o(\ell^{-s})$$

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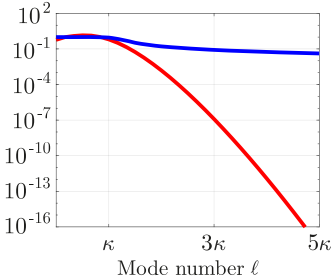

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Theorem: For every $0 \leq |m| \leq \ell, P \in \mathbb{N}, \mu \in \mathbb{C}^P$, and $0 < \eta \leq 1$

$$\left\|b_\ell^m - \tilde{u}[\boldsymbol{\mu}]\right\| \leq \eta \implies \|\boldsymbol{\mu}\| \geq (1-\eta)\beta_\ell/2\sqrt{\pi(2\ell+1)}.$$

Approximation of spherical waves by PPWs

Find $\boldsymbol{\xi}_{S,\epsilon} := (\xi_p)_{p=1}^{P}$ s.t.

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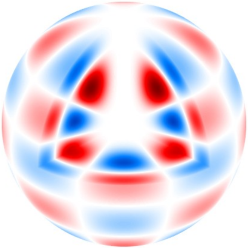

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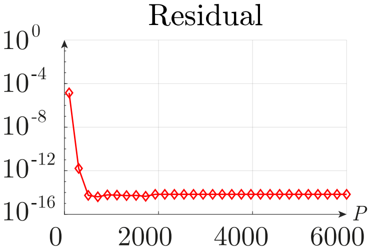

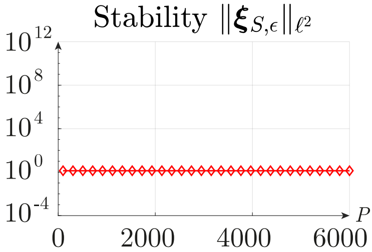

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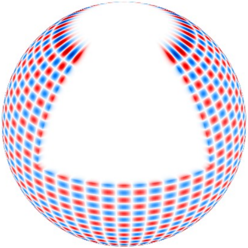

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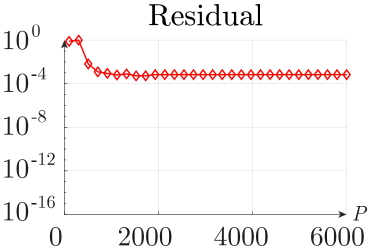

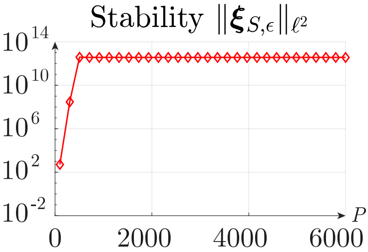

Stability of evanescent plane waves (EPWs)

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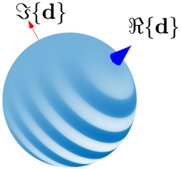

EPWs are exponential Helmholtz solutions again.

Let $\boldsymbol{\theta} := (\theta_1, \theta_2, \theta_3) \in \Theta := [0,\pi] \times [0,2\pi) \times [0,2\pi)$ be the Euler angles and $R_{\boldsymbol{\theta}}$ the associated rotation matrix. The wave direction is given by

$$\mathbf{d} = \mathbf{d}(\boldsymbol{\theta}, \zeta) := R_{\boldsymbol{\theta}}\, \mathbf{d}_{\uparrow}(\zeta/2\kappa + 1) \in \mathbb{C}^3, \qquad \forall\, (\boldsymbol{\theta}, \zeta) \in \Theta \times [0,+\infty),$$

where $\mathbf{d}_{\uparrow}$ is the reference upward complex direction vector defined by

$$\mathbf{d}_{\uparrow}(z) := (i\sqrt{z^2 - 1}, 0, z), \qquad \forall\, z \geq 1.$$

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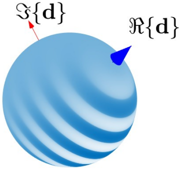

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Influence of the evanescence parameters θ_3 and ζ:

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = e^{i\left(\frac{\zeta}{2}+\kappa\right)\mathbf{d}_{\text{prop}}(\theta_1,\theta_2)\cdot\mathbf{x}}\; e^{-\left(\zeta\left(\frac{\zeta}{4}+\kappa\right)\right)^{1/2}\mathbf{d}^{\perp}_{\text{decay}}(\boldsymbol{\theta})\cdot\mathbf{x}},$$

where the directions $\mathbf{d}_{\text{prop}}(\theta_1,\theta_2)$ and $\mathbf{d}^{\perp}_{\text{decay}}(\boldsymbol{\theta})$ are real and orthogonal.

