Stable approximation of Helmholtz solutions using evanescent plane waves: an application to conforming Trefftz methods

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Helmholtz equation and Trefftz methods

Let u be a solution of the Helmholtz equation (wavenumber $\kappa > 0$):

$$-\Delta u - \kappa^2 u = 0, \qquad \text{in a bounded domain } \Omega \subset \mathbb{R}^d, \quad d \in \{2, 3\}$$

Goal: Computing approximation of u using Trefftz methods

$$u \approx \sum_{n=1}^{N} \xi_n \phi_n, \qquad \text{where} \quad -\Delta \phi_n - \kappa^2 \phi_n = 0 \quad \text{(locally)}$$

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Strengths:

- Spectral accuracy
- Many formulations (LS, TDG, UWVF)

Weaknesses:

- Limited to piecewise-constant coefficients & homogeneous PDEs
- High numerical instability from redundancy in approximation sets, e.g. propagative plane waves, convergence stalls in finite precision

Outline

- **Propagative plane waves (PPWs)**
- **Evanescent plane waves (EPWs)**
- **EPW-Trefftz Continuous Galerkin Methods**

Propagative plane waves (PPWs)

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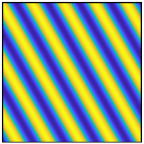

Propagative plane wave (defined in $\mathbb{R}^d$)

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}}, \qquad \text{where} \quad \mathbf{d} \in \mathbb{R}^d, \quad \mathbf{d}\cdot\mathbf{d} = 1$$

- Exact solution of $(-\Delta - \kappa^2)\, u = 0$, since $\mathbf{d}\cdot\mathbf{d} = 1$
- Simple parametrization of propagation direction $\mathbf{d}(\boldsymbol{\theta}) \in \mathbb{S}^{d-1}$:
  - $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [0, 2\pi)$ in 2D
  - $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [0, 2\pi) \times [0, \pi]$ in 3D
- Easy to manipulate: closed-form integration on flat submanifold
- Simple discretization: evenly-distributed $\mathbf{d}(\boldsymbol{\theta}_n) \in \mathbb{S}^{d-1}$

$$u(\mathbf{x}) \approx \sum_{n=1}^{N} \xi_n e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_n)\cdot\mathbf{x}}$$

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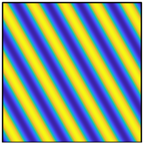

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Approximation results: better rates than polynomial w.r.t # DOFs

- h-estimates: Taylor expansions [Cessenat, Després 1998]
- hp-estimates: Vekua theory, wavenumber-explicit [Melenk 1995], [Moiola, Hiptmair, Perugia 2011]

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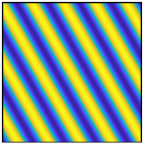

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$$u(\mathbf{x}) \approx \textstyle\sum_{n=1}^{N} \xi_n e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_n)\cdot\mathbf{x}}$$

In practice (finite-precision arithmetic) → convergence stagnates:

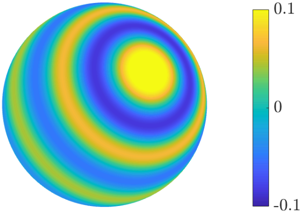
- Redundant approximation set if $\mathbf{d}(\boldsymbol{\theta}_n)\cdot\mathbf{d}(\boldsymbol{\theta}_m) \approx 1$ for $n \neq m$
- Ill-conditioned linear system [Hiptmair, Moiola, Perugia 2016]
- Requires regularization [Barucq, Bendali, Diaz, Tordeux 2021]

Motivating numerical experiment (PPWs)

Approximation in the unit ball Ω of the 3D fundamental solution, i.e.

$$\mathbf{x} \mapsto \frac{1}{4\pi} \frac{e^{i\kappa|\mathbf{x}-\mathbf{s}|}}{|\mathbf{x}-\mathbf{s}|}, \qquad \mathbf{s} \in \mathbb{R}^3 \setminus \overline{\Omega}$$

with $\kappa = 10$ and $\operatorname{dist}(\mathbf{s}, \Omega)/\lambda = 1$



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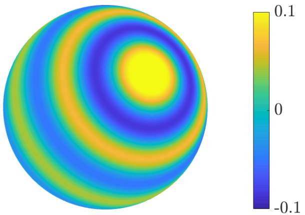

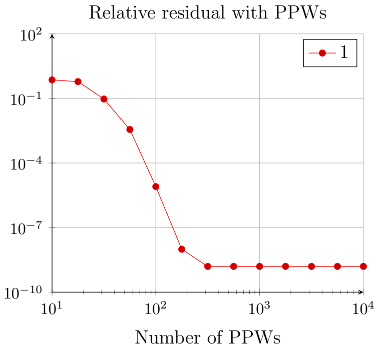

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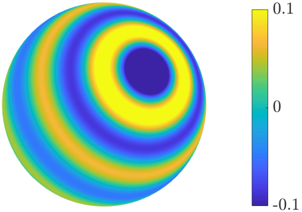

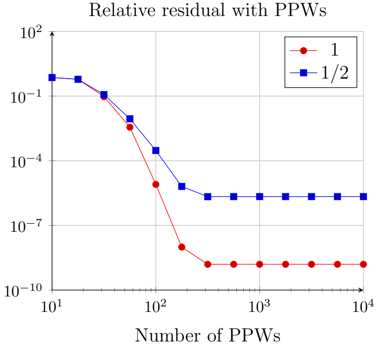

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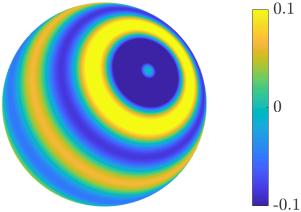

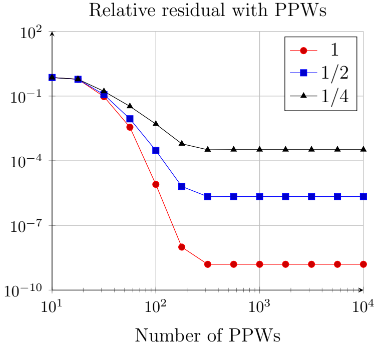

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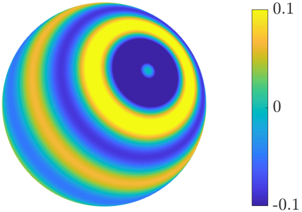

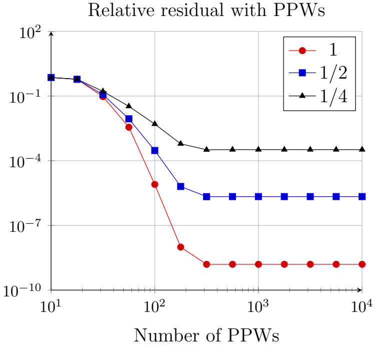

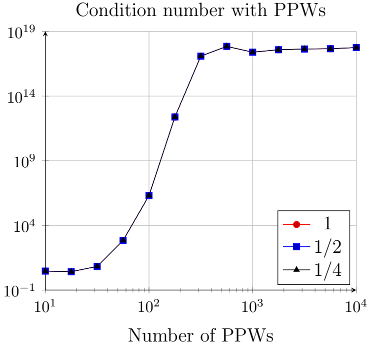

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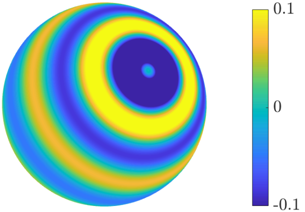

