A conforming Trefftz method for stable approximations of Helmholtz solutions

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Helmholtz equation and Trefftz methods

Let u be a solution of the Helmholtz equation (wavenumber $\kappa > 0$):

$$-\Delta u - \kappa^2 u = 0, \qquad \text{in a bounded domain } \Omega \subset \mathbb{R}^d, \quad d \in \{2, 3\}$$

Goal: Computing approximation of u using Trefftz methods

$$u \approx \sum_{n=1}^{N} \xi_n \phi_n, \qquad \text{where} \quad -\Delta \phi_n - \kappa^2 \phi_n = 0 \quad \text{(locally)}$$

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Strengths:

- Spectral accuracy
- Many formulations (LS, TDG, UWVF)

Weaknesses:

- Limited to piecewise-constant coefficients & homogeneous PDEs
- High numerical instability from redundancy in approximation sets, e.g. propagative plane waves, convergence stalls in finite precision

Outline

- **Propagative plane waves (PPWs)**
- **Evanescent plane waves (EPWs)**
- **EPW-Trefftz Continuous Galerkin Methods**

Propagative plane waves (PPWs)

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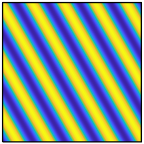

Propagative plane wave (defined in $\mathbb{R}^d$)

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}}, \qquad \text{where} \quad \mathbf{d} \in \mathbb{R}^d, \quad \mathbf{d}\cdot\mathbf{d} = 1$$

- Exact solution of $(-\Delta - \kappa^2)u = 0$, since $\mathbf{d}\cdot\mathbf{d} = 1$
- Single parameter family: $\mathbf{d} \in \mathbb{S}^{d-1}$ propagation direction
- Simple parametrization of $\mathbf{d}(\boldsymbol{\theta}) \in \mathbb{S}^{d-1}$, where:
  - $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [0, 2\pi)$ in 2D
  - $\boldsymbol{\theta} \in \boldsymbol{\Theta} := [0, 2\pi) \times [0, \pi]$ in 3D
- Easy to manipulate: closed-form integration on flat submanifold
- Simple discretization: evenly-distributed $\mathbf{d}(\boldsymbol{\theta}_n) \in \mathbb{S}^{d-1}$

$$u(\mathbf{x}) \approx \sum_{n=1}^{N} \xi_n e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_n)\cdot\mathbf{x}}$$

Discrete approximation by PPWs

Can we construct accurate approximations $u(\mathbf{x}) \approx \sum_n \xi_n e^{i\kappa \mathbf{d}(\boldsymbol{\theta}_n)\cdot\mathbf{x}}$?

In theory, yes: better rates w.r.t #DOFs than polynomial spaces:

- *h*-estimates [Cessenat, Després 1998]
- *hp*-estimates [Melenk 1995], [Moiola Hiptmair Perugia 2011]

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In pratice (finite-precision arithmetic), not always:

The issue is "instability"

Increasing #PPWs, at some point convergence stagnates

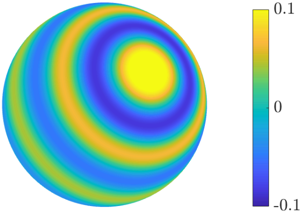
- Instability, commonly observed in PPW-based Trefftz methods, is usually described as an issue of linear system ill-conditioning
- Redundant approximation set if $\mathbf{d}(\boldsymbol{\theta}_n) \cdot \mathbf{d}(\boldsymbol{\theta}_m) \approx 1$ for $n \neq m$
- Requires regularization [Barucq, Bendali, Diaz, Tordeux 2021]

Motivating numerical experiment (PPWs)

Approximation in the unit ball Ω of the 3D fundamental solution, i.e.

$$\mathbf{x} \mapsto \frac{1}{4\pi}\frac{e^{i\kappa|\mathbf{x}-\mathbf{s}|}}{|\mathbf{x}-\mathbf{s}|}, \qquad \mathbf{s} \in \mathbb{R}^3 \setminus \overline{\Omega}$$

with $\kappa = 10$ and $\operatorname{dist}(\mathbf{s}, \Omega)/\lambda = 1$



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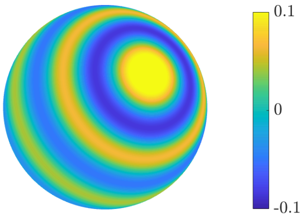