Modal analysis — EPW stability

The Jacobi–Anger identity holds also for EPWs:

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[4\pi i^{\ell} \beta_{\ell}^{-1} \overline{\mathbf{D}_{\ell}^{m}(\boldsymbol{\theta}) \cdot \mathbf{P}_{\ell}(\zeta)} \right] b_{\ell}^{m}(\mathbf{x}).$$

- $\mathbf{D}_{\ell}^{m}(\boldsymbol{\theta}) \in \mathbb{C}^{2\ell+1}$ is the $(\ell+m+1)$-column of the Wigner D-matrix,
- $\mathbf{P}_{\ell}(\zeta) := \left(\sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-n)!}{(\ell+n)!}} i^{n} P_{\ell}^{n} \left(\frac{\zeta}{2\kappa} + 1 \right) \right)_{n=-\ell}^{\ell} \in \mathbb{C}^{2\ell+1}$

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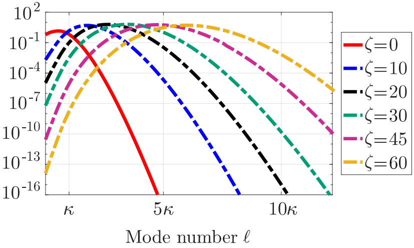

It looks promising! But how to choose the evanescence parameters?

Recipe for choosing the EPWs

Integral representation via EPWs

We want to represent $u \in \mathcal{B}$ as continuous superposition of EPWs:

$$u(\mathbf{x}) = \int_0^{+\infty}\int_\Theta v(\boldsymbol{\theta},\zeta)\; e^{i\kappa \mathbf{d}(\boldsymbol{\theta},\zeta)\cdot\mathbf{x}}\; w(\theta_1,\zeta)\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\zeta, \quad \forall \mathbf{x} \in B_1,$$

with density $v \in \mathcal{A} = \overline{\operatorname{span}\{a_\ell^m\}_{(\ell,m)}} \subsetneq L^2(\Theta\times[0,+\infty);w)$ and weight w.

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Theorem: The operator $T : \mathcal{A} \to \mathcal{B}$ is bounded and invertible,

$$Ta_\ell^m = \tau_\ell b_\ell^m, \qquad \tau_-\|v\|_\mathcal{A} \le \|Tv\|_\mathcal{B} \le \tau_+\|v\|_\mathcal{A}, \quad \forall v \in \mathcal{A},$$

where $\tau_\ell \in \mathbb{C}$ and $0 < \tau_- \le |\tau_\ell| \le \tau_+ < +\infty$ for every $\ell \ge 0$.

Hence, every Helmholtz solution $u \in \mathcal{B}$ is a (continuous) linear combination of EPWs with bounded coefficients: $\|v\|_\mathcal{A} \le \tau_-^{-1}\|u\|_\mathcal{B}$.

Sampling in the parametric domain

We seek suitable discretizations of the previous integral representation $u(\mathbf{x}) \approx \sum_{p=1}^{P} \xi_p e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_p,\zeta_p)\cdot\mathbf{x}} \omega_p$ with bounded coefficients $\boldsymbol{\xi}_{S,\epsilon} := (\xi_p)_p$.

- [Cohen, Migliorati 2017] and [Migliorati, Nobile 2022].

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We approximate $u = Tv$ by $u_L = Tv_L$, where v_L is the orthogonal projection in $\mathcal{A}_L := \mathrm{span}\{a_\ell^m\}_{\ell \le L}$. The $P \in \mathbb{N}$ cubature nodes $\{(\boldsymbol{\theta}_p, \zeta_p)\}_{p=1}^{P}$ distribute according to the probability density

$$\rho_L := \frac{w\mu_L}{(L+1)^2}, \qquad \text{where } \mu_L := \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2.$$

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↓

We expect $u_L \in \mathcal{B}_L := \text{span}\{b_\ell^m\}_{\ell \leq L}$ to be approximated by

$$\left\{ \mathbf{x} \mapsto \frac{1}{\sqrt{P\mu_L(\boldsymbol{\theta}_p, \zeta_p)}} e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_p,\zeta_p)\cdot\mathbf{x}} \right\}_{p=1}^{P} \subset \mathcal{B},$$

with bounded coefficients.