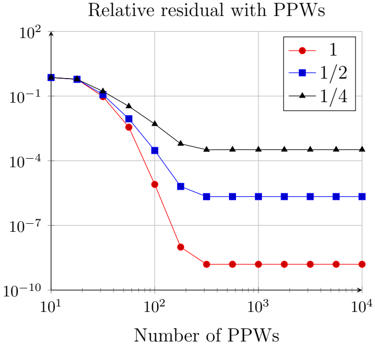

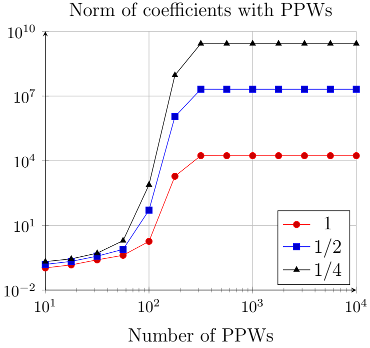

Continuous superposition of PPWs

For a bounded Lipschitz domain Ω, let $T_{\mathrm{P}} : L^2(\boldsymbol{\Theta}) \to H^1(\Omega)$ such that

$$\left(T_{\mathrm{P}} v\right)(\mathbf{x}) := \int_{\boldsymbol{\Theta}} v(\boldsymbol{\theta}) e^{i\kappa \mathbf{d}(\boldsymbol{\theta})\cdot\mathbf{x}} \mathrm{d}\sigma(\boldsymbol{\theta}), \qquad \mathbf{x} \in \Omega$$

- T_{P} is bounded, and $u = T_{\mathrm{P}} v$ is an Helmholtz solution in Ω called Herglotz function [Colton, Kress 2013]
- T_{P} has H^1-dense image in the Helmholtz space in Ω [Weck 2004]
- T_{P} is a Hilbert–Schmidt operator $\Longrightarrow$ not boundedly invertible

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[Adcock, Huybrechs 2020] Ill-conditioning can be solved, provided accurate approximations with **bounded coefficients** exist

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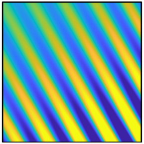


Evanescent plane wave (defined in $\mathbb{R}^d$)

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}}, \qquad \text{where} \quad \mathbf{d} \in \mathbb{C}^d, \quad \mathbf{d}\cdot\mathbf{d} = 1$$

- Complex-valued direction vector $\mathbf{d} \in \mathbb{C}^d$:
  - Propagation direction $= \Re(\mathbf{d})$
  - Evanescence direction $= \Im(\mathbf{d})$
- Still exact solution of $(-\Delta - \kappa^2)\, u = 0$, since $\mathbf{d}\cdot\mathbf{d} = 1$
- Still easy and cheap to evaluate, differentiate, integrate, etc.
- Localization effect in a bounded domain: requires normalization

EPW direction parametrization

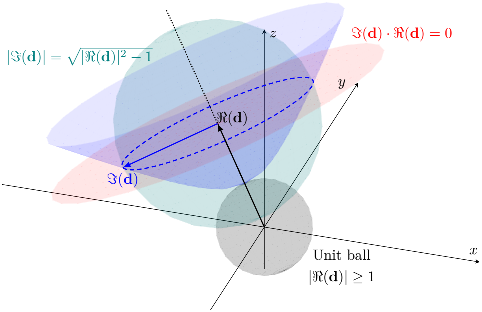
Evanescent plane wave directions $\mathbf{d} \in \mathbb{C}^d$ can be parametrized as

$$\mathbf{d}(\mathbf{y}) = \cosh(\zeta)\mathbf{d}^{\parallel}(\boldsymbol{\theta}) + i\sinh(\zeta)\mathbf{d}^{\perp}_{\boldsymbol{\theta}}(\varphi), \qquad \mathbf{y} := (\zeta, \boldsymbol{\theta}, \varphi) \in \mathbb{R}^+ \times \boldsymbol{\Theta} \times \Phi$$

- Decay strength $|\Im(\mathbf{d})|$ parametrized by $\zeta \in \mathbb{R}^+$
- Propagation direction $\mathbf{d}^{\parallel}$ parametrized by $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Evanescence direction $\mathbf{d}^{\perp}_{\boldsymbol{\theta}}$ parametrized by $\varphi \in \Phi$,

$$\Phi := \begin{cases} \{\pm 1\} & \text{in 2D} \\ [0, 2\pi) & \text{in 3D} \end{cases}$$

- PPWs recovered for $\zeta = 0$



Continuous superposition of EPWs

If Ω is a disk in 2D or a ball in 3D, let $T_{\mathrm{E}} : L^2_{w^2}(\mathbb{R}^+ \times \boldsymbol{\Theta} \times \Phi) \to H^1(\Omega)$

$$(T_{\mathrm{E}} v)(\mathbf{x}) := \int_{\mathbb{R}^+} \int_{\boldsymbol{\Theta}} \int_{\Phi} v(\zeta, \boldsymbol{\theta}, \varphi)\, e^{i\kappa \mathbf{d}(\zeta, \boldsymbol{\theta}, \varphi) \cdot \mathbf{x}}\, w^2(\zeta) \mathrm{d}\sigma(\varphi) \mathrm{d}\sigma(\boldsymbol{\theta}) \mathrm{d}\zeta$$

[Parolin, Huybrechs, Moiola 2023] and [G., Moiola, Parolin 2024] provide a weight w such that T_{E} is boundedly invertible, namely

$$\forall u \text{ Helmholtz solution}, \quad v = T_{\mathrm{E}}^{-1} u, \qquad \|v\|_{L^2_{w^2}} \lesssim \|u\|_{H^1}$$

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In PPW setting, the operator $T_{\mathrm{P}} : L^2(\boldsymbol{\Theta}) \to H^1(\Omega)$ is compact with H^1-dense image in the Helmholtz solution space in Ω, thus

$$\forall u \notin \operatorname{range} T_{\mathrm{P}} \quad \exists\, (v_n)_n \subset L^2(\boldsymbol{\Theta}) : \quad \begin{cases} \|u - T_{\mathrm{P}} v_n\|_{H^1} \to 0 \\ \|v_n\|_{L^2} \to +\infty \end{cases}$$

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Approximation in the unit ball Ω of the 3D fundamental solution, i.e.

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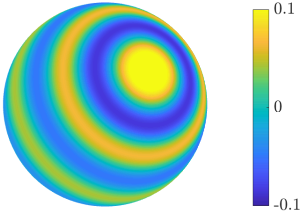
with $\kappa = 10$ and $\mathrm{dist}\,(\mathbf{s}, \Omega)\,/\lambda = 1$



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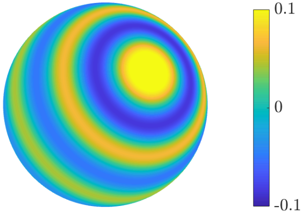

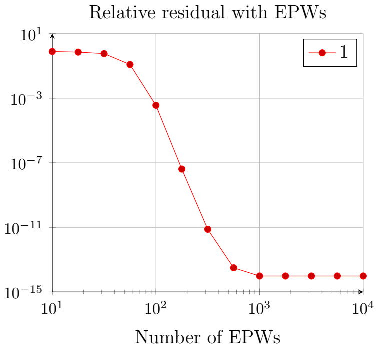

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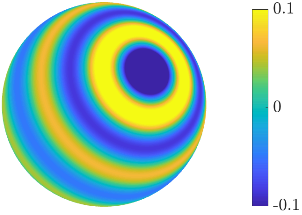

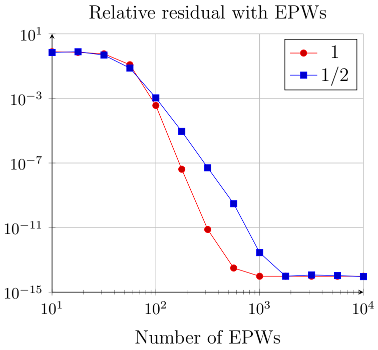

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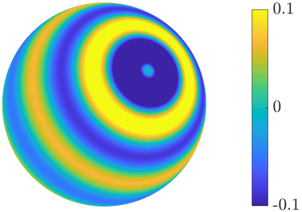

