Multi-scale methods for highly oscillatory differential equations

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The general context: The modelling and the numerical simulation of plasma (a gas of charged particles) is of great importance from a physical and mathematical point of view. In this context, the Vlasov-Maxwell equations provide a kinetic modelling approach of the dynamics of charged particles under the influence of an electro-magnetic field. Difficulties in solving such equations come from the existence of several scales in space and time of the solutions.

Proposed research work:

The generic problem to be treated is the Vlasov-Poisson system with an additional strong external magnetic field, which has several applications in plasma physics, for example the confinement of particles. The modelling relies on a particle distribution function $f(t, x, v)$ depending on time $t \geq 0$ and on the phase space variables $(x, v) \in \mathbb{R}^6$, which is solution to

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} \left(E + \frac{1}{\varepsilon}(v \times B) \right) \cdot \nabla_v f = 0, \tag{1}$$

where m is the mass of particles, q their charge and $(E, B) = (E(t, x), B(t, x)) \in (\mathbb{R}^3)^2$ denotes the electro-magnetic field. This equation takes into account the self-consistent electric (mean-)field generated by the charged particles themselves, whereas B is given. Then, the coupling with the Poisson equation has to be considered

$$\nabla_x \cdot E = \frac{1}{\epsilon_0} q \int_{\mathbb{R}^3} f \mathrm{d}v,$$

with ϵ_0 the permittivity. We underline the presence in (1) of a typical parameter arising in magnetized plasmas, $\varepsilon = T_c/T$ where T is an observation time and T_c is the cyclotronic period which is related to the intensity of the external magnetic field.

We stress that ε can take arbitrary small values so that multiple scales coexist in the system whereas B is a given function. It is well known that when ε is small, the solution f exhibits high oscillations in time related to the cyclotronic motion of charged particles around the magnetic field lines. This multi-scale behaviour imposes tiny time steps to the discretizations and therefore, the computational cost of long time simulations is prohibitive (see [2] for a numerical approach to overcome this difficulty).

A solution for avoiding this problem is to use instead of equation (1), reduced models based on averaging. More precisely, by homogenization in time, we can approach model (1) when $\varepsilon \to 0$ by a simpler one, whose solutions are free of high oscillations. An example is the two-scale limit model (see [1]).

Nevertheless, in some applications this model does not cover the general situation where the rapid motion is a combination of several small scales which cannot be separated. A first direction of research will be the generalization of the two-scale convergence to several scales without scale separation, with a projection method [3].

In a second step, we aim at the development of new reduced models, of zero-order and of first-order, for Vlasov-Poisson problems.

A third objective is to efficiently implement the previous models in order to perform simulations for several applications in plasma physics and data assimilation. We therefore aim to develop, analyze and implement a parareal method (see [4]) for solving the previous equations. This algorithm is an efficient method which performs real time simulations by means of parallel-in-time integration. We will follow a strategy where a different (reduced) model than the original one is used for the coarse solver. This method was successfully applied in [5] for solving highly oscillatory partial differential equations with plasma physics applications.

References:

1. E. Frénod, "Two-scale approach to oscillatory singularly perturbed transport equations", Springer, vol. 2190, 2017.
2. E. Frénod, S.A. Hirstoaga, M. Lutz, E. Sonnendrücker, "Long time behaviour of an exponential integrator for a Vlasov-Poisson system with strong magnetic field", Communications in Computational Physics, 18 (2), 263–296, 2015.
3. N. Crouseilles, E. Frenod, S. A. Hirstoaga, A. Mouton, "Two-scale macro–micro decomposition of the Vlasov equation with a strong magnetic field", Mathematical Models and Methods in Applied Sciences, 23 (08), 1527–1559, 2013.
4. J.-L. Lions, Y. Maday, and G. Turinici: "A parareal in time discretization of PDE's", Comptes Rendus de l'Académie des Sciences - Series I - Mathematics, 332, 661–668, 2001.

5 L. Grigori, S.A. Hirstoaga, J. Salomon: "A Parareal algorithm for a highly oscillating Vlasov-Poisson system with reduced models for the coarse solving", Computers & Mathematics with Applications, 130, 137–148, 2023.