Numerical results

Approximation of spherical waves by EPWs

Find $\boldsymbol{\xi}_{S,\epsilon} := (\xi_p)_p$ such that

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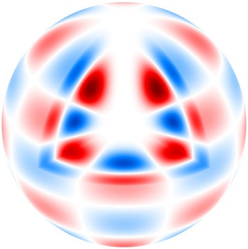

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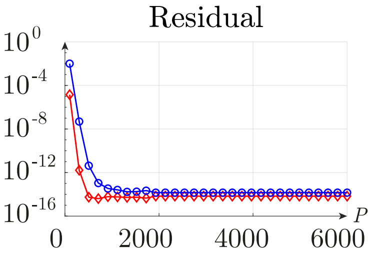

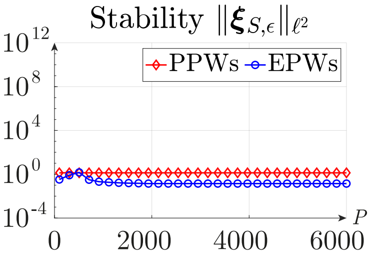

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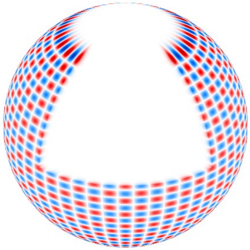
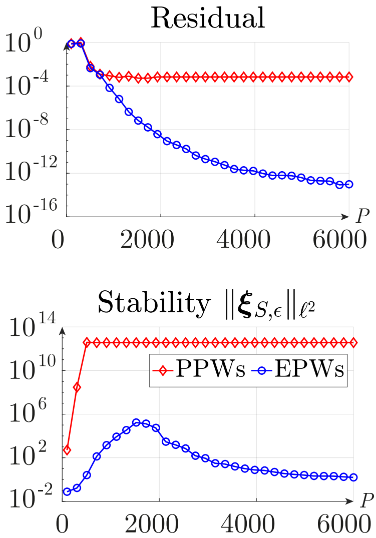


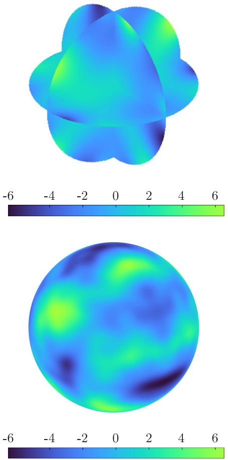
Evanescent mode
$\ell = 2m = 3\kappa = 48$



Solution and error plots

Approximation of $u = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \hat{u}_\ell^m b_\ell^m \in \mathcal{B}_L$ with random $(\hat{u}_\ell^m)_{(\ell,m)}$,

$\kappa = 5, \qquad L = 25, \qquad \dim \mathcal{B}_L = 676, \qquad P = 6084.$



Real part solution $\Re u$

Solution and error plots

Approximation of $u = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \hat{u}_\ell^m b_\ell^m \in \mathcal{B}_L$ with random $(\hat{u}_\ell^m)_{(\ell,m)}$,

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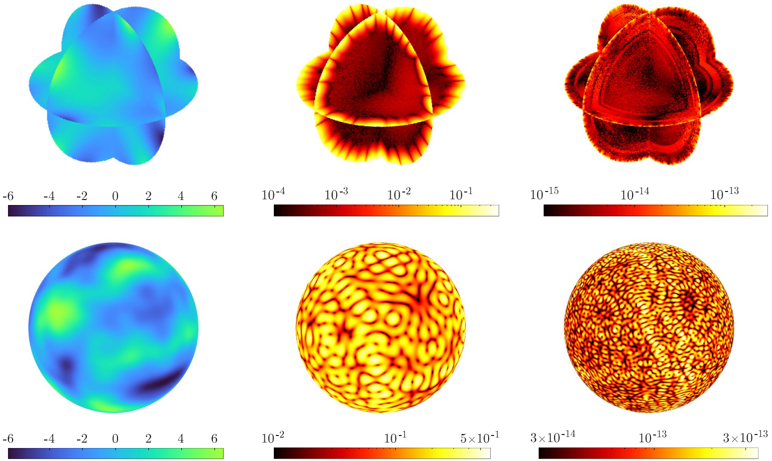


Real part solution $\Re u$ | Error $|u - \tilde{u}|$ PPWs | Error $|u - \tilde{u}|$ EPWs

Conclusions

Summary

Ill-conditioning can be overcome (via regularization) if there exist accurate and stable approximations (bounded coefficients).

To approximate Helmholtz solutions with Trefftz methods

- PPWs give accurate but **unstable** results,
- EPWs give accurate and **stable** results.

$\rightarrow$ Key result is the stable integral representation.

EPWs parameters are chosen by sampling the parametric domain according to some explicit probability density.

Next steps:

- Prove the EPW stability conjecture
- Time-harmonic Maxwell/Elasticity
- Extend to general geometries
- Tailor to Trefftz–DG schemes

Thank you for your attention!

- GitHub repository (code written in MATLAB):
  https://github.com/Nicola-Galante/evanescent-plane-wave-approximation

Reconstruction from Dirichlet sampling data

How to construct $\tilde{u}[\boldsymbol{\xi}](\mathbf{x}) := \sum_{p=1}^{P} \xi_p \, e^{i\kappa \mathbf{d}_p \cdot \mathbf{x}}$ approximation of u?