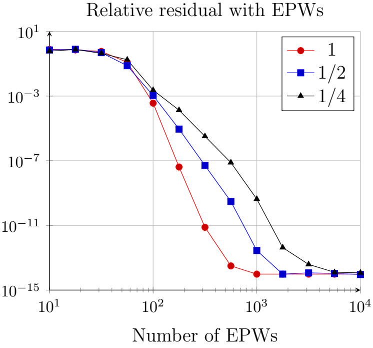

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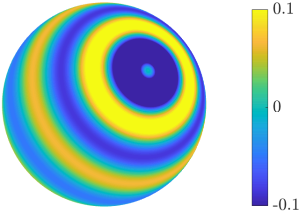

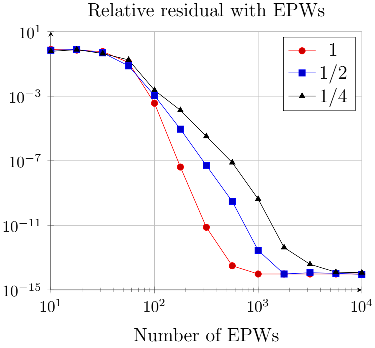

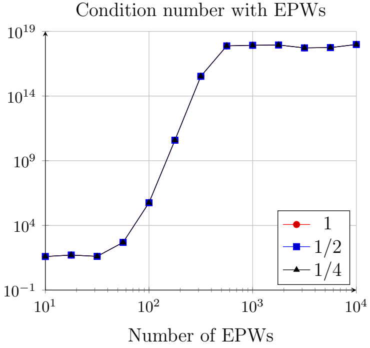

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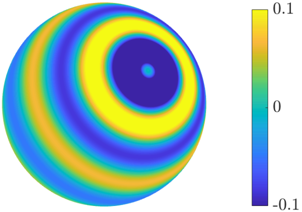

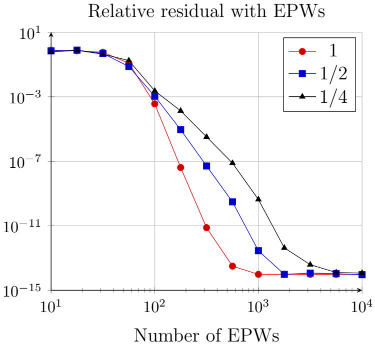

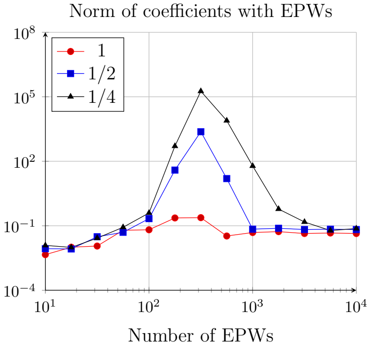

EPW approximation sets and Trefftz methods

- In general, it is difficult to construct EPW discrete approximation sets

$$\left\{\mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\mathbf{y}_n)\cdot \mathbf{x}}\right\}_{n=1}^{N}, \qquad \text{where} \qquad \mathbf{y}_n = (\zeta_n, \boldsymbol{\theta}_n, \varphi_n)$$

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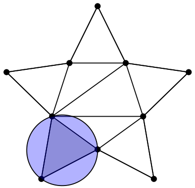

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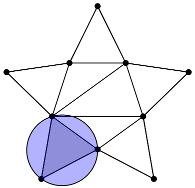
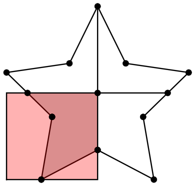


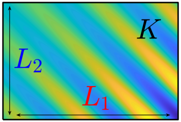
- Different recipe can be built for rectangular geometry
- The approximation set is explicit and easy to construct
- Applying the recipe to each rectangle enables the construction of **Trefftz Continuous Galerkin methods**



EPW-Trefftz Continuous Galerkin Methods

Rectangular cell symmetries

- Let K be a rectangle and $\phi(\mathbf{y})$ a normalized EPW centered in K



Goal: Construct a family of (linear combinations of) EPWs whose trace forms a $L^2(\partial K)$ Hilbert basis. To achieve this, we exploit K's symmetries

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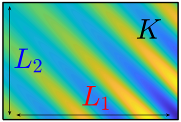

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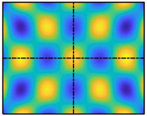
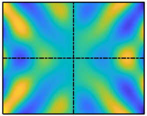
- Let S_i denote the reflection operator that flips the i-th coordinate
- For any $\mathbf{j} = (j_1, j_2) \in \{0,1\}^2$ we define the orthogonal projections $\Pi_{\mathbf{j}}$

$$\Pi_{00} = \frac{\mathrm{Id} + \mathrm{S}_1 + \mathrm{S}_2 + \mathrm{S}_1\mathrm{S}_2}{4}$$



$$\Pi_{10} = \frac{\mathrm{Id} - \mathrm{S}_1 + \mathrm{S}_2 - \mathrm{S}_1\mathrm{S}_2}{4}$$



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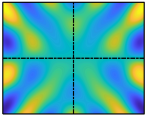

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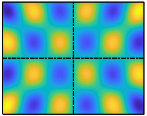

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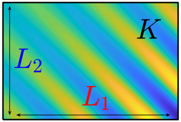

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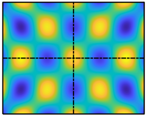

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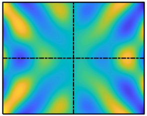

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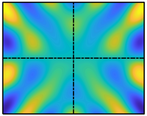

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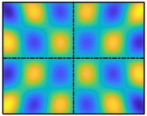
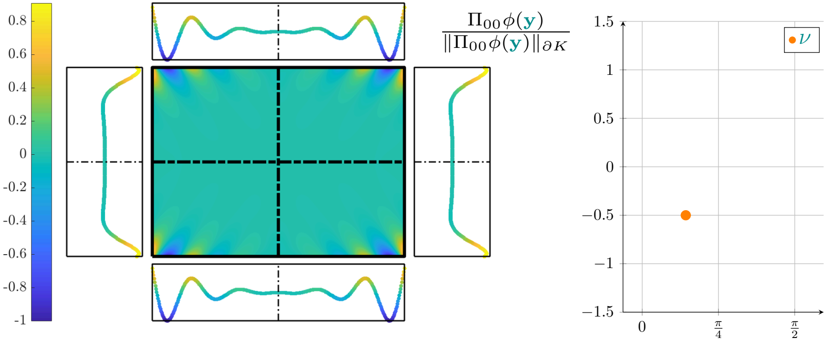


- $\Pi_\mathbf{j}\phi(\mathbf{y})$ is a linear combination of EPWs $\implies$ solves Helmholtz

An orthogonal EPW basis

- Let $\mathbf{y} = (\zeta = |\Im(\nu)|, \theta = \Re(\nu), \varphi = \operatorname{sign}\Im(\nu)) \in \mathbb{R}^+ \times \boldsymbol{\Theta} \times \Phi$



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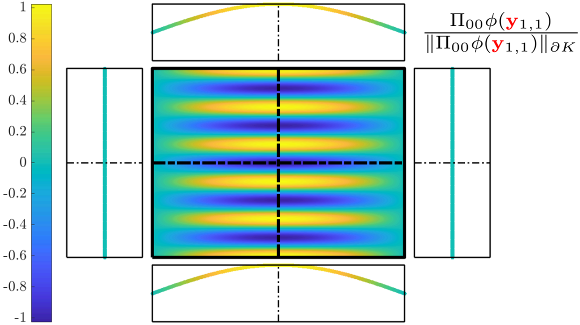
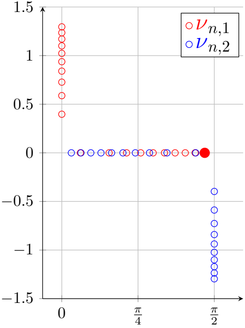
- Let $\boxed{\mathbf{y}_{n,i} = (|\Im(\nu_{n,i})|, \Re(\nu_{n,i}), \operatorname{sign}\Im(\nu_{n,i})) \in \mathbb{R}^{+} \times \boldsymbol{\Theta} \times \Phi}$





