A posteriori analysis

of the penalty method

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A posteriori analysis has become the basic tool for automatic mesh adaptivity in finite elements and finite volumes. However many other applications have recently appeared.

- **About multi-step discretizations**

- **Optimization of the penalty parameter for the Stokes equations**
  with V. Girault, and F. Hecht

- **Other applications of the penalty method**

About multi-step discretizations

Let X be a Banach space, and assume that A is a continuous mapping from X into X. For a given element f of X, we are interested in the discretization of the equation :

Find u in X such that

$$A(u) = f.$$

δ : positive parameter

We consider the problem :

Find u_δ in X_δ such that

$$A_\delta(u_\delta) = f_\delta,$$

where X_δ is a finite-dimensional subspace of X, A_δ denotes an approximation of A defined on X_δ and f_δ an approximation of f in X_δ.

A priori estimates

$$\|u - u_\delta\|_X \le F(\delta, u) + H(\delta, f).$$

The quantity $F(\delta, u)$ is usually equal to some power of δ times some norm of u, so it involves the regularity of u (which is most often unknown).

$\Longrightarrow$ proves the convergence of the method.

$\Longrightarrow$ useful for the choice of the discretization.

A posteriori estimates

$$\|u - u_\delta\|_X \le G(\delta, f_\delta, u_\delta) + K(\delta, f).$$

The quantity $G(\delta, f_\delta, u_\delta)$ can be computed explicitly once the discrete solution u_δ is known.

First application : mesh adaptivity

A new application : multi-step discretizations

In a large number of cases, the discretization of partial differential equations involves one or several intermediate (non discrete) problems before the final discrete problem. However error indicators only depending on the discrete solution can be constructed for each step of the discretization.

Initial problem : Find u in X such that $A(u) = f$.

Intermediate problem : Find u_ε in X such that $A_\varepsilon(u_\varepsilon) = f$.

Discrete problem : Find $u_{\varepsilon\delta}$ in X_δ such that $A_{\varepsilon\delta}(u_{\varepsilon\delta}) = f_\delta$.

Several parameters ε and δ are involved in the discretization. The aim of a posteriori analysis is to optimize simultaneously the choice of both ε and δ.

- Parabolic equation

Intermediate problem : Time semi-discrete problem
Discretization parameters : Time step and mesh size

A. Bergam, C.B., Z. Mghazli

- Penalization (or regularization) methods

C.B., V. Girault, F. Hecht

Intermediate problem : Continuous penalized problem
Discretization parameters : Penalty parameter and mesh size

- Automatic coupling of models

M. Braack, A. Ern

Several independent parameters appear in the discretization. This leads to use several families of error indicators.

The aim is to uncouple as much as possible the errors linked to the different parameters. But each indicator only requires the knowledge of the fully discrete solution (in order that it can be computed explicitly). So this is not completely possible for the estimates.

The idea of local representation of the error must be given up (or at least modified) for some families of indicators. But the optimization of the other parameters must remain compatible with mesh adaptivity.

An optimization criterion : The error due to ε must be of the same order as the discretization error due to δ.

Optimization of the penalty parameter for the Stokes equations

Let Ω be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary.

$$\begin{aligned} -\Delta \mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

The unknowns are the velocity u and the pressure p.

Variational formulation

Find $(\mathbf{u}, p)$ *in* $H_0^1(\Omega)^d \times L_0^2(\Omega)$ *satisfying*

$$\forall v \in H_0^1(\Omega)^d, \quad \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\, d\mathbf{x} - \int_\Omega (\operatorname{div} \mathbf{v})(\mathbf{x})\, p(\mathbf{x})\, d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x},$$

$$\forall q \in L_0^2(\Omega), \quad -\int_\Omega (\operatorname{div} \mathbf{u})(\mathbf{x})\, q(\mathbf{x})\, d\mathbf{x} = 0.$$

For any data f in $H^{-1}(\Omega)^d$, this problem has a unique solution.

The penalized problem

Let ε be the penalty parameter, $0 < \varepsilon \leq 1$.

Find $(\mathbf{u}_\varepsilon, p_\varepsilon)$ *in* $H_0^1(\Omega)^d \times L_0^2(\Omega)$ *satisfying*

$$\forall v \in H_0^1(\Omega)^d, \quad \int_\Omega \mathbf{grad}\,\mathbf{u}_\varepsilon : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} - \int_\Omega (\operatorname{div}\mathbf{v})(\mathbf{x})\,p_\varepsilon(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})\,d\mathbf{x},$$

$$\forall q \in L_0^2(\Omega), \quad -\int_\Omega (\operatorname{div}\mathbf{u}_\varepsilon)(\mathbf{x})\,q(\mathbf{x})\,d\mathbf{x} - \varepsilon \int_\Omega p_\varepsilon(\mathbf{x})q(\mathbf{x})\,d\mathbf{x} = 0.$$

The second equation is equivalent to

$$p_\varepsilon = -\varepsilon^{-1}\operatorname{div}\mathbf{u}_\varepsilon.$$

As a consequence, the penalized problem can equivalently be written

Find $\mathbf{u}_\varepsilon$ *dans* $H_0^1(\Omega)^d$ *satisfying*

$$\forall v \in H_0^1(\Omega)^d, \quad \int_\Omega \mathbf{grad}\,\mathbf{u}_\varepsilon : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} + \varepsilon^{-1} \int_\Omega (\mathrm{div}\,\mathbf{u}_\varepsilon)(\mathbf{x})\,(\mathrm{div}\,\mathbf{v})(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,d\mathbf{x},$$

Find p_ε *dans* $L_0^2(\Omega)$ *satisfying*

$$\forall q \in L_0^2(\Omega), \quad \varepsilon \int_\Omega p_\varepsilon(\mathbf{x}) q(\mathbf{x})\,d\mathbf{x} = -\int_\Omega (\mathrm{div}\,\mathbf{u}_\varepsilon)(\mathbf{x})\,q(\mathbf{x})\,d\mathbf{x}.$$

For any data f **in** $H^{-1}(\Omega)^d$**, this problem has a unique solution.**

This does not require any inf-sup condition !