Collocation method with $S \in \mathbb{N}$ Dirichlet data:

$$\tilde{u}[\boldsymbol{\xi}_S](\mathbf{x}_s) = u(\mathbf{x}_s), \quad \forall s = 1, ..., S \quad \rightarrow \quad A\boldsymbol{\xi}_S = \mathbf{b},$$

where $\{\mathbf{x}_s\}_{s=1}^{S}$ are (almost) evenly-spaced points on ∂B_1.

- A is ill-conditioned [Moiola, Hiptmair, Perugia 2011].

Oversampling ($S \gg P$) & SVD ϵ-regularization:

$$A = U\Sigma V^* \quad \rightarrow \quad A \approx U\Sigma_\epsilon V^* \quad \rightarrow \quad \boldsymbol{\xi}_{S,\epsilon} := V\Sigma_\epsilon^\dagger U^* \mathbf{b},$$

where the singular values below ϵ have been trimmed in Σ_ϵ.

- Well-defined if $u \in C^0(\overline{B_1})$ and $\kappa^2 \neq \Delta$-Dirichlet eigenvalue.

Definition: Ferrers functions and Legendre polynomials

For every $(\ell, m) \in \mathcal{I}$, the Ferrers functions are defined as:

$$\mathsf{P}_\ell^m(x) := \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell, \qquad |x| \leq 1,$$

so that

$$\mathsf{P}_\ell^{-m}(x) = (-1)^m\frac{(\ell-m)!}{(\ell+m)!}\mathsf{P}_\ell^m(x), \qquad |x| \leq 1.$$

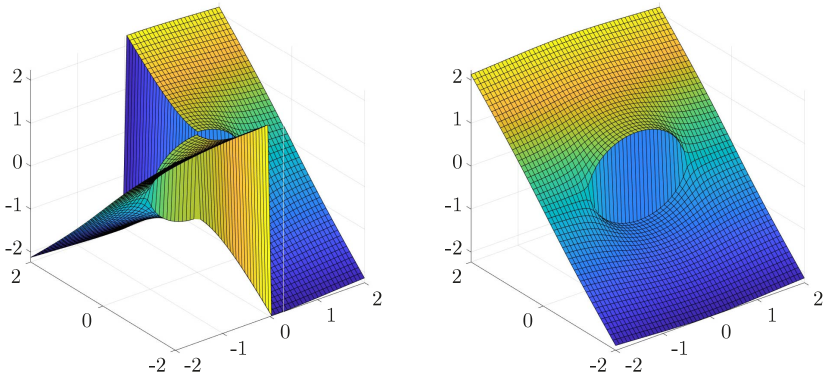
The associated Legendre polynomials are defined as:

$$P_\ell^m(z) := \frac{1}{2^\ell \ell!}(z^2-1)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}z^{\ell+m}}(z^2-1)^\ell, \qquad \forall z \in \mathbb{C},$$

so that

$$P_\ell^{-m}(z) = \frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(z), \qquad \forall z \in \mathbb{C}.$$

Definition: associated Legendre polynomials



We use the convention

$$(z^2-1)^{m/2} := \mathcal{P}\left[(z+1)^{m/2}\right]\mathcal{P}\left[(z-1)^{m/2}\right], \qquad \forall z \in \mathbb{C},$$

where $\mathcal{P}[\,\cdot\,]$ indicates that the principal branch is chosen.

Definition: Wigner matrices

Let $\boldsymbol{\theta} := (\theta_1, \theta_2, \theta_3)$ be the Euler angles. The Wigner D-matrix is the unitary matrix $D_\ell(\boldsymbol{\theta}) = (D_\ell^{m,m'}(\boldsymbol{\theta}))_{m,m'} \in \mathbb{C}^{(2\ell+1)\times(2\ell+1)}$, where $|m|, |m'| \leq \ell$, whose elements are defined by

$$D_\ell^{m,m'}(\boldsymbol{\theta}) := e^{im'\theta_2} d_\ell^{m,m'}(\theta_1) e^{im\theta_3}.$$

In turn, the matrix $d_\ell(\theta) := (d_\ell^{m,m'}(\theta))_{m,m'} \in \mathbb{R}^{(2\ell+1)\times(2\ell+1)}$, where $|m|, |m'| \leq \ell$, is called Wigner d-matrix , its elements are

$$d_\ell^{m,m'}(\theta) := \sum_{k=k_{\min}}^{k_{\max}} w_{\ell,k}^{m,m'} \left(\cos\frac{\theta}{2}\right)^{2(\ell-k)+m'-m} \left(\sin\frac{\theta}{2}\right)^{2k+m-m'}$$

where

$$w_{\ell,k}^{m,m'} := \frac{(-1)^k \left[(\ell+m)!(\ell-m)!(\ell+m')!(\ell-m')!\right]^{1/2}}{(\ell-m-k)!(\ell+m'-k)!(k+m-m')!\, k!}$$