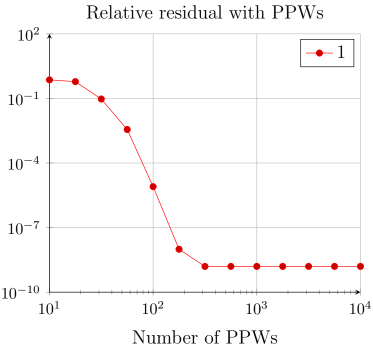

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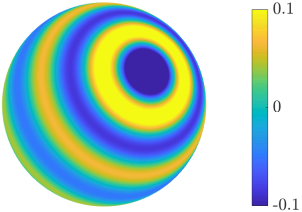

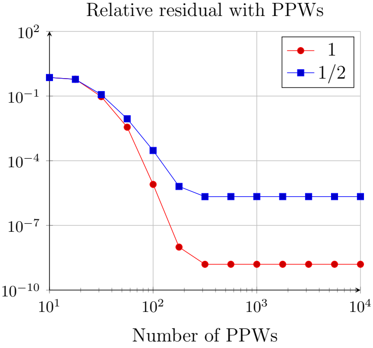

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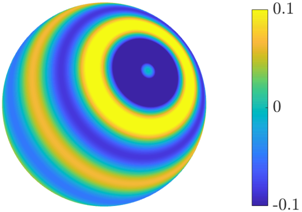

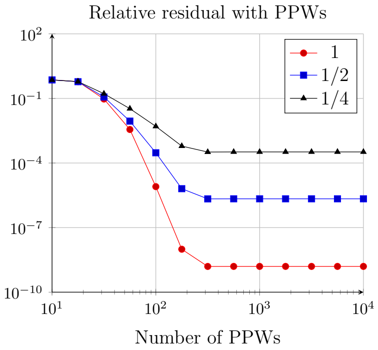

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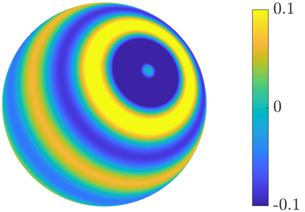

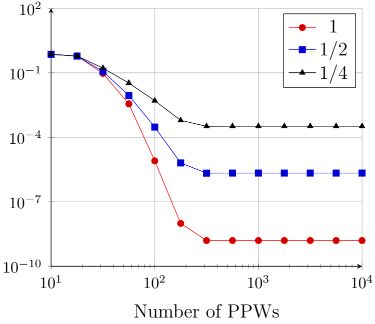

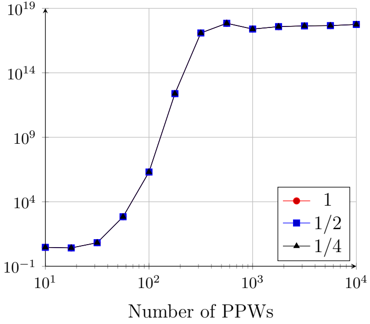

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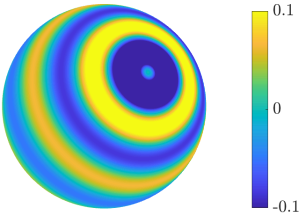

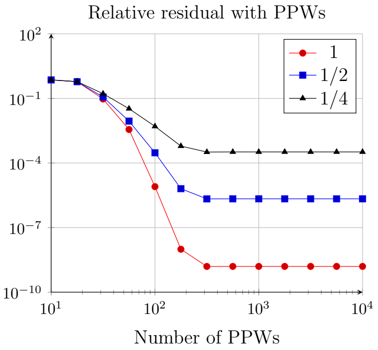

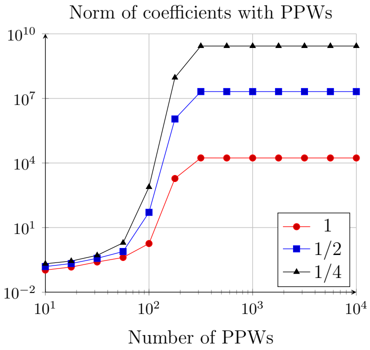

Stability condition

For some $\mathbf{x} \in \Omega$, consider approximations of $u(\mathbf{x})$ of the form

$$\widetilde{u}[\boldsymbol{\mu}](\mathbf{x}) := \sum_{n=1}^{N} \mu_n e^{i\kappa \mathbf{d}_n \cdot \mathbf{x}}, \qquad \text{where} \quad \boldsymbol{\mu} := (\mu_n)_n$$

Ill-conditioning can be solved, provided that accurate approximations $\widetilde{u}[\boldsymbol{\mu}]$ with bounded coefficients $\|\boldsymbol{\mu}\|$ exist

[Adcock, Huybrechs 2020] An approximate solution vector $\boldsymbol{\xi}$ to an ill-conditioned linear system, computed using an ϵ-regularized backward stable algorithm, satisfies:

$$\|u - \widetilde{u}[\boldsymbol{\xi}]\| \lesssim \inf_{\boldsymbol{\mu}} \left(\|u - \widetilde{u}[\boldsymbol{\mu}]\| + \epsilon \|\boldsymbol{\mu}\| \right)$$