$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right)$$

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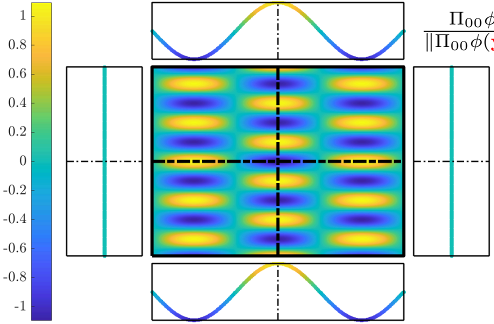
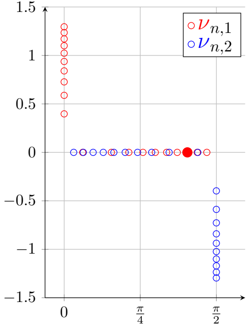


$$\frac{\Pi_{00}\phi(\mathbf{y}_{3,1})}{\|\Pi_{00}\phi(\mathbf{y}_{3,1})\|_{\partial K}}$$



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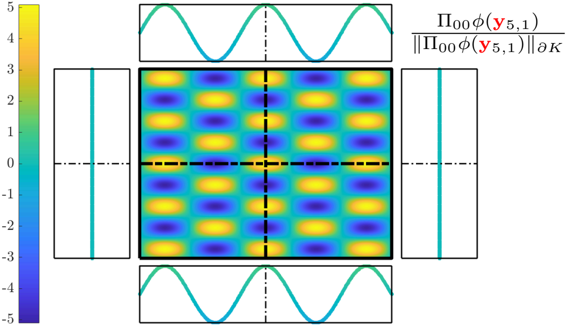

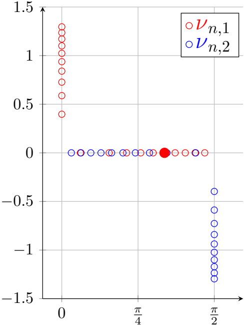

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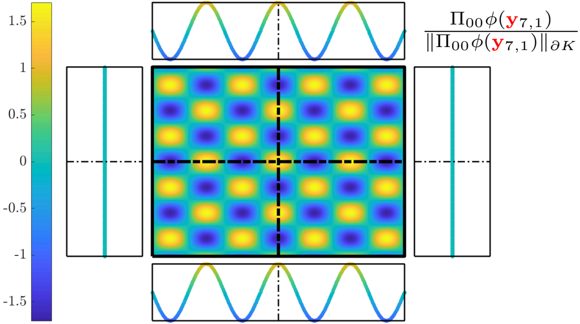

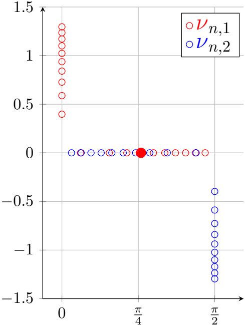

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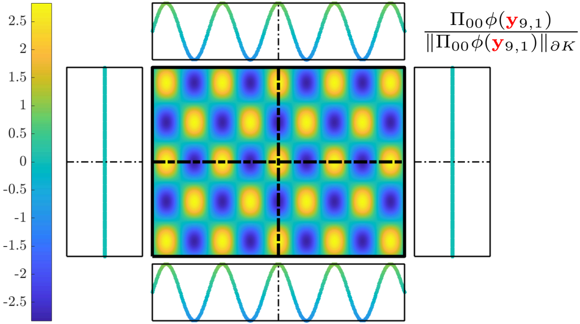

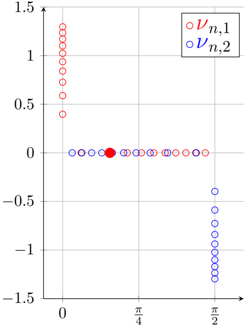

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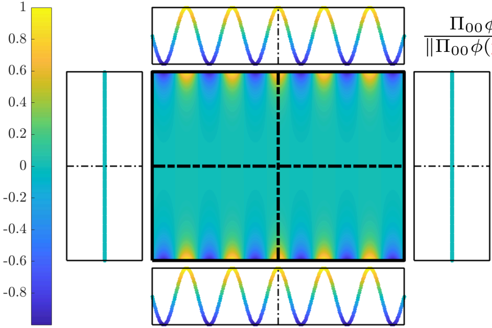

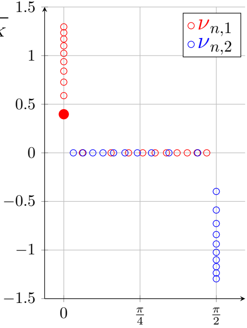

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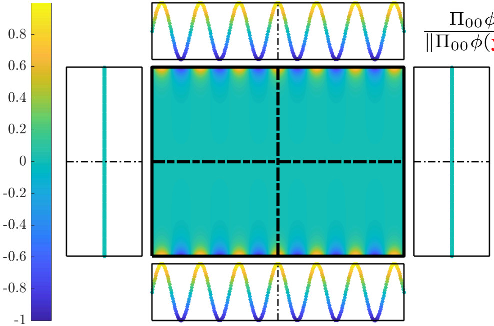

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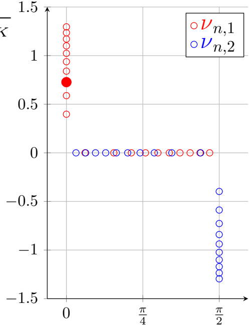

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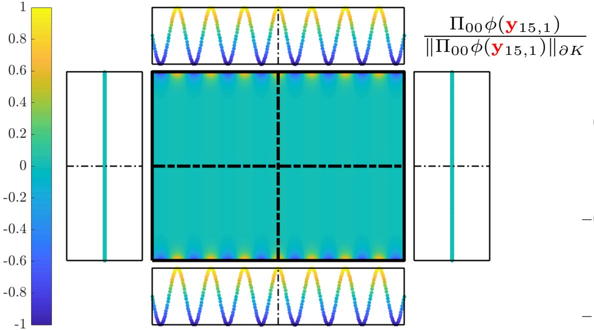

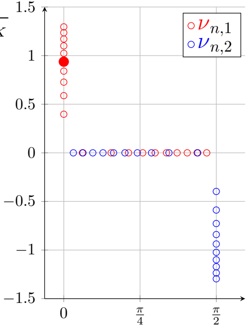

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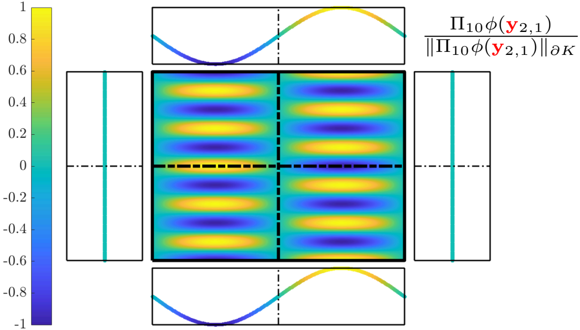