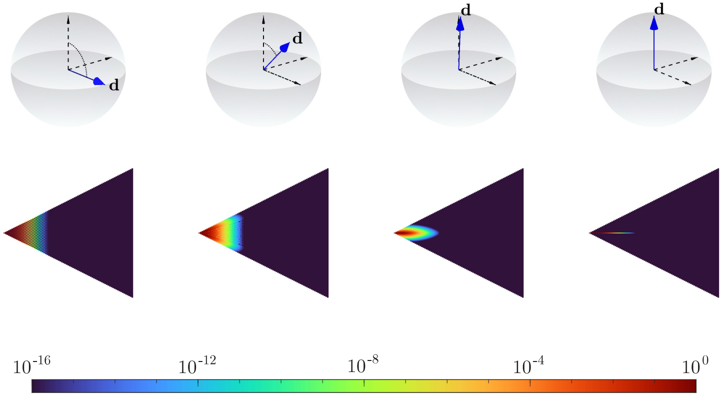
with $k_{\min} := \max\{0, m'-m\}$ and $k_{\max} := \max\{\ell-m, \ell+m'\}$.

Modal analysis—PPWs

For any $\mathbf{d} = \mathbf{d}(\theta_1, \theta_2) \in \mathbb{S}^2$, if $\phi_{\mathbf{d}}(\mathbf{x}) := e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$, we have

$$\hat{\phi}_\ell^m(\theta_1) := |(\phi_{\mathbf{d}}, b_\ell^m)_{\mathcal{B}}| = \frac{4\pi}{\beta_\ell}\gamma_\ell^m \, |\mathsf{P}_\ell^m(\cos\theta_1)|$$

$\hat{\phi}_\ell^m(\pi/2)$ $\hat{\phi}_\ell^m(\pi/4)$ $\hat{\phi}_\ell^m(\pi/64)$ $\hat{\phi}_\ell^m(0)$

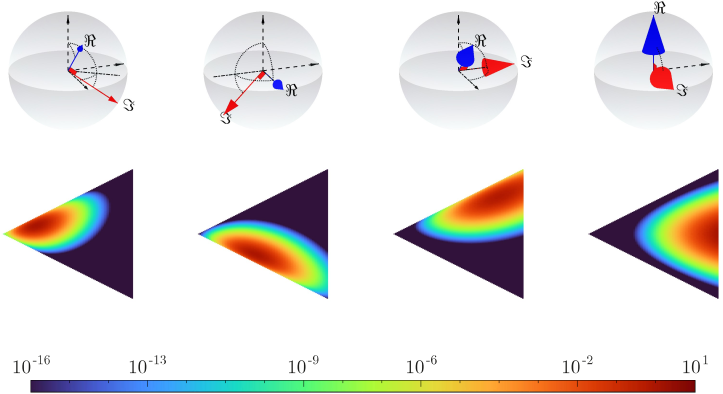


Modal analysis—EPWs

For any $\mathbf{d} = \mathbf{d}(\theta_1, \theta_2, \theta_3, \zeta) \in \mathbb{S}^2$, if $\phi_{\mathbf{d}}(\mathbf{x}) := e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$, we have

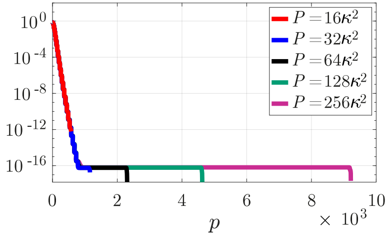
$$\hat{\phi}_\ell^m(\theta_1, \theta_3, \zeta) := |(\phi_{\mathbf{d}}, b_\ell^m)_{\mathcal{B}}| = \frac{4\pi}{\beta_\ell} \left| \sum_{m'=-\ell}^{\ell} \gamma_\ell^{m'} i^{-m'} d_\ell^{m',m}(\theta_1) e^{-im'\theta_3} P_\ell^{m'}\left(\frac{\zeta}{2\kappa} + 1\right) \right|$$

$\hat{\phi}_\ell^m(\frac{\pi}{4}, \frac{\pi}{4}, 30)$ $\quad$ $\hat{\phi}_\ell^m(\frac{\pi}{2}, \frac{7\pi}{4}, 60)$ $\quad$ $\hat{\phi}_\ell^m(\frac{\pi}{4}, \frac{\pi}{2}, 120)$ $\quad$ $\hat{\phi}_\ell^m(0, 0, 180)$

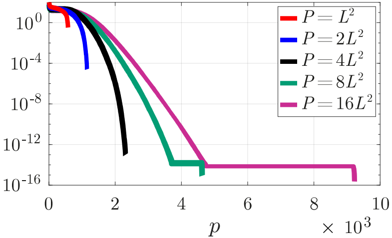


Singular values $\{\sigma_p\}_p$ of the matrix A

PPWs



EPWs (Sobol, $L = 4\kappa$)