A priori error estimates can be proven between the solutions (u, p) **and** $(\mathbf{u}_\varepsilon, p_\varepsilon)$**. The error behaves like** ε**.**

V. Girault, P.-A. Raviart

The discrete problem

$(\mathcal{T}_h)_h$: regular family of triangulations of Ω by triangles or tetrahedra.

$$X_h \subset H_0^1(\Omega)^d, \qquad M_h \subset L_0^2(\Omega).$$

The spaces X_h et M_h are mede of piecewise polynomial functions on the triangulation $\mathcal{T}_h$.

Find $(\mathbf{u}_{\varepsilon h}, p_{\varepsilon h})$ *in* $X_h \times M_h$ *satisfying*

$$\forall \mathbf{v}_h \in X_h, \quad \int_\Omega \mathbf{grad}\,\mathbf{u}_{\varepsilon h} : \mathbf{grad}\,\mathbf{v}_h \, d\mathbf{x} - \int_\Omega (\mathrm{div}\,\mathbf{v}_h)(\mathbf{x})\, p_{\varepsilon h}(\mathbf{x})\, d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}_h(\mathbf{x})\, d\mathbf{x},$$

$$\forall q_h \in M_h, \quad -\int_\Omega (\mathrm{div}\,\mathbf{u}_{\varepsilon h})(\mathbf{x})\, q_h(\mathbf{x})\, d\mathbf{x} - \varepsilon \int_\Omega p_{\varepsilon h}(\mathbf{x}) q_h(\mathbf{x})\, d\mathbf{x} = 0.$$

Let Π_h denote the orthogonal projection operator from $L_0^2(\Omega)$ onto M_h. The second equation is equivalent to

$$p_{\varepsilon h} = -\varepsilon^{-1}\, \Pi_h(\mathrm{div}\,\mathbf{u}_{\varepsilon h}).$$

The discrete problem can equivalently be written

Find $\mathbf{u}_{\varepsilon h}$ *in* X_h *satisfying*

$$\forall v_h \in X_h, \quad \int_\Omega \mathbf{grad}\,\mathbf{u}_{\varepsilon h} : \mathbf{grad}\,\mathbf{v}_h\,d\mathbf{x} \;+\; \varepsilon^{-1}\int_\Omega \Pi_h(\mathrm{div}\,\mathbf{u}_{\varepsilon h})(\mathbf{x})\,\Pi_h(\mathrm{div}\,\mathbf{v}_h)(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{v}_h(\mathbf{x})\,d\mathbf{x},$$

Find $p_{\varepsilon h}$ *in* M_h *satisfying*

$$\forall q_h \in M_h, \quad \varepsilon\int_\Omega p_{\varepsilon h}(\mathbf{x})q_h(\mathbf{x})\,d\mathbf{x} = -\int_\Omega (\mathrm{div}\,\mathbf{u}_{\varepsilon h})(\mathbf{x})\,q_h(\mathbf{x})\,d\mathbf{x}.$$

For any f **in** $H^{-1}(\Omega)^d$**, this problem has a unique solution.**

This does not require any inf-sup condition !

Provides an efficient algorithm for solving the discrete problem, since the two unknowns are now uncoupled !

If an inf-sup condition exists between the spaces X_h **and** M_h**, a priori error estimates can be proven between the solutions** (u, p) **and** $(\mathrm{u}_{\varepsilon h}, p_{\varepsilon h})$**.**

Two families of error indicators

- **Error indicator related to the penalty term**

$$\eta_\varepsilon = \varepsilon\, \|p_{\varepsilon h}\|_{L^2(\Omega)}.$$

- **Error indicators related to the finite elements**

$\mathcal{E}_K$: set of edges or faces of K which are not contained in $\partial\Omega$.

For any element K of $\mathcal{T}_h$,

$$\eta_K = h_K\, \|\mathbf{f}_h + \Delta \mathbf{u}_{\varepsilon h} - \mathbf{grad}\, p_{\varepsilon h}\|_{L^2(K)^d} + \frac{1}{2} \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}}\, \|[\partial_{\mathbf{n}} \mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\, \mathbf{n}]\|_{L^2(e)^d} + \|\mathrm{div}\, \mathbf{u}_{\varepsilon h}\|_{L^2(K)},$$

where f_h denotes a piecewise polynomial approximation of f.

C.B., B. Métivet, R. Verfürth

An a posteriori error estimate

The aim is to bound the error between (u, p) and $(\mathrm{u}_{\varepsilon h}, p_{\varepsilon h})$ as a function of η_ε, the η_K and the data.

$$|\mathbf{u} - \mathbf{u}_{\varepsilon h}|_{H^1(\Omega)^d} \leq |\mathbf{u} - \mathbf{u}_\varepsilon|_{H^1(\Omega)^d} + |\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(\Omega)^d},$$

$$\|p - p_{\varepsilon h}\|_{L^2(\Omega)} \leq \|p - p_\varepsilon\|_{L^2(\Omega)} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)}.$$

Idea : A residual equation to bound each term.