Continuous superposition of PPWs

For a bounded Lipschitz domain Ω, let $T_{\mathrm{P}} : L^2(\boldsymbol{\Theta}) \to H^1(\Omega)$ such that

$$\left(T_{\mathrm{P}} v\right)(\mathbf{x}) := \int_{\boldsymbol{\Theta}} v(\boldsymbol{\theta}) e^{i\kappa \mathbf{d}(\boldsymbol{\theta})\cdot \mathbf{x}} \mathrm{d}\sigma(\boldsymbol{\theta}), \qquad \mathbf{x} \in \Omega$$

- T_{P} is bounded, and $u = T_{\mathrm{P}} v$ is an Helmholtz solution in Ω called Herglotz function [Colton, Kress 2013]
- $u = T_{\mathrm{P}} v$ is an entire function independent of Ω
- T_{P} has H^1-dense image in the Helmholtz solution space in Ω [Weck 2004]
- T_{P} is a Hilbert–Schmidt operator $\implies$ not boundedly invertible

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[Parolin, Huybrechs, Moiola 2023] Any Helmholtz solution in Ω not in the range of T_{P} can be arbitrarily well approximated by PPWs, but with unbounded coefficients (asymptotically)

Some considerations

In presence of redundant approximation sets:

- Numerical instability:

  → ill-conditioning does not imply inaccurate approximations
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- Some available remedies by modifying the approximation set:
  [Antunes 2018] (change of basis)
  [Congreve, Gedicke, Perugia 2019] (basis orthogonalization)
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$\Longrightarrow$ We propose to enrich the approximation set

Evanescent plane waves (EPWs)

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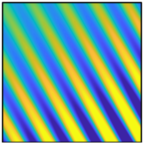


Evanescent plane wave (defined in $\mathbb{R}^d$)

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}\cdot\mathbf{x}}, \qquad \text{where} \quad \mathbf{d} \in \mathbb{C}^d, \quad \mathbf{d}\cdot\mathbf{d} = 1$$

- Complex direction $\mathbf{d} \in \mathbb{C}^d$:
  - Propagation direction $\Re(\mathbf{d})$
  - Evanescence direction $\Im(\mathbf{d})$
- Still exact solution of $(-\Delta - \kappa^2)u = 0$, since $\mathbf{d}\cdot\mathbf{d} = 1$
- Still easy and cheap to evaluate, differentiate, integrate, etc.
- Localization effect in a bounded domain: requires normalization
- Scarcer use in the literature: Wave-Based Method (W. Desmet) [Deckers et al 2014]

EPW direction parametrization

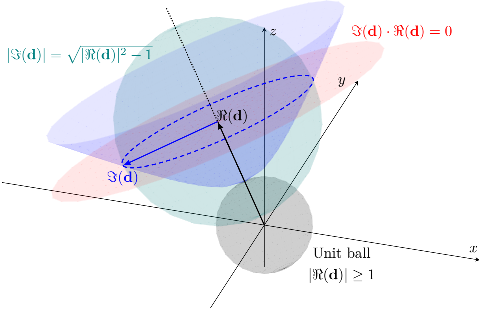
Evanescent plane wave directions $\mathbf{d} \in \mathbb{C}^d$ can be parametrized as

$$\mathbf{d}(\mathbf{y}) = \cosh(\zeta)\mathbf{d}^{\parallel}(\boldsymbol{\theta}) + i\sinh(\zeta)\mathbf{d}^{\perp}_{\boldsymbol{\theta}}(\varphi), \qquad \mathbf{y} = (\zeta, \boldsymbol{\theta}, \varphi) \in Y := \mathbb{R}^{+} \times \boldsymbol{\Theta} \times \Phi$$

- Decay strength $|\Im(\mathbf{d})|$ parametrized by $\zeta \in \mathbb{R}^+$
- Propagation direction $\mathbf{d}^{\parallel}$ parametrized by $\boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Evanescence direction $\mathbf{d}^{\perp}_{\boldsymbol{\theta}}$ parametrized by $\varphi \in \Phi$,

$$\Phi := \begin{cases} \{\pm 1\} & \text{in 2D} \\ [0, 2\pi) & \text{in 3D} \end{cases}$$

- PPWs recovered for $\zeta = 0$



Continuous superposition of EPWs

For a bounded Lipshitz domain Ω, let $T_{\mathrm{E}} : L^2_{w^2_\Omega}(Y) \to H^1(\Omega)$ s.t.