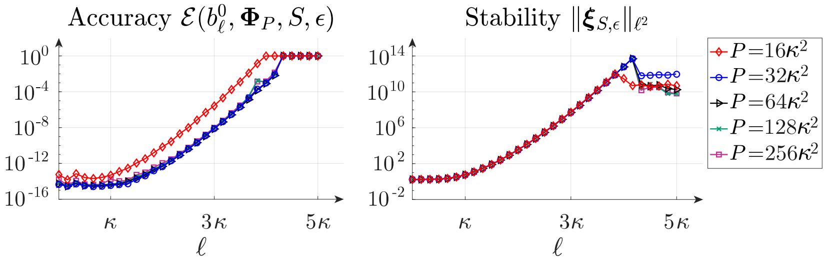


- If P is large enough, the condition number of the matrix A is comparable for both PPWs and EPWs
- The improved accuracy for evanescent modes is not due to a better conditioning of the linear system, but rather to an increase of the ϵ-rank
- Raising the truncation parameter L allows to increase the ϵ-rank

Approximation of spherical waves by PPWs

Approximation of spherical waves b_ℓ^0 by PPWs.

$$\kappa = 6, \quad \epsilon = 10^{-14}, \quad S = 2P, \quad \text{Residual } \mathcal{E} := \frac{\|A\boldsymbol{\xi}_{S,\epsilon} - \mathbf{b}\|}{\|\mathbf{b}\|}.$$

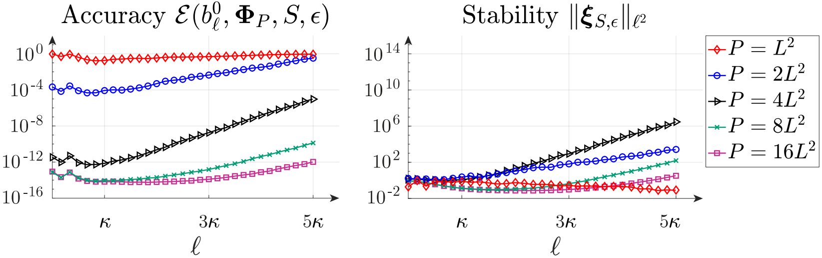


- Propagative modes $\ell \lesssim \kappa$: $\mathcal{O}(\epsilon)$ error, $\mathcal{O}(1)$ coefficients $\forall P$,
- Evanescent modes $\ell \gtrsim 4\kappa$: $\mathcal{O}(1)$ error, large coefficients $\forall P$.

Approximation of spherical waves by EPWs

Approximation of spherical waves b_ℓ^0 by EPWs (Sobol sampling).

$$\kappa = 6, \quad \epsilon = 10^{-14}, \quad S = 2P, \quad L = 4\kappa.$$



- If $P \in \mathbb{N}$ is large enough, the discrete EPW space approximates all b_ℓ^m for $0 \le |m| \le \ell \le L = 4\kappa$. The accuracy and stability properties do not differ significantly varying the order $|m| \le \ell$.

Weighted L^2 space $\mathcal{A}$

The weight function is:

$$w(\theta_1, \zeta) := \sin(\theta_1)\, \zeta^{1/2} e^{-\zeta}, \qquad \forall (\boldsymbol{\theta}, \zeta) \in \Theta \times [0, +\infty).$$

The w-weighted L^2 Hermitian product and the associated norm are:

$$(u, v)_{\mathcal{A}} := \int_0^{+\infty} \int_{\Theta} u(\boldsymbol{\theta}, \zeta) \overline{v(\boldsymbol{\theta}, \zeta)} w(\theta_1, \zeta) \mathrm{d}\boldsymbol{\theta}\, \mathrm{d}\zeta, \qquad \|u\|_{\mathcal{A}}^2 := (u, u)_{\mathcal{A}},$$

where $u, v \in L^2(\Theta \times [0, +\infty); w)$.

For every $(\ell, m) \in \mathcal{I}$, the Herglotz densities are defined as:

$$a_\ell^m(\boldsymbol{\theta}, \zeta) := \alpha_\ell\, \mathbf{D}_\ell^m(\boldsymbol{\theta}) \cdot \mathbf{P}_\ell(\zeta), \quad \forall (\boldsymbol{\theta}, \zeta) \in \Theta \times [0, +\infty),$$

where α_ℓ is a L^2_w-normalization constant. Furthermore:

$$\mathcal{A} := \overline{\mathrm{span}\{a_\ell^m\}_{(\ell,m) \in \mathcal{I}}}^{\|\cdot\|_{\mathcal{A}}} \subsetneq L^2(\Theta \times [0, +\infty); w).$$

The reproducing kernel property

We seek suitable discretizations of the previous integral representation $u(\mathbf{x}) \approx \sum_{p=1}^{P} \xi_p e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_p, \zeta_p)\cdot \mathbf{x}} \omega_p$, with bounded coefficients $\boldsymbol{\xi}_{S,\epsilon} := (\xi_p)_p$.