Residual equation for the first term :

$$\forall v \in H_0^1(\Omega)^d,\ \int_\Omega \mathbf{grad}\,(\mathbf{u} - \mathbf{u}_\varepsilon) : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} - \int_\Omega (\mathrm{div}\,\mathbf{v})(\mathbf{x})\,(p - p_\varepsilon)(\mathbf{x})\,d\mathbf{x} = 0,$$

$$\forall q \in L_0^2(\Omega), \qquad -\int_\Omega \Big(\mathrm{div}\,(\mathbf{u} - \mathbf{u}_\varepsilon)\Big)(\mathbf{x})\,q(\mathbf{x})\,d\mathbf{x} = \varepsilon \int_\Omega p_\varepsilon(\mathbf{x}) q(\mathbf{x})\,d\mathbf{x}.$$

The estimate results from the stability properties of the Stokes operator and a triangle inequality.

Proposition : The following a posteriori estimate holds

$$|\mathbf{u} - \mathbf{u}_\varepsilon|_{H^1(\Omega)^d} + \|p - p_\varepsilon\|_{L^2(\Omega)} \leq c\,\Big(\eta_\varepsilon + \varepsilon\,\|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)}\Big).$$

Residual equation for the second term :

$$\forall v \in H_0^1(\Omega)^d, \quad \int_\Omega \mathbf{grad}\,(\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}) : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} + \dots$$

The estimate is derived via the standard arguments for the Stokes problem (and requires the introduction of an approximation $\mathbf{v}_h$ of the function $\mathbf{v}$ in X_h).

Proposition : Let us assume that the space X_h contains

$$Y_h = \left\{\mathbf{v}_h \in H_0^1(\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_{h\,|K} \in \mathcal{P}_1(K)^d\right\}.$$

The following a posteriori estimate holds

$$|\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(\Omega)^d} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} \leq c\left(\eta_\varepsilon + \Big(\sum_{K\in\mathcal{T}_h} (\eta_K^2 + h_K^2\,\|\mathbf{f} - \mathbf{f}_h\|^2_{L^2(K)^d}\Big)^{\frac{1}{2}}\right).$$

An upper bound for the indicators

Idea : In the first residual equation,

$$\forall q \in L_0^2(\Omega),$$
$$\varepsilon \int_\Omega p_{\varepsilon h}(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x} = -\varepsilon \int_\Omega (p_\varepsilon - p_{\varepsilon h})(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x} - \int_\Omega \Big(\mathrm{div}\,(\mathbf{u} - \mathbf{u}_\varepsilon)\Big)(\mathbf{x})\, q(\mathbf{x})\, d\mathbf{x},$$

we take q equal to $p_{\varepsilon h}$.

Proposition : The following estimate holds

$$\eta_\varepsilon \le c\, |\mathbf{u} - \mathbf{u}_\varepsilon|_{H^1(\Omega)^d} + \varepsilon\, \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)}.$$

Idea : In the second residual equation, we take v successively equal to

$$v_K = (\mathbf{f}_h + \Delta\,\mathbf{u}_{\varepsilon h} - \mathbf{grad}\, p_{\varepsilon h})\,\psi_K \quad \text{on } K, \qquad 0 \quad \text{elsewhere},$$

$$v_e = \mathcal{L}\Big([\partial_\nu \mathbf{u}_{\varepsilon h} - p_{\varepsilon h}\,\mathbf{n}]\,\psi_e) \quad \text{on } K \cup K', \qquad 0 \quad \text{elsewhere},$$

next q equal to

$$q_K = \operatorname{div} \mathbf{u}_{\varepsilon h} \quad \text{on } K, \qquad 0 \quad \text{elsewhere}.$$

Proposition : The following estimate holds, for any element K of $\mathcal{T}_h$,

$$\eta_K \le c\Big(|\mathbf{u} - \mathbf{u}_\varepsilon|_{H^1(K)^d} + |\mathbf{u}_\varepsilon - \mathbf{u}_{\varepsilon h}|_{H^1(\omega_K)^d} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\omega_K)} + h_K\,\|\mathbf{f} - \mathbf{f}_h\|_{L^2(\omega_K)^d}\Big),$$

where ω_K stands for the union of elements of $\mathcal{T}_h$ that share at least an edge or a face with K.

Optimal estimates, the second one is local in space !