$$\left(T_{\mathrm{E}} v\right)(\mathbf{x}) := \int_Y v(\mathbf{y}) e^{i\kappa \mathbf{d}(\mathbf{y}) \cdot \mathbf{x}} w^2_\Omega(\mathbf{y}) \mathrm{d}\sigma(\mathbf{y}), \qquad \mathbf{x} \in \Omega$$

- T_{E} is a bounded operator for a suitable weight w_Ω
- $u = T_{\mathrm{E}} v$ is a Helmholtz solution in Ω that can be singular on $\partial\Omega$
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If Ω is a disk in 2D or a ball in 3D, [Parolin, Huybrechs, Moiola 2023] and [G. , Moiola, Parolin 2024] provide a weight w_Ω, only dependent on ζ, such that T_{E} is boundedly invertible, namely

$$\forall u \text{ Helmholtz solution}, \quad v = T_{\mathrm{E}}^{-1} u, \quad \|v\|_{L^2_{w^2_\Omega}(Y)} \sim \|u\|_{H^1(\Omega)}$$

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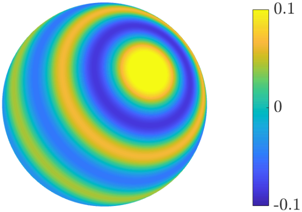

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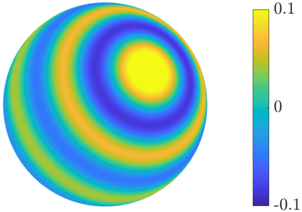

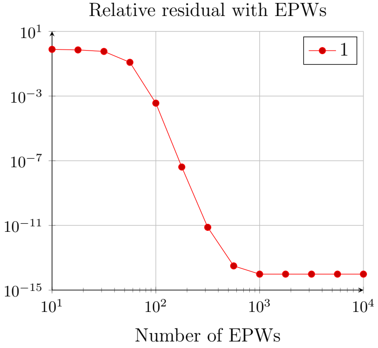

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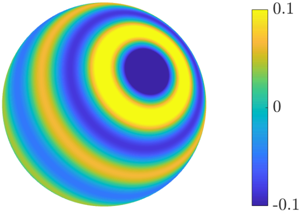

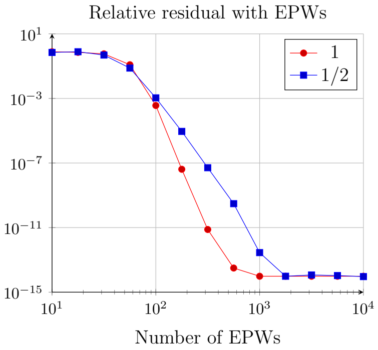

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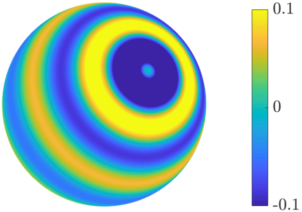

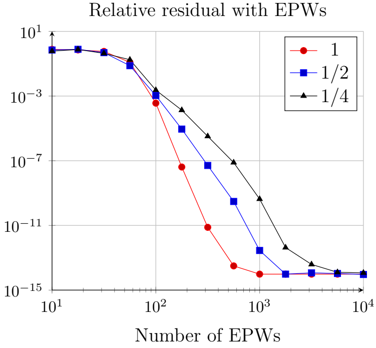

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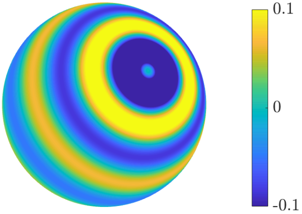

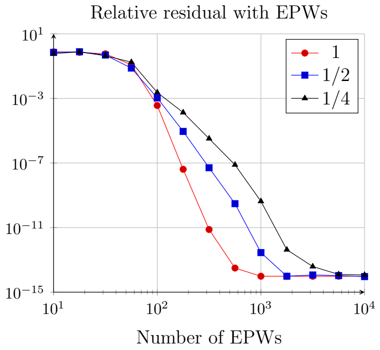

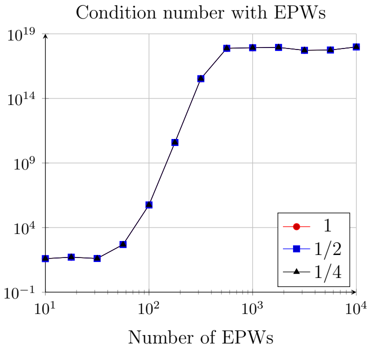

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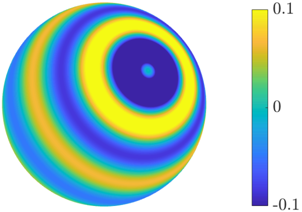