Corollary: The space $\mathcal{A}$ has the reproducing kernel property

$$|v(\boldsymbol{\theta}, \zeta)| \leq C(\boldsymbol{\theta}, \zeta) \|v\|_{\mathcal{A}}, \qquad \forall v \in \mathcal{A},$$

$$\overset{\text{Riesz Th.}}{\Longrightarrow} \qquad \exists!\, K_{\boldsymbol{\theta},\zeta} \in \mathcal{A} : v(\boldsymbol{\theta}, \zeta) = (v, K_{\boldsymbol{\theta},\zeta})_{\mathcal{A}}, \qquad \forall v \in \mathcal{A}.$$

Moreover, the EPWs are the images under T of $K_{\boldsymbol{\theta},\zeta}$, namely

$$K_{\boldsymbol{\theta},\zeta} \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad \mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\boldsymbol{\theta},\zeta)\cdot \mathbf{x}}$$

$\Longrightarrow$ Equivalence between two approximation problems:

in the parametric space $\mathcal{A}$ — in the physical space $\mathcal{B}$

$$v \approx \sum_{p=1}^{P} \xi_p\, K_{\boldsymbol{\theta}_p,\zeta_p} \omega_p \quad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \quad u(\mathbf{x}) \approx \sum_{p=1}^{P} \xi_p\, e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_p,\zeta_p)\cdot \mathbf{x}}\, \omega_p$$

Sampling in the parametric domain

- [Cohen, Migliorati 2017] and [Migliorati, Nobile 2022].

Fix $L \geq 0$, set $\mathcal{A}_L := \mathrm{span}\{a_\ell^m\}_{\ell \leq L}$, define the probability density

$$\rho_L := \frac{w\mu_L}{(L+1)^2}, \qquad \text{where } \mu_L := \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} |a_\ell^m|^2,$$

generate $P \in \mathbb{N}$ nodes $\{(\boldsymbol{\theta}_p, \zeta_p)\}_{p=1}^P$ distributed according to ρ_L.

↓

We expect that $v \in \mathcal{A}_L := \mathrm{span}\{a_\ell^m\}_{\ell \leq L}$ can be approximated by

$$\left\{ (\boldsymbol{\theta}, \zeta) \mapsto \frac{1}{\sqrt{P\mu_L(\boldsymbol{\theta}_p, \zeta_p)}} K_{(\boldsymbol{\theta}_p, \zeta_p)}(\boldsymbol{\theta}, \zeta) \right\}_{p=1}^{P} \subset \mathcal{A},$$

with bounded coefficients.

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generate $P \in \mathbb{N}$ nodes $\{(\boldsymbol{\theta}_p, \zeta_p)\}_{p=1}^{P}$ distributed according to ρ_L.

$\downarrow$

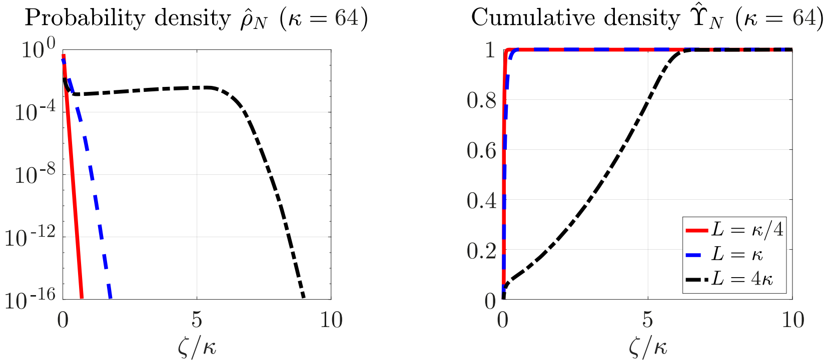
We expect that $u \in \mathcal{B}_L := \mathrm{span}\{b_\ell^m\}_{\ell \leq L}$ can be approximated by

$$\left\{ \mathbf{x} \mapsto \frac{1}{\sqrt{P\mu_L(\boldsymbol{\theta}_p, \zeta_p)}} e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_p, \zeta_p)\cdot \mathbf{x}} \right\}_{p=1}^{P} \subset \mathcal{B},$$

with bounded coefficients.