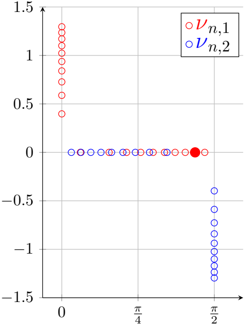

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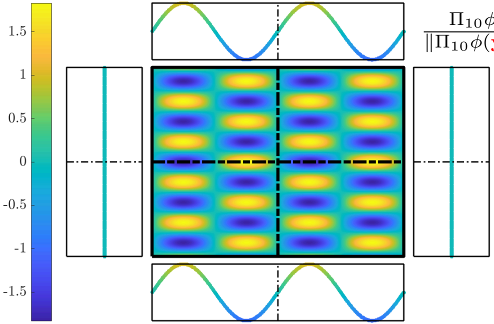

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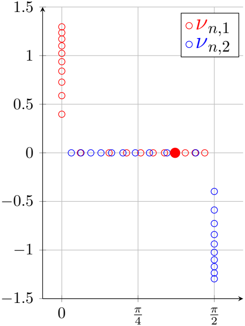

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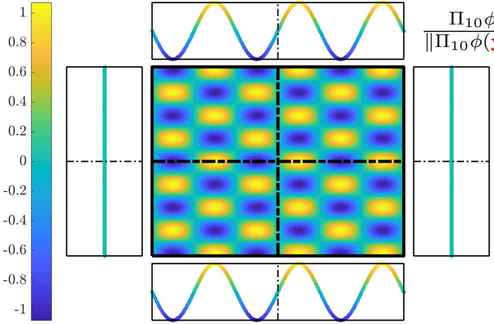

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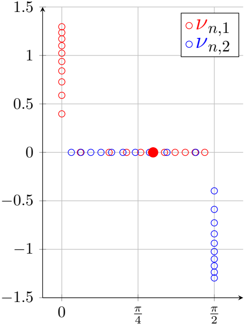

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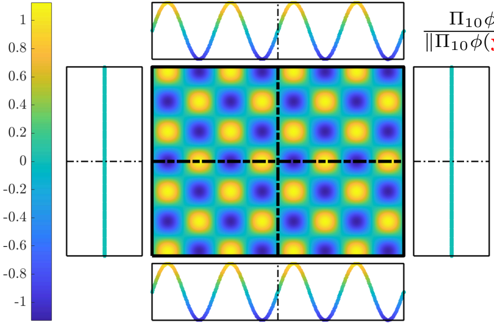

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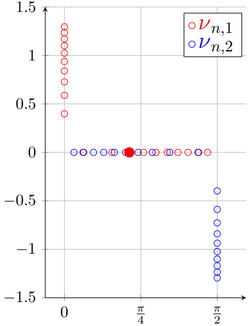

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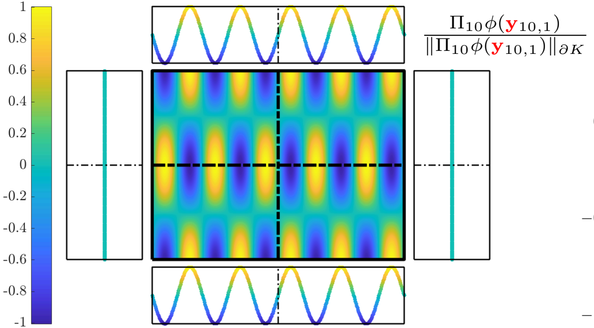

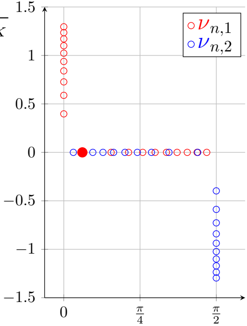

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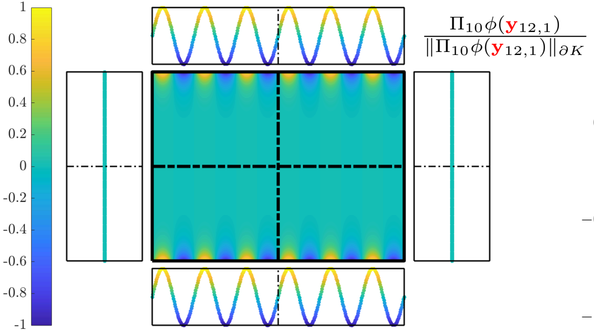

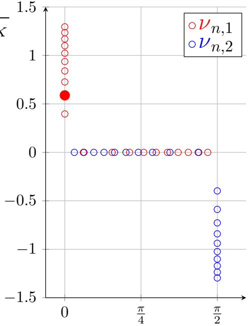

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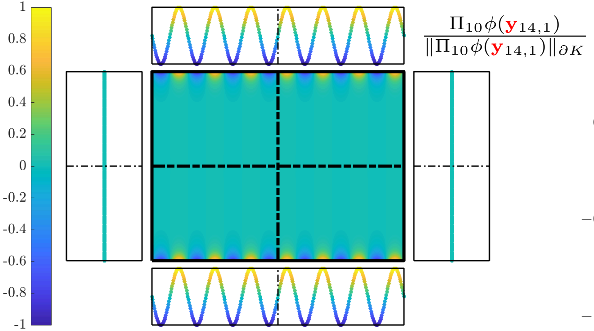
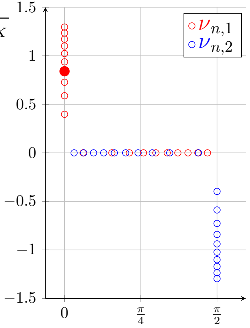
$$\nu_{n,2} = \sin^{-1}\left(\frac{n\pi}{\kappa L_2}\right)$$

An orthogonal EPW basis

- Let $\mathbf{y}_{n,i} = (|\Im(\nu_{n,i})|, \Re(\nu_{n,i}), \operatorname{sign}\Im(\nu_{n,i})) \in \mathbb{R}^+ \times \boldsymbol{\Theta} \times \Phi$





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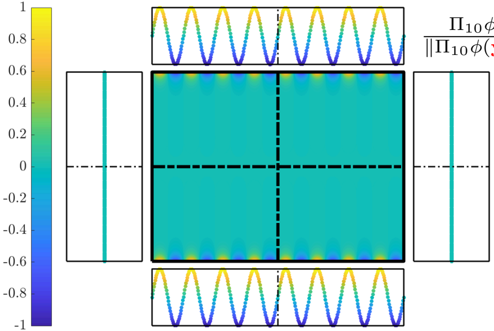
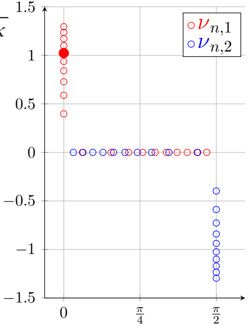


$$\frac{\Pi_{10}\phi(\mathbf{y}_{16,1})}{\|\Pi_{10}\phi(\mathbf{y}_{16,1})\|_{\partial K}}$$



- On 2 opposite sides:
  $\Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})|_{\partial K} \propto \Delta$-Dirichlet eigenfunctions
- On the other 2 sides: $\Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})|_{\partial K} = 0$

$\Longrightarrow \{\Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})|_{\partial K}\}_{\mathbf{j},n,i}$ is a $L^2(\partial K)$ Hilbert basis

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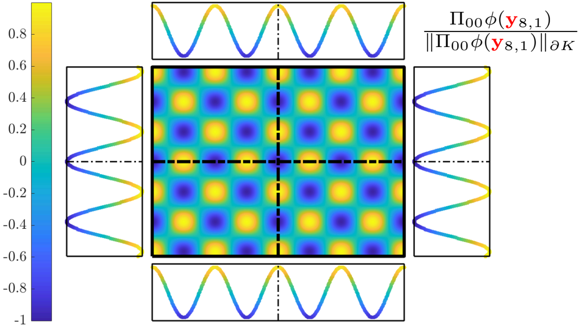