Some numerical experiments

realized with the code FreeFem++

F. Hecht, O. Pironneau

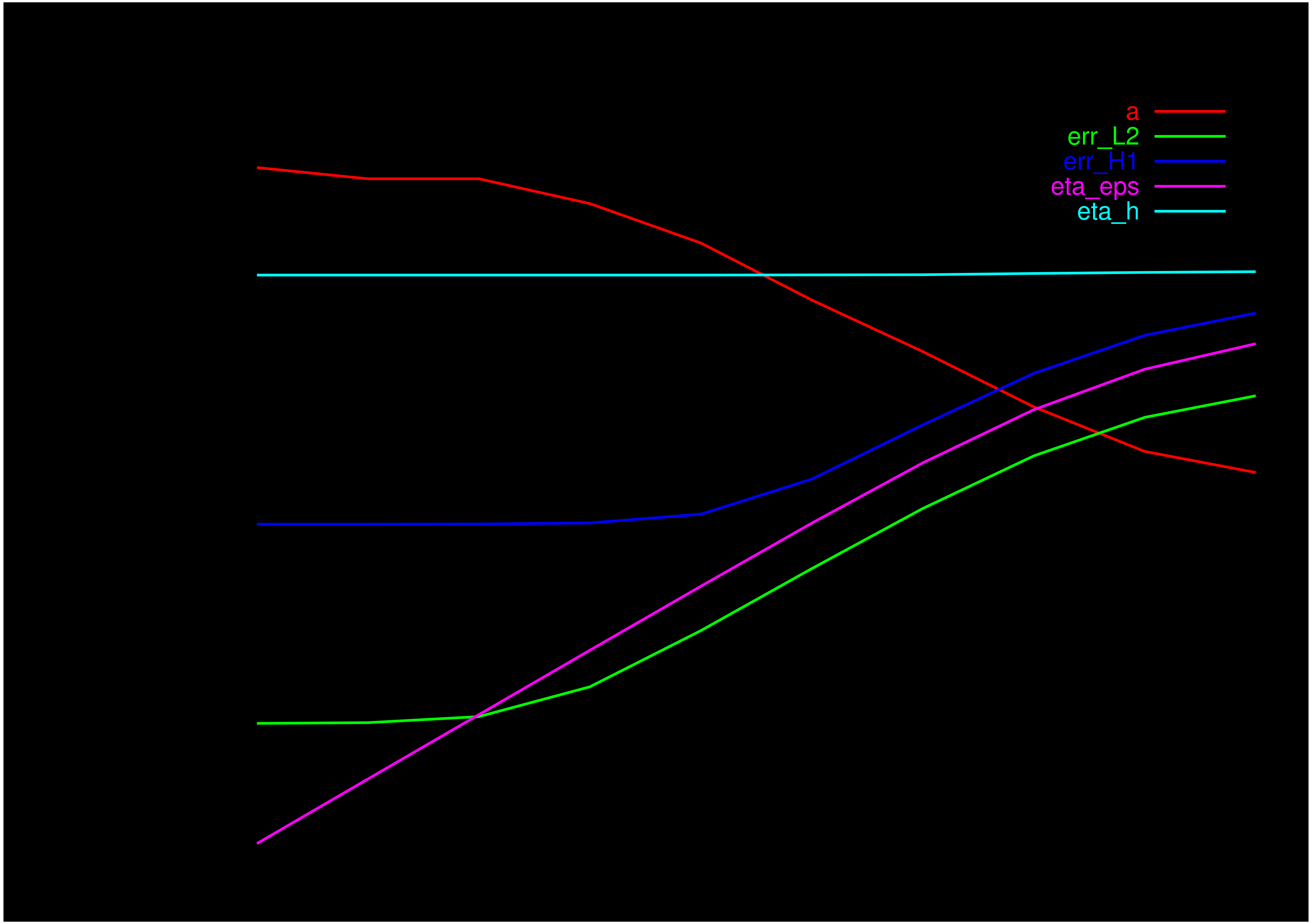
Taylor–Hood finite element, uniform mesh

Given smooth solution in a square

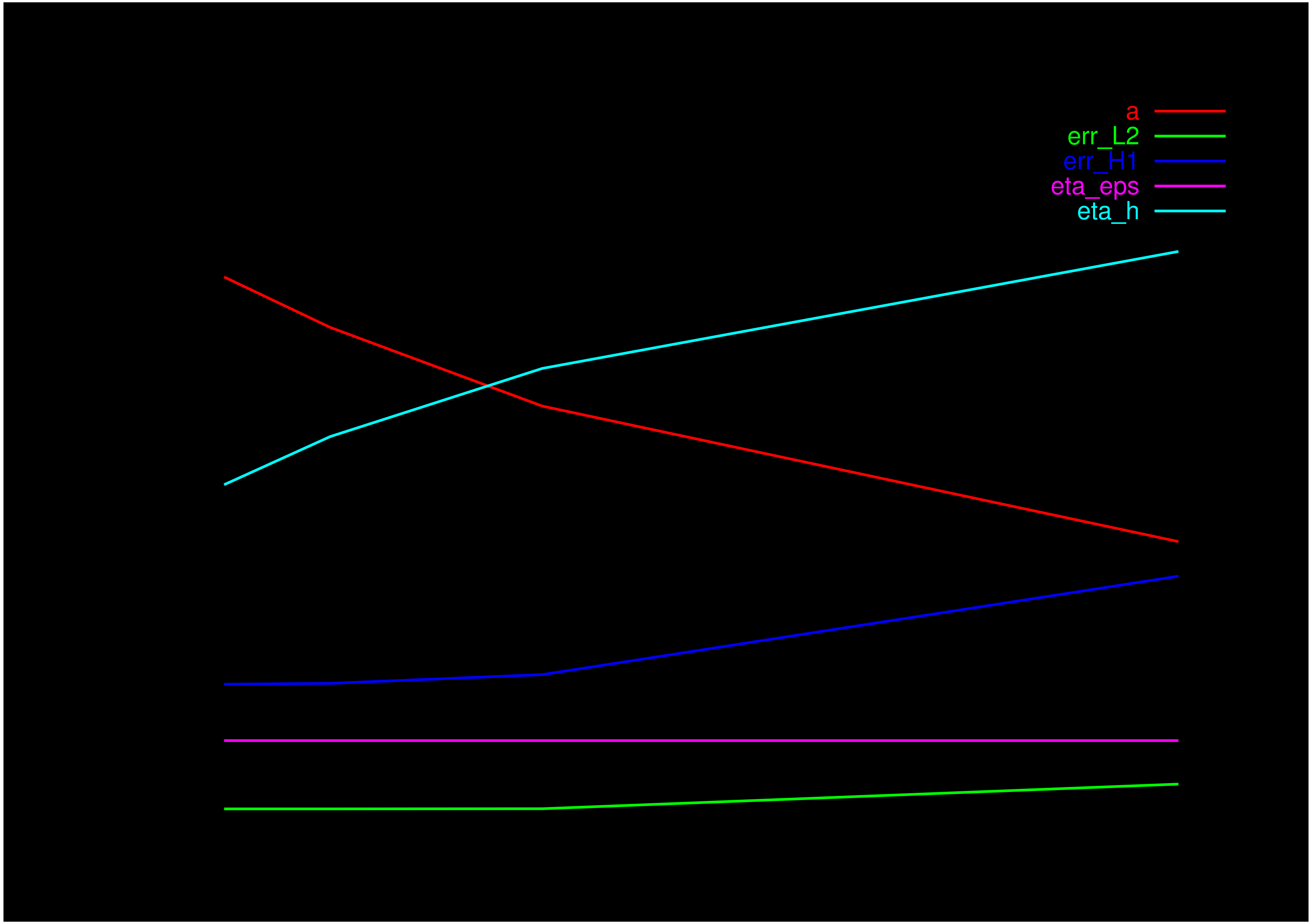
- Influence of the penalty parameter
- Influence of the mesh