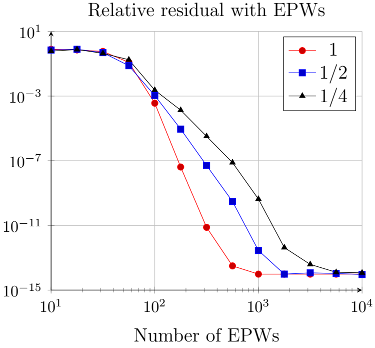

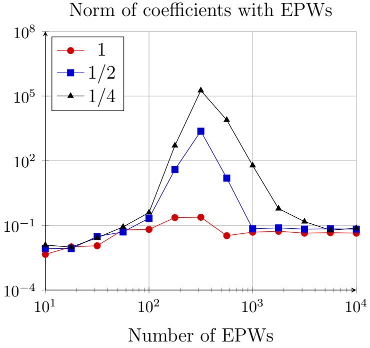

EPW approximation sets and Trefftz methods

- How to construct EPW discrete approximation sets

$$\left\{\mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\mathbf{y}_n)\cdot \mathbf{x}}\right\}_{n=1}^{N} ?$$

If Ω is a disk in 2D or a ball in 3D:

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EPW approximation sets and Trefftz methods

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Trefftz Discontinuous Galerkin

- Polytopal cell mesh
- Discrete recipe tailored for each circumscribed ball
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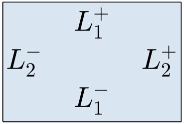
Trefftz Discontinuous Galerkin	Trefftz Continuous Galerkin
Polytopal cell mesh	Rectangular cell mesh
Discrete recipe tailored for each circumscribed ball	Discrete recipe tailored for each rectangular cell
Sampling strategy	Deterministic strategy

EPW-Trefftz Continuous Galerkin Methods

Rectangular cell symmetries



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- Consider a rectangle K with side lengths L_1 and L_2, label its sides as $L_1^{\pm}$ and $L_2^{\pm}$

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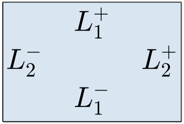

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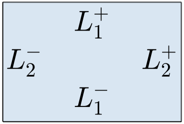

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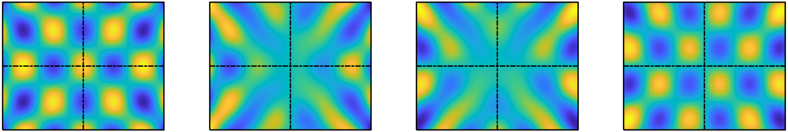

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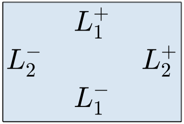

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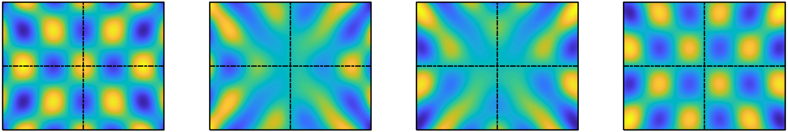


- $\Pi_{\mathbf{j}}\phi(\mathbf{y})$ is a linear combination of EPWs $\implies$ solves Helmholtz

An orthogonal EPW basis

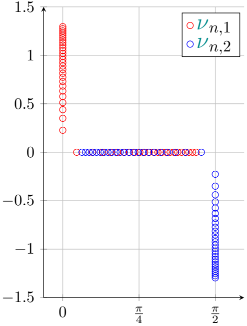
- Assume that κ^2 is not in the eigenvalue set

$$\sigma_K(-\Delta) := \{\pi^2\left(n^2/L_1^2 + m^2/L_2^2\right) : n, m \in \mathbb{N}\}$$

- Consider the parameters $\{\mathbf{y}_{n,i}\}_{n\in\mathbb{N},i=1,2} \subset Y$

$$\mathbf{y}_{n,i} := (|\Im(\nu_{n,i})|, \Re(\nu_{n,i}), \text{sign}\,\Im(\nu_{n,i}))$$

$$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right), \qquad \nu_{n,2} = \sin^{-1}\left(\frac{n\pi}{\kappa L_2}\right)$$



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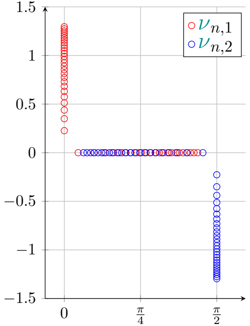