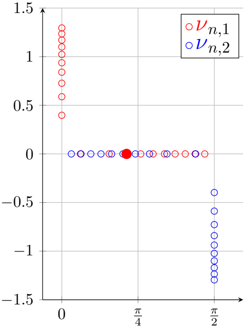

- On 2 opposite sides:
  $\partial_{\mathbf{n}}\Pi_{\mathbf{1}-\mathbf{j}}\phi(\mathbf{y}_{n,i}) \propto \Delta$-Neumann eigenfunctions
- On the other 2 sides: $\partial_{\mathbf{n}}\Pi_{\mathbf{1}-\mathbf{j}}\phi(\mathbf{y}_{n,i}) = 0$

$\Longrightarrow \{\partial_{\mathbf{n}}\Pi_{\mathbf{1}-\mathbf{j}}\phi(\mathbf{y}_{n,i})\}_{\mathbf{j},n,i}$ is $L^2(\partial K)$ Hilbert basis

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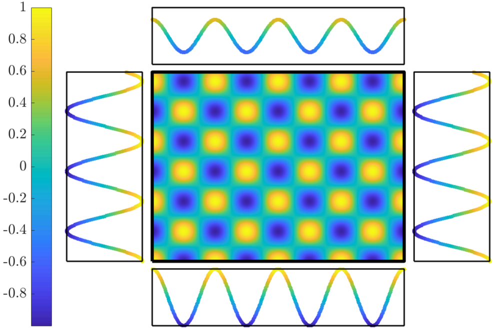

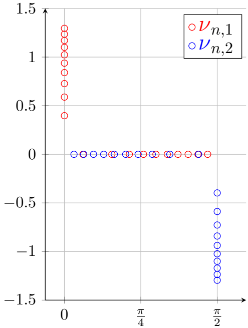

Some resemblances to the Wave-Based method [Deckers et al 2014], whose trace families:

- Still form a complete basis for $L^2(\partial K)$
- But lack the orthogonality property

$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right)$$

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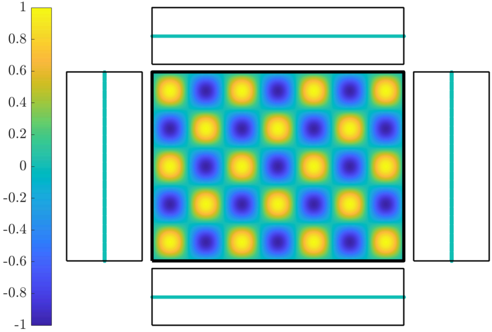

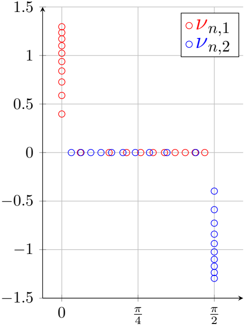

We need to assume that κ is not an exceptional frequency, i.e. κ^2 is not in the eigenvalue set

$$\sigma_K(-\Delta) := \left\{\pi^2\left(\frac{n^2}{L_1^2} + \frac{m^2}{L_2^2}\right) : n, m \in \mathbb{N}\right\}$$

$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right)$$

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One-edge Dirichlet basis

Goal: Compactly supported H^1-conforming basis

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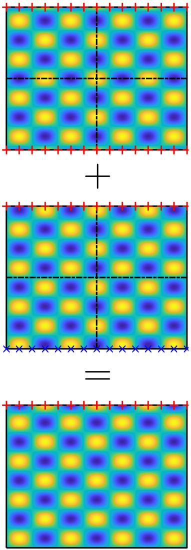
$$\Psi^{\pm}_{n,1} := \frac{1}{2}\left(\frac{\Pi_{00}\phi(\mathbf{y}_{n,1})}{\|\Pi_{00}\phi(\mathbf{y}_{n,1})\|_{\partial K}} \pm \frac{\Pi_{01}\phi(\mathbf{y}_{n,1})}{\|\Pi_{01}\phi(\mathbf{y}_{n,1})\|_{\partial K}}\right) \quad n \text{ odd}$$



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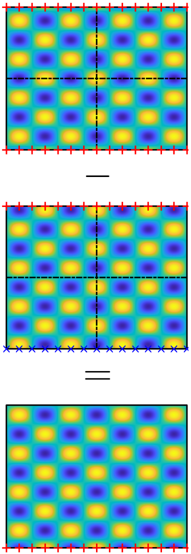

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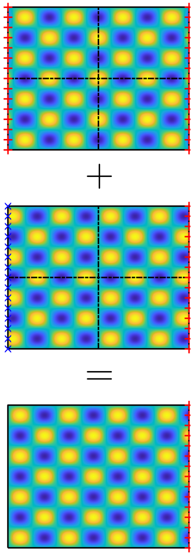
$$\Psi_{n,1}^{\pm} := \frac{1}{2}\left(\frac{\Pi_{00}\phi(\mathbf{y}_{n,1})}{\|\Pi_{00}\phi(\mathbf{y}_{n,1})\|_{\partial K}} \pm \frac{\Pi_{01}\phi(\mathbf{y}_{n,1})}{\|\Pi_{01}\phi(\mathbf{y}_{n,1})\|_{\partial K}}\right) \quad n \text{ odd}$$

$$\Psi_{n,2}^{\pm} := \frac{1}{2}\left(\frac{\Pi_{00}\phi(\mathbf{y}_{n,2})}{\|\Pi_{00}\phi(\mathbf{y}_{n,2})\|_{\partial K}} \pm \frac{\Pi_{10}\phi(\mathbf{y}_{n,2})}{\|\Pi_{10}\phi(\mathbf{y}_{n,2})\|_{\partial K}}\right) \quad n \text{ odd}$$



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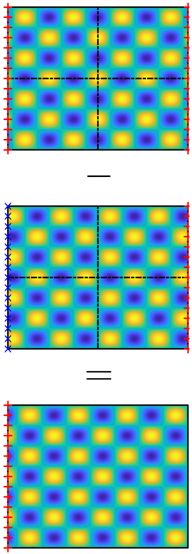

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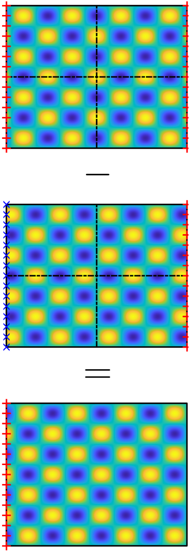
$$\Psi^{\pm}_{n,1} := \frac{1}{2}\left(\frac{\Pi_{10}\phi(\mathbf{y}_{n,1})}{\|\Pi_{10}\phi(\mathbf{y}_{n,1})\|_{\partial K}} \pm \frac{\Pi_{11}\phi(\mathbf{y}_{n,1})}{\|\Pi_{11}\phi(\mathbf{y}_{n,1})\|_{\partial K}}\right) \quad n \text{ even}$$

$$\Psi^{\pm}_{n,2} := \frac{1}{2}\left(\frac{\Pi_{01}\phi(\mathbf{y}_{n,2})}{\|\Pi_{01}\phi(\mathbf{y}_{n,2})\|_{\partial K}} \pm \frac{\Pi_{11}\phi(\mathbf{y}_{n,2})}{\|\Pi_{11}\phi(\mathbf{y}_{n,2})\|_{\partial K}}\right) \quad n \text{ even}$$



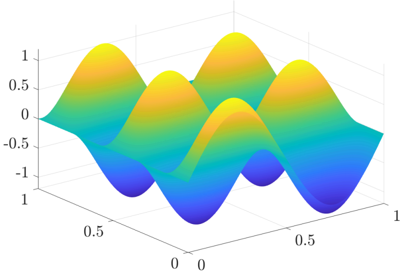
- On 2 opposite sides:
  $\Psi^{\pm}_{n,i}|_{\partial K} = \Delta$-Dirichlet eigenfunctions
- On the other 2 sides: $\Psi^{\pm}_{n,i}|_{\partial K} = 0$