$$\eta_h = \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{\frac{1}{2}}.$$

$h = \sqrt{2}\, 10^{-1}$



$\varepsilon = 10^{-2}$



Extensions

- **Local penalization**

We have tried to replace $\varepsilon \int_\Omega p(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}$ by

$$\sum_{K\in\mathcal{T}_h} \varepsilon_K \int_K p(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}.$$

$+$ two types of local error indicators.

Some restrictions on the discretization, no special interest.

- **Navier–Stokes equations**

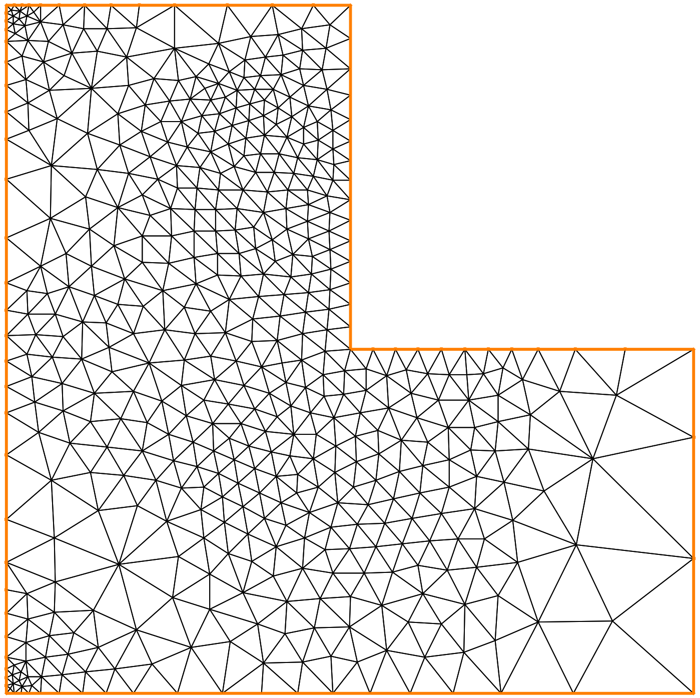
J. Pousin, J. Rappaz

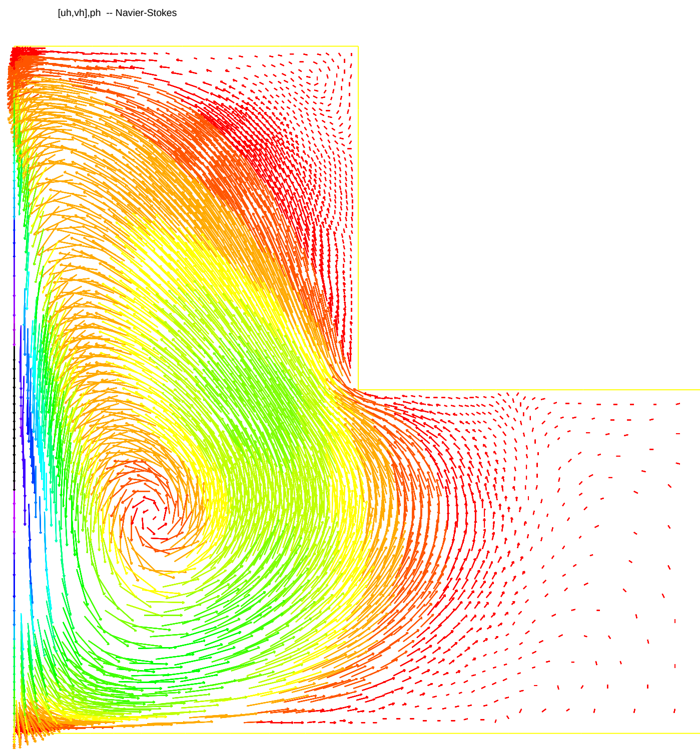
R. Verfürth

No further difficulty.

Data : viscosity $\nu = 10^{-2}$ and non zero tangential velocity on the left boundary.

$$\varepsilon = 0.14 \times 10^{-2}$$





Other applications of the penalty method

- **Spectral element discretization of the Stokes problem**

with **A. Blouza**, **N. Chorfi**, and **N. Kharrat**

Let Ω be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary.

$$\begin{aligned} -\Delta \mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned}$$

The unknowns are the velocity $\mathbf{u}$ and the pressure p.