$\sin^{-1}/\cos^{-1}$ are the principal branches of the complex inverse sin/cos

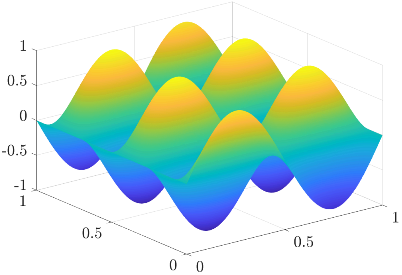
The family $\{\Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})\}_{\mathbf{j},n,i}$ is a complete orthogonal basis for $L^2(\partial K)$. Moreover, denoting with $\{\psi_{n,i}\}_n$ the Δ-Dirichlet eigenfunctions on $L_i^{\pm}$,

$$\Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})|_{L_i^{\pm}} \propto \psi_{n,i}, \qquad \Pi_{\mathbf{j}}\phi(\mathbf{y}_{n,i})|_{\partial K\setminus(L_i^+ \cup L_i^-)} = 0$$

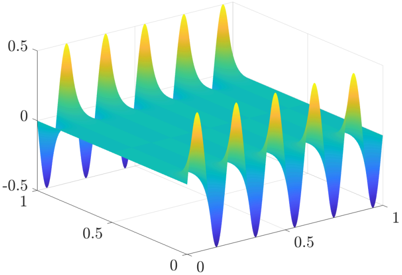
Example

Plots of $\Pi_{(1,1)}\phi(\mathbf{y}_{n,1})$ with $L_1 = L_2 = 1$ and $\kappa = 16$

Propagative Wave (n=2)



Evanescent Wave (n=10)



$\nu_{n,1} = \cos^{-1}\left(\frac{n\pi}{\kappa L_1}\right) \in \mathbb{R}$

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One-edge basis

- **Goal**: Compactly supported H^1-conforming basis for sparsity

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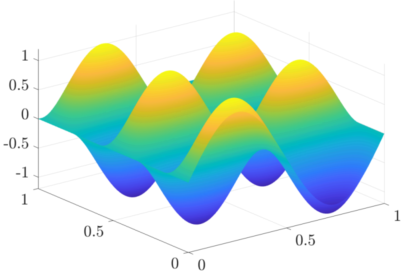
> If $\kappa^2 \notin \sigma_K(-\Delta)$, there exist $c_{\mathbf{j}}^{\pm} \in \mathbb{C}$ such that, defining
>
> $$\Psi_{n,i}^{\pm} := \sum_{\mathbf{j}} c_{\mathbf{j}}^{\pm} \Pi_{\mathbf{j}} \phi(\mathbf{y}_{n,i}), \qquad \left\| \Psi_{n,i}^{\pm} \right\|_{L^\infty(\partial K)} = 1$$
>
> we have
>
> $$\Psi_{n,i}^{\pm}|_{L_i^{\pm}} = \psi_{n,i}, \qquad \Psi_{n,i}^{\pm}|_{\partial K \setminus L_i^{\pm}} = 0$$
>
> Hence, $\{\Psi_{n,i}^{\pm}\}_{n,i}$ is a complete orthogonal basis for $L^2(\partial K)$

- The functions $\Psi_{n,i}^{\pm}$ are Helmholtz solutions with two main regimes
- Their trace is zero on 3 edges and equal to $\psi_{n,i}$ on the remaining one
- The trace of $\Phi_{n,i}^{+}$ on L_i^{+} coincides with the trace of $\Phi_{n,i}^{-}$ on L_i^{-}

Example

Plots of $\Psi^{-}_{n,1}$ with $L_1 = L_2 = 1$ and $\kappa = 16$

Propagative Wave (n=2)



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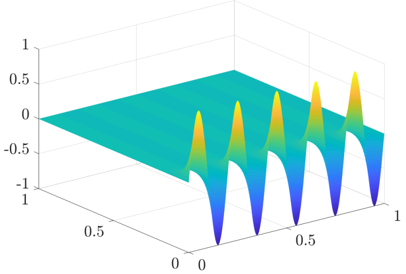

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A single mesh method

A single mesh method

- Let Ω be a bounded domain discretized by a mesh $\mathcal{T}_h := \{K\}$ composed of rectangular cells. Moreover, assume that

$$\kappa^2 \notin \bigcup_{K \in \mathcal{T}_h} \sigma_K(-\Delta)$$

This is not restrictive up to a (local) rescaling of the mesh $\mathcal{T}_h$

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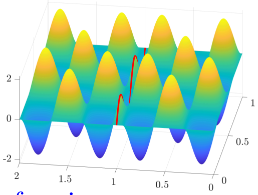
- Glue two functions along the non-zero interface to ensure C^0-continuity

- The resulting function has compact support, solves Helmholtz in each cell



- The generated Trefftz space $V_N(\mathcal{T}_h)$ is conforming

- We can rely on the Galerkin projection onto the conforming Trefftz space $V_N(\mathcal{T}_h)$ to approximate any Helmholtz BVP