Probability measures

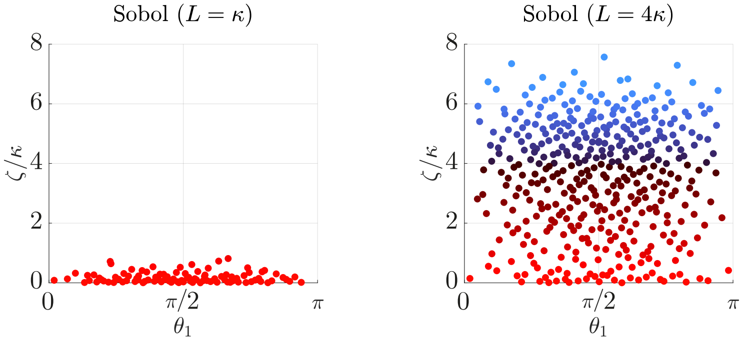
Probability density ρ_N and cumulative density Υ_N as functions of ζ:



They depend on L: target functions in $\mathcal{B}_L := \text{span}\{b_\ell^m\}_{\ell \leq L}$.

Sampling nodes in the parametric domain

Samples $\{(\theta_{1,p}, \zeta_p)\}_{p=1}^{P}$ in $[0,\pi] \times [0,+\infty)$:

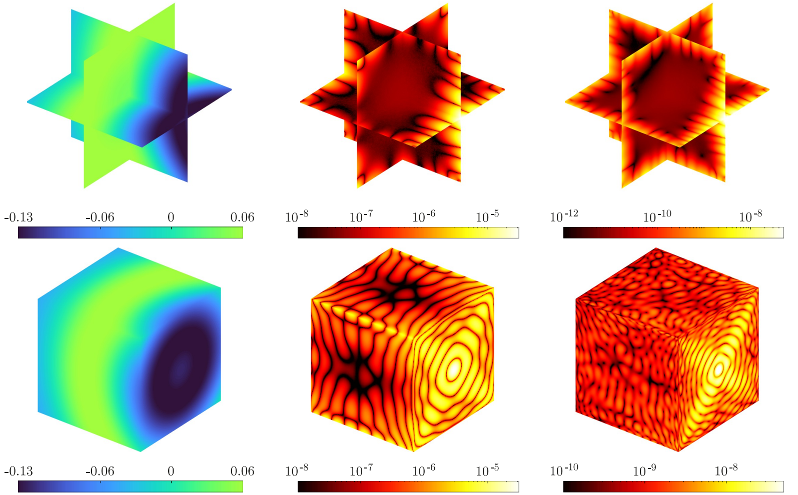


The initial samples in $[0,1]^4$, here corresponding to Sobol sequences (quasi-random low-discrepancy sequences), are generated according to the product of four uniform distributions $\mathcal{U}_{[0,1]}$ and then are mapped back to the parametric domain $\Theta \times [0,+\infty)$ by Υ_N^{-1}.

Fundamental solution and different geometries

Approximation of $u(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}-\mathbf{s}|}}{4\pi|\mathbf{x}-\mathbf{s}|}$ with $\mathbf{s} \in \mathbb{R}^3 \setminus \overline{\Omega}$ so that $\mathrm{dist}(\mathbf{s}, \partial\Omega) = \frac{\lambda}{3}$,

$\kappa = 5$, $L = 16$, $P = 2704$.



Real part solution $\Re u$ | Error $|u - \tilde{u}|$ PPWs | Error $|u - \tilde{u}|$ EPWs

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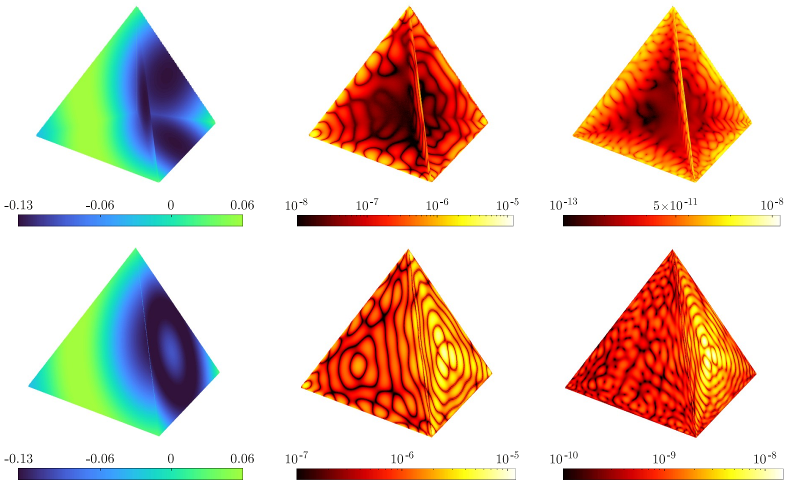

Real part solution $\Re u$ | Error $|u - \tilde{u}|$ PPWs | Error $|u - \tilde{u}|$ EPWs

Enhanced accuracy near singularities