The penalized problem

Let ε be the penalty parameter, $0 < \varepsilon \leq 1$.

Find $(\mathbf{u}, p)$ *in* $H_0^1(\Omega)^d \times L_0^2(\Omega)$ *such that*

$$\forall v \in H_0^1(\Omega)^d, \quad \int_\Omega \mathbf{grad}\,\mathbf{u} : \mathbf{grad}\,\mathbf{v}\,d\mathbf{x} - \int_\Omega (\mathrm{div}\,\mathbf{v})(\mathbf{x})\,p(\mathbf{x})\,d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,d\mathbf{x},$$

$$\forall q \in L_0^2(\Omega), \quad -\int_\Omega (\mathrm{div}\,\mathbf{u})(\mathbf{x})\,q(\mathbf{x})\,d\mathbf{x} - \varepsilon \int_\Omega p_\varepsilon(\mathbf{x}) q(\mathbf{x})\,d\mathbf{x} = 0.$$

We assume that Ω admits a decomposition without overlap

$$\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k \qquad \textbf{et} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k < k' \leq K,$$

where :

- Each Ω_k is a rectangle ($d = 2$) or a rectangular parallelepiped($d = 3$),
- The intersection of two different Ω_k is either empty or a vertex or a whole edge or a whole face of booth subdomains.

The discrete spaces

$\mathbb{P}_n(\Omega_k)$: space of restrictions to Ω_k of polynomials with degree $\leq n$ with respect to each variable.

Let λ be a fixed parameter, $0 < \lambda \leq 1$. For any integer $N \geq 2$, we define the spaces

$$X_N = \Big\{ \mathbf{v}_N \in H_0^1(\Omega)^d;\ \mathbf{v}_{N|\Omega_k} \in \mathbb{P}_N(\Omega_k)^d,\ 1 \leq k \leq K \Big\},$$

$$M_N = \Big\{ q_N \in L_0^2(\Omega);\ q_{N|\Omega_k} \in \mathbb{P}_{N-2}(\Omega_k) \cap \mathbb{P}_{\lambda N}(\Omega_k),\ 1 \leq k \leq K \Big\}.$$

Numerical integrationn

$$\xi_0 = -1, \qquad \xi_N = 1.$$

Gauss-Lobatto quadrature formula : there exis nodes ξ_j, $1 \le j \le N-1$, and weights ρ_j, $0 \le j \le N$, such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j.$$

If F_k denotes one of the affine mappings that maps the reference domain $]-1,1[^d$ onto Ω_k, we define a discrete product, for instance in dimension $d = 2$ by

$$(u,v)_N = \sum_{k=1}^{K} \frac{\text{meas}(\Omega_k)}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} u \circ F_k(\xi_i, \xi_j) v \circ F_k(\xi_i, \xi_j)\, \rho_i \rho_j, .$$

The discrete problem

Find $(\mathbf{u}_{\varepsilon N}, p_{\varepsilon N})$ *in* $X_N \times M_N$ *such that*

$$\forall \mathbf{v}_N \in X_N, \quad \left(\mathbf{grad}\, \mathbf{u}_{\varepsilon N}, \mathbf{grad}\, \mathbf{v}_N\right)_N - \left(\mathrm{div}\, \mathbf{v}_N, p_{\varepsilon N}\right)_N = (\mathbf{f}, v_N)_N,$$

$$\forall q_N \in M_N, \quad -\left(\mathrm{div}\, \mathbf{u}_{\varepsilon N}, q_N\right)_N - \varepsilon\,(p_{\varepsilon N}, q_N)_N = 0.$$

Let Π_N be the orthogonal projection operator from $L^2_0(\Omega)$ onto M_N. The second equation reads

$$p_{\varepsilon N} = -\varepsilon^{-1}\,\Pi_N(\mathrm{div}\, \mathbf{u}_{\varepsilon N}).$$

The discrete problem can equivalently be written as

Find $\mathbf{u}_{\varepsilon N}$ *in* X_N *such that*

$$\forall v_N \in X_N, \quad \Big(\mathbf{grad}\,\mathbf{u}_{\varepsilon N}, \mathbf{grad}\,\mathbf{v}_N\Big)_N \;+\; \varepsilon^{-1}\Big(\Pi_N(\mathrm{div}\,\mathbf{u}_{\varepsilon N}), \Pi_N(\mathrm{div}\,\mathbf{v}_N)\Big)_N = (\mathbf{f}, v_N)_N,$$

Find $p_{\varepsilon N}$ *in* M_N *such that*

$$\forall q_N \in M_N, \quad \varepsilon\,(p_{\varepsilon N}, q_N)_N = -\Big(\mathrm{div}\,\mathbf{u}_{\varepsilon N}, q_N\Big)_N.$$

For any continuous data f on $\overline{\Omega}$, this problem has a unique solution.

This does not require any inf-sup condition !

Provides an efficient algorithm for solving the discrete problem, since the two unknowns are now uncoupled !

If an inf-sup condition exists between the spaces X_N and M_N, a priori error estimates can be proven between the solutions (u, p) and $(\mathrm{u}_{\varepsilon N}, p_{\varepsilon N})$.