$\Longrightarrow \{\Psi^{\pm}_{n,i}|_{\partial K}\}_{n,i}$ is a Hilbert basis for $L^2(\partial K)$

Example

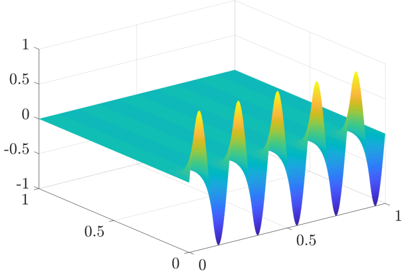
- Plots of $\Psi^-_{n,1}$ with $L_1 = L_2 = 1$ and $\kappa = 16$

Propagative Wave (n=2)



$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right) \in \mathbb{R}$$

Evanescent Wave (n=10)



$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right) \in i\mathbb{R}$$

- $\kappa\,\mathrm{diam}(K) \to 0$: $\{\Psi^{\pm}_{n,i}\}_{n,i}$ contains only evanescent waves
- $\kappa\,\mathrm{diam}(K) \to +\infty$: $\{\Psi^{\pm}_{n,i}\}_{n,i}$ contains more propagative waves

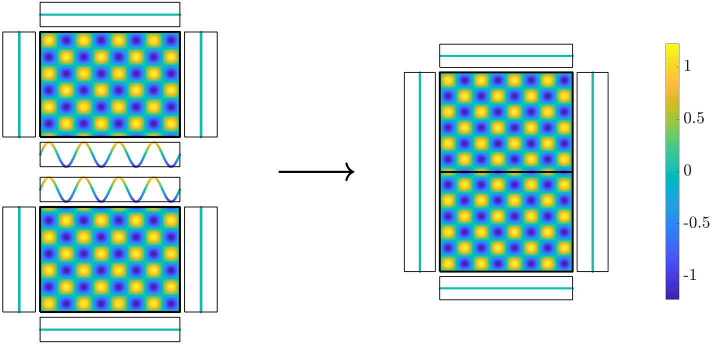
A single-mesh method

A single-mesh method

- Let Ω be a bounded domain discretized by a mesh $\mathcal{T}_h := \{K\}$ composed of rectangular cells. Moreover, assume that

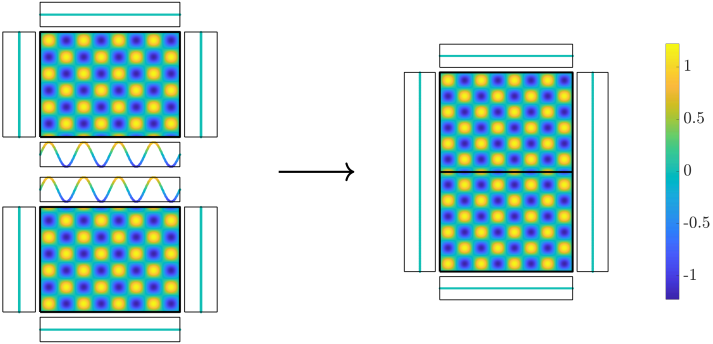
$$\kappa^2 \notin \bigcup_{K \in \mathcal{T}_h} \sigma_K(-\Delta)$$

This is not restrictive up to a (local) resizing of the mesh cells



A single-mesh method

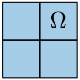
- Glue two functions along the non-zero interface to ensure C^0-continuity
- The glued function has compact support, solves Helmholtz in each cell
- The Trefftz space generated by these functions is conforming
- We can rely on the Galerkin projection onto the conforming Trefftz space to approximate any Helmholtz BVP



Example – PPW approximation

Consider $\Omega = [0,1]^2$ and $\kappa = 32$. We want to approximate the PPW

$$\phi_{\theta} : \mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\theta)\cdot\mathbf{x}}, \qquad \text{where} \qquad \theta = \frac{\pi}{4}$$

Mesh = 

We take $N = 32$, and a 12-edge mesh $\rightarrow$ #DOFs$= 32 \times 12 = 384$

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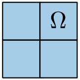

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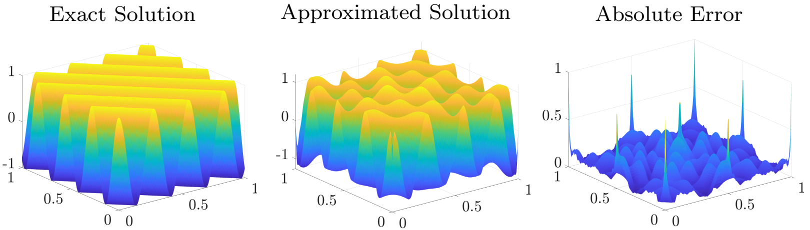


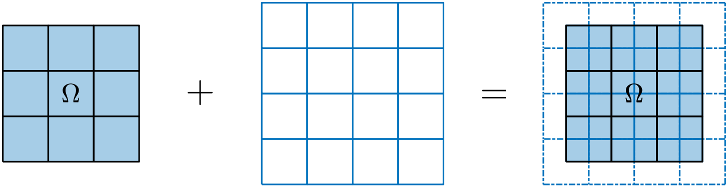
The approximation is poor: all functions in the discrete Trefftz space vanish at the mesh nodes. For a fixed mesh size $h > 0$ and any $\theta \in \boldsymbol{\Theta}$,

$$\inf_{v_{N,h}} \|\phi_\theta - v_{N,h}\|_{H^1(\Omega)} \gtrsim N^{-1/2}, \qquad N \in \mathbb{N}$$

An interlaced-mesh method

An interlaced-mesh method

- We take a shifted second grid to patch the nodes of the first mesh



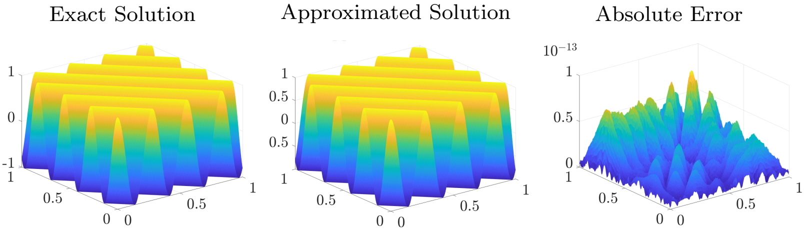
- The new interlaced-mesh Trefftz space remains conforming
- Its functions solve the Helmholtz equation in each cell intersection
- Let us try to approximate the propagative plane wave $\phi_{\pi/4}$, using the Galerkin projection onto this new Trefftz space

Example – PPW approximation

Consider again the previous test, namely the approximation of

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\theta)\cdot \mathbf{x}}, \qquad \text{where} \qquad \theta = \frac{\pi}{4}$$

For $N = 16$, a 4-edge mesh + a 12-edge shifted mesh $\rightarrow$ #DOFs$= 256$



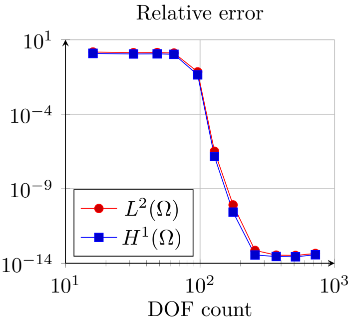
The approximation is really good !

Example – PPW approximation

Consider again the previous test, namely the approximation of

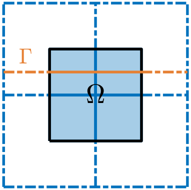
$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\theta)\cdot\mathbf{x}}, \qquad \text{where} \qquad \theta = \frac{\pi}{4}$$

Given a 4-edge mesh + a 12-edge shifted mesh (wavenumber $\kappa = 32$)



The convergence seems spectral !