Example – PPW approximation

Consider $\Omega = [0,1]^2$ and $\kappa = 16$. We want to approximate the PPW

$$\phi_\theta : \mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\theta)\cdot \mathbf{x}}, \qquad \text{where} \qquad \theta = \frac{\pi}{4}$$

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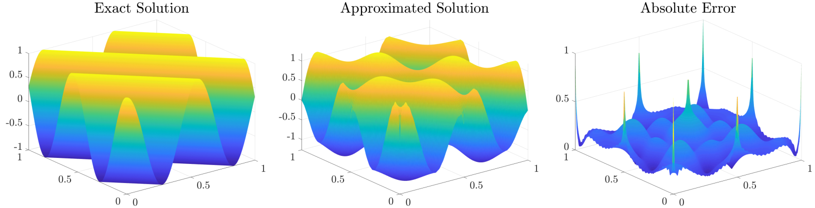


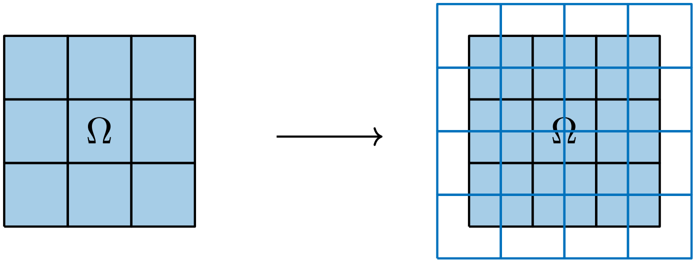
The approximation is poor: all functions in the discrete space $V_N(\mathcal{T}_h)$ vanish at the mesh nodes. In fact, for a fixed mesh $\mathcal{T}_h$ and any $\theta \in \mathbf{\Theta}$,

$$\inf_{v_{N,h} \in V_N(\mathcal{T}_h)} \|\phi_\theta - v_{N,h}\|_{H^1(\Omega)} \gtrsim N^{-1/2}, \qquad N \in \mathbb{N}$$

An interweaved mesh method

An interweaved mesh method

- We take a shifted second grid to patch the nodes of the first mesh
- The generated Trefftz space $W_N(\mathcal{T}_h)$ will still be conforming
- Functions in $W_N(\mathcal{T}_h)$ solve Helmholtz in each cell intersection



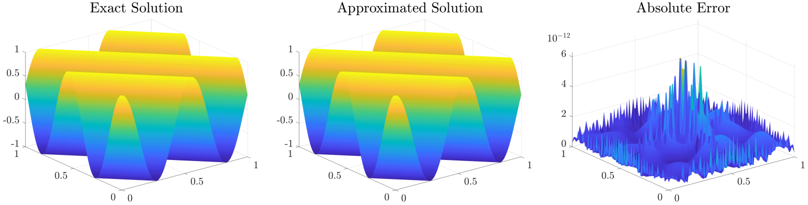
- Let us try to approximate the propagative plane wave $\phi_{\frac{\pi}{4}}$, using the Galerkin projection onto the Trefftz space $W_N(\mathcal{T}_h)$

Example – PPW approximation

Consider again the previous test, namely the approximation of

$$\mathbf{x} \mapsto e^{i\kappa \mathbf{d}(\theta)\cdot \mathbf{x}}, \qquad \text{where} \qquad \theta = \frac{\pi}{4}$$

For $N = 16$, a 4-edge mesh, and a 12-edge shifted mesh, #DOFs$= 256$



The approximation is really good !

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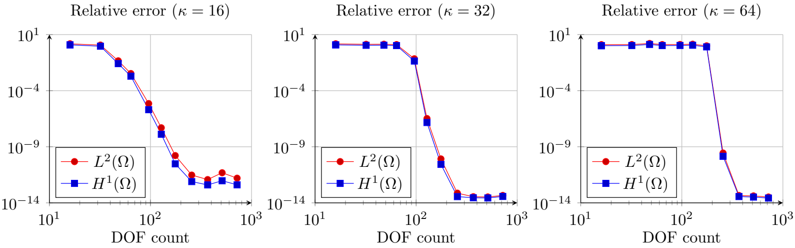
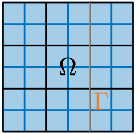
Given a 4-edge mesh, and a 12-edge shifted mesh, we let κ vary



The convergence is spectral !