Two families of error indicators

- **Error indicator related to the penalty term**

$$\eta_\varepsilon = \varepsilon \, \|p_{\varepsilon N}\|_{L^2(\Omega)}.$$

- **Error indicators related to the spectral discretization**

$\mathcal{E}_k$ **: set of edges or faces of Ω_k which are not contained in $\partial\Omega$.**

For $1 \leq k \leq K$,

$$\eta_k = N^{-1} \, \|\mathcal{I}_N \mathbf{f} + \Delta \mathbf{u}_{\varepsilon N} - \mathbf{grad}\, p_{\varepsilon N}\|_{L^2(K)^d} + \sum_{e \in \mathcal{E}_k} N^{-\frac{1}{2}} \, \|[\partial_\nu \mathbf{u}_{\varepsilon N} - p_{\varepsilon N} \, \mathbf{n}]\|_{L^2(e)^d} + \|\mathrm{div}\, \mathbf{u}_{\varepsilon N}\|_{L^2(K)},$$

where $\mathcal{I}_N$ stands for the interpolation operator at all nodes $F_k(\xi_i, \xi_j)$ or $F_k(\xi_i, \xi_j, \xi_p)$.

A posteriori error estimate

Proposition : The following a posteriori error estimate holds

$$|\mathbf{u}-\mathbf{u}_\varepsilon|_{H^1(\Omega)^d}+\|p-p_\varepsilon\|_{L^2(\Omega)}\leq c\left(\eta_\varepsilon+\varepsilon\,\|p_\varepsilon-p_{\varepsilon N}\|_{L^2(\Omega)}\right).$$

Proposition : The following a posteriori error estimate homds

$$|\mathbf{u}_\varepsilon-\mathbf{u}_{\varepsilon N}|_{H^1(\Omega)^d}+\|p_\varepsilon-p_{\varepsilon N}\|_{L^2(\Omega)}\leq c\left(\eta_\varepsilon+\rho_\Omega\Big(\sum_{k=1}^{K}\eta_k^2\Big)^{\frac{1}{2}}+\|\mathbf{f}-\mathcal{I}_N f\|_{L^2(K)^d}^2\Big)^{\frac{1}{2}}\right),$$

where ρ_Ω is equal to 1 in dimension $d=2$ or if Ω is convex, to $N^{-\frac{1}{2}}$ otherwise.

An upper bound for the indicators

Proposition : The following estimate holds

$$\eta_\varepsilon \leq |\mathbf{u} - \mathbf{u}_\varepsilon|_{H^1(\Omega)^d} + \varepsilon \, \|p_\varepsilon - p_{\varepsilon N}\|_{L^2(\Omega)}.$$

We do not state an upper bound for the η_k (which would be nonoptimal) since we do not intend to adapt the N.

Some numerical experiments

Adaptivity strategy : Let us choose $\varepsilon^0 \leq 1$ and a parameter μ, $0 < \mu < 1$. Next, assuming that ε^m is known,
(i) we compute the solution $(\mathrm{u}_{\varepsilon N}, p_{\varepsilon N})$ forr $\varepsilon = \varepsilon^m$,t he indicator η^{ε^m} and the Hilbertian sum $\eta(N) = \left(\sum_{k=1}^{K} \eta_k^2\right)^{\frac{1}{2}}$.
(ii) if

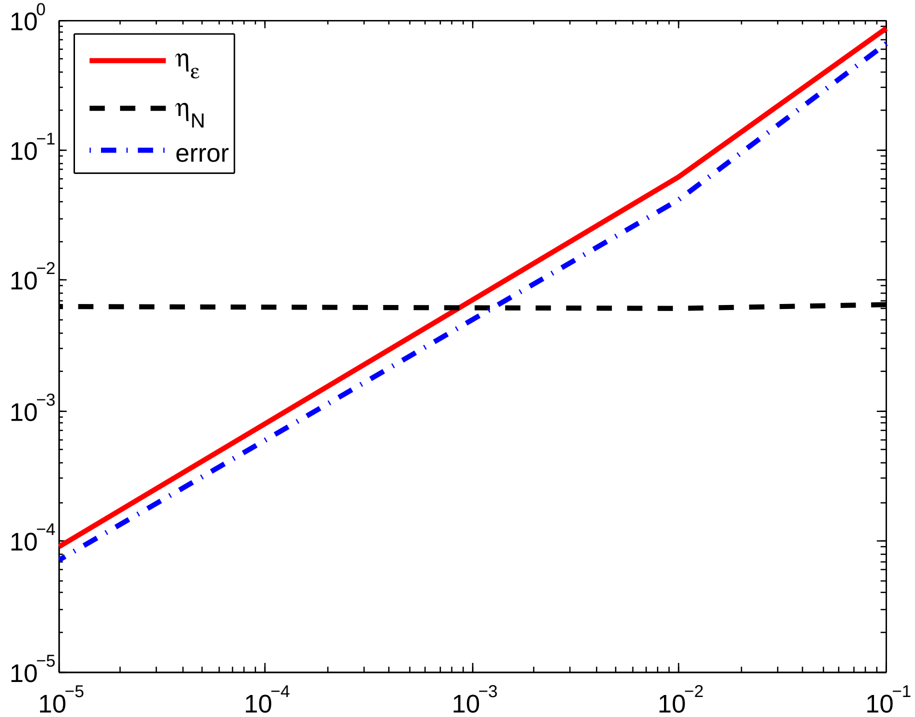
$$\mu\, \eta(N) \leq \eta^{\varepsilon^m} \leq \frac{1}{\mu}\, \eta(N),$$

we stop the process.
(iii) otherwise, we take ε^{m+1} equal to $\varepsilon^m\, \eta(N)/\eta^{\varepsilon^m}$ and we go back to step (i).