A first error estimate

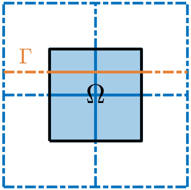


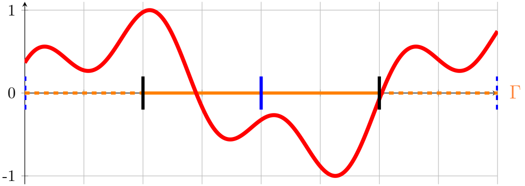
- Consider any vertical/horizontal segment Γ that cuts through the domain Ω

For any $m \in \mathbb{N}$ and $u \in H^m(\Gamma)$, there exists $C_{m,h} > 0$ such that

$$\inf_{v_{N,h}} \|u - v_{N,h}\|_{H^1(\Gamma)} \leq C_{m,h}\, N^{3/2-m} \|u\|_{H^m(\Gamma)}$$

A first error estimate





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- The trace Trefftz space contains functions vanishing on Γ except on one cell restriction, where they match the Δ-Dirichlet eigenfunctions

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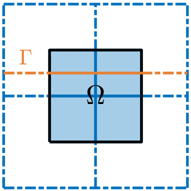

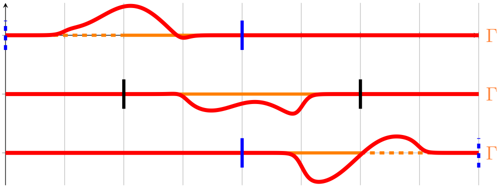

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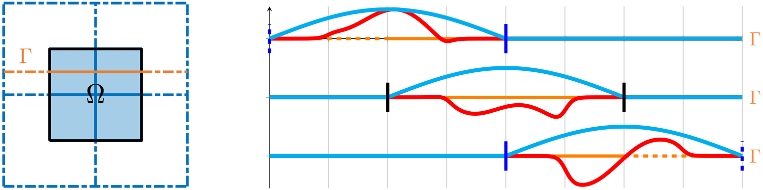

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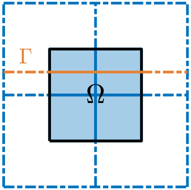

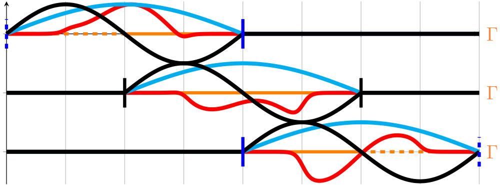

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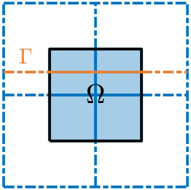

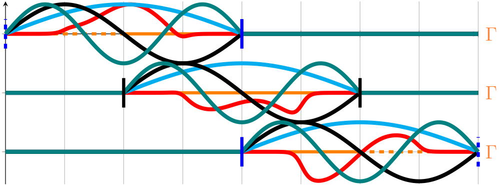

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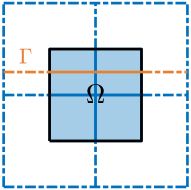
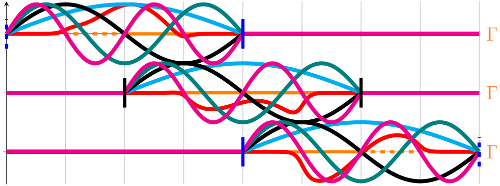
A first error estimate





For any $m \in \mathbb{N}$ and $u \in H^m(\Gamma)$, there exists $C_{m,h} > 0$ such that

$$\inf_{v_{N,h}} \|u - v_{N,h}\|_{H^1(\Gamma)} \leq C_{m,h}\, N^{3/2-m} \|u\|_{H^m(\Gamma)}$$

- The trace Trefftz space contains functions vanishing on Γ except on one cell restriction, where they match the Δ-Dirichlet eigenfunctions

In particular:

- EPWs needed for p-convergence → only PPWs gives #modes $< +\infty$
- EPWs needed for h-convergence → no PPW-conforming Trefftz as $h \to 0$

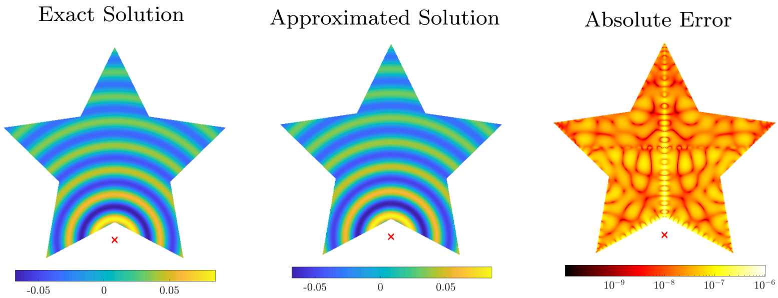
More numerical experiments

- If Ω is a general Lipshitz domain, consider the space spanned by basis functions on a mesh of rectangular cells covering Ω
- For polygonal domain Ω, the basis $\Psi^{\pm}_{n,i}$ (EPW combinations) enable exact matrix assembly via closed-form integration

Approximation of the 2D fundamental solution with $\kappa = 30$, namely

$$\mathbf{x} \mapsto \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x} - \mathbf{s}|), \qquad \mathbf{s} \in \mathbb{R}^2 \setminus \overline{\Omega}$$

For $N = 32$, a 4-edge mesh + a 12-edge shifted mesh $\rightarrow$ #DOFs$=512$



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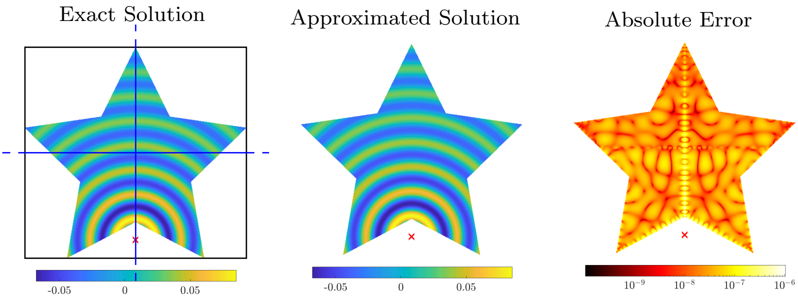

Conclusions

Summary

Ill-conditioning can be overcome (via regularization) if there exist accurate and stable approximations (bounded coefficients)

$$u = \int v e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$$

- PPW: $v \mapsto u$ has dense image but is compact
- EPW: $u \mapsto v$ is bounded (for the disk/ball)

$$u \approx \sum_n \xi_n e^{i\kappa \mathbf{d}_n \cdot \mathbf{x}}$$

- PPW: numerical instability
- EPW: much better approximation results

We developed a Trefftz scheme that numerically exhibits **spectral accuracy**, preserves the **conformity** of classical FEM methods, and ensures **stability** in high-resolution Trefftz spaces using EPWs

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Next steps:

- Extend the bounded invertibility of $T_{\mathrm{E}} : L^2_{w^2_\Omega}(Y) \to H^1(\Omega)$ from the disk/ball to a broader class of domains (WIP for convex domains)
- Derive error estimates in 2D & 3D for the EPW-Trefftz scheme

References:

- E. Parolin, D. Huybrechs and A. Moiola
  Stable approximation in the disk of Helmholtz solutions using evanescent plane waves

  ESAIM Math. Model. Numer. Anal. 57.6 (2023)
- N. Galante, A. Moiola and E. Parolin
  Stable approximation in the ball of Helmholtz solutions using evanescent plane waves
  `arXiv:2401.04016`

Thank you for your attention!