A first error estimate



- Consider any vertical/horizontal segment Γ that cuts through the domain Ω

Let $s \in [0, 1]$ and $m \in \mathbb{N}$ such that $s < m$. For any $u \in H^m(\Gamma)$, there exists a constant $C_{m,h} > 0$ such that

$$\inf_{v_{N,h} \in W_N(\mathcal{T}_h)} \|u - v_{N,h}\|_{H^s(\Gamma)} \leq C_{m,h} N^{s-m+1/2} \|u\|_{H^m(\Gamma)}$$

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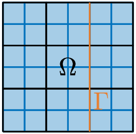

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- The space $W_N(\mathcal{T}_h)|_\Gamma$ contains functions vanishing on Γ except on one cell restriction, where they match the Δ-Dirichlet eigenfunctions

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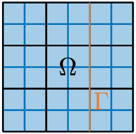

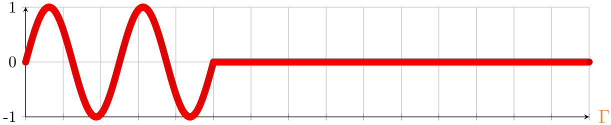

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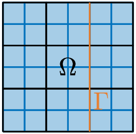

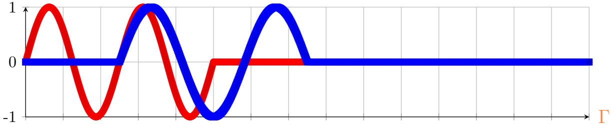

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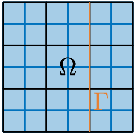

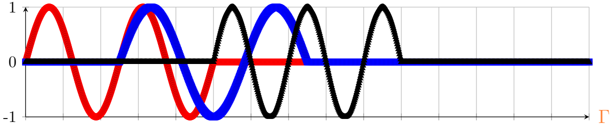

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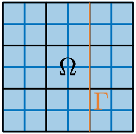

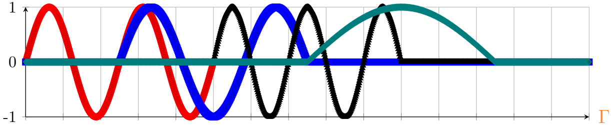

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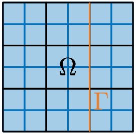

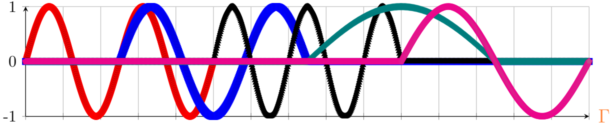

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Eigenfunction interlacing $\Longrightarrow$ Spectral convergence

More numerical experiments

- If Ω is a general Lipshitz domain, consider the space spanned by basis functions on a mesh of rectangular cells covering Ω
- For polygonal domain Ω, the basis $\Psi^{\pm}_{n,i}$ (EPW combinations) enable exact matrix assembly via closed-form integration

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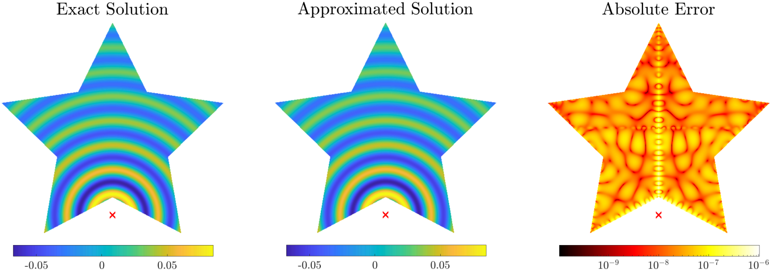
Approximation of the 2D fundamental solution with $\kappa = 30$, namely

$$\mathbf{x} \mapsto \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x} - \mathbf{s}|), \qquad \mathbf{s} \in \mathbb{R}^2 \setminus \overline{\Omega}$$

For $N = 32$, a 4-edge mesh, a 12-edge shifted mesh, #DOFs = 512



Conclusions

Summary

Ill-conditioning can be overcome (via regularization) if there exist accurate and stable approximations (bounded coefficients)

$$u = \int v e^{i\kappa \mathbf{d}\cdot\mathbf{x}}$$

- PPW: $v \mapsto u$ has dense image but is compact
- EPW: $u \mapsto v$ is bounded (for the disk/ball)

$$u \approx \sum_n \xi_n e^{i\kappa \mathbf{d}_n \cdot \mathbf{x}}$$

- PPW: numerical instability
- EPW: much better approximation results

We developed a Trefftz scheme that numerically exhibits **spectral accuracy**, preserves the **conformity** of classical FEM methods, and ensures **stability** in high-resolution Trefftz spaces using EPWs

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Next steps:

- Extend the bounded invertibility of $T_{\mathrm{E}} : L^2_{w^2_\Omega}(Y) \to H^1(\Omega)$ from the disk/ball to a broader class of domains (WIP for convex domains)
- Derive error estimates in 2D & 3D for the EPW-Trefftz scheme

References:

- E. Parolin, D. Huybrechs and A. Moiola
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  `arXiv:2401.04016`

Thank you for your attention!