For a given solution in a L-shaped domain divided into three squares,

N	5	10	15	20	30
$\varepsilon_{\mathrm{opt}}$	0.0375	0.0069	0.0024	0.0007	0.0006

$N = 20$



10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
η_ε
η_N
error
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}

- **Domain decomposition methods**
  with **T. Chacón Rebollo, E. Chacón Vera, and D. Franco Coronil**

When a decomposition of the domain without overlap is cosidered, the matching conditions on the interfaces are most often handled via the introduction of a Lagrange multiplier. And of course it is possible to use a penalty method for the corresponding mixed problem.

Let Ω be a bounded open domain in $I\!R^d$, $d = 2$ or 3, with a Lipschitz– continuous boundary.

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

We assume that Ω admits a decomposition without overlap

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \qquad \text{and} \qquad \Omega_1 \cap \Omega_2 = \emptyset,$$

and also that, if Γ denotes the interface $\partial\Omega_1 \cap \partial\Omega_2$, Γ is Lipschitz– continuous and t$\partial\Gamma$ is contained in $\partial\Omega$.

Variational formulation

$$H^1_*(\Omega_i) = \Big\{ v \in H^1(\Omega_i);\; v = 0 \text{ **on** } \partial\Omega_i \setminus \Gamma \Big\}.$$

Findr (u_1, u_2, λ) *in* $H^1_*(\Omega_1) \times H^1_*(\Omega_2) \times H^{\frac{1}{2}}_{00}(\Gamma)$ *such that*

$$\forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2),$$

$$\sum_{i=1}^{2} \int_{\Omega_i} \mathbf{grad}\, u_i \cdot \mathbf{grad}\, v_i \, d\mathbf{x} + [[v_1 - v_2, \lambda]]_\Gamma = \sum_{i=1}^{2} \int_{\Omega_i} f(\mathbf{x}) \cdot v_i(\mathbf{x}) \, d\mathbf{x},$$

$$\forall \mu \in H^{\frac{1}{2}}_{00}(\Gamma), \qquad [[u_1 - u_2, \mu]]_\Gamma = 0,$$

where $[[\cdot,\cdot]]_\Gamma$ **denotes the scalar product of** $H^{\frac{1}{2}}_{00}(\Gamma)$**.**

The equivalence of this formulation with the initial problem is readily checked.

The penalized problem

$$\textit{Find } (u_1^\varepsilon, u_2^\varepsilon, \lambda^\varepsilon) \textit{ in } H^1_*(\Omega_1) \times H^1_*(\Omega_2) \times H^{\frac{1}{2}}_{00}(\Gamma) \textit{ such that}$$

$$\forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2),$$
$$\sum_{i=1}^{2} \int_{\Omega_i} \mathbf{grad}\, u_i^\varepsilon \cdot \mathbf{grad}\, v_i \, d\mathbf{x} + [[v_1 - v_2, \lambda^\varepsilon]]_\Gamma = \sum_{i=1}^{2} \int_{\Omega_i} f(\mathbf{x}) \cdot v_i(\mathbf{x}) \, d\mathbf{x},$$

$$\forall \mu \in H^{\frac{1}{2}}_{00}(\Gamma), \qquad [[u_1^\varepsilon - u_2^\varepsilon, \mu]]_\Gamma = \varepsilon \, [[\lambda^\varepsilon, \mu]]_\Gamma.$$

A priori error estimates of order ε between the solutions (u_1, u_2, λ) and $(u_1^\varepsilon, u_2^\varepsilon, \lambda^\varepsilon)$ are easily derived.

But when replacing the scalar product $[[\cdot,\cdot]]_\Gamma$ by the scalar product of $L^2(\Gamma)$, the convergence is only of order $\sqrt{\varepsilon}$.

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From now on, we only consider the reduced problem

Find $(u_1^\varepsilon, u_2^\varepsilon)$ *in* $H^1_*(\Omega_1) \times H^1_*(\Omega_2)$ *such that*

$$\forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2),$$
$$\sum_{i=1}^{2} \int_{\Omega_i} \mathbf{grad}\, u_i^\varepsilon \cdot \mathbf{grad}\, v_i \, d\mathbf{x} + \varepsilon^{-1} [[u_1^\varepsilon - u_2^\varepsilon, v_1 - v_2]]_\Gamma = \sum_{i=1}^{2} \int_{\Omega_i} f(\mathbf{x}) \cdot v_i(\mathbf{x}) \, d\mathbf{x},$$

The discrete problem

$(\mathcal{T}_h)_h$: regular family of trianngulations of Ω by triangles or tetrahedra such that Γ is contained in edges ($d = 2$) or faces ($d = 3$) of elements of $\mathcal{T}_h$.

We consider discrete spaces built from triangulation $\mathcal{T}_h$ such that

$$X_{ih} \subset H^1_*(\Omega_i).$$

Find $(u^\varepsilon_{1h}, u^\varepsilon_{2h})$ in $X_{1h} \times X_{2h}$ such that

$$\forall (v_{1h}, v_{2h}) \in X_{1h} \times X_{2h},$$
$$\sum_{i=1}^{2} \int_{\Omega_i} \mathbf{grad}\, u^\varepsilon_{ih} \cdot \mathbf{grad}\, v_{ih}\, d\mathbf{x} + \varepsilon^{-1}\, [[u^\varepsilon_{1h} - u^\varepsilon_{2h}, v_{1h} - v_{2h}]]_\Gamma = \sum_{i=1}^{2} \int_{\Omega_i} f(\mathbf{x}) \cdot v_{ih}(\mathbf{x})\, d\mathbf{x}.$$

This problem has a unique solution.

Two families of error indicators

- **Error indicator related to the penalty term**

$$\eta_\varepsilon = \varepsilon \, \| u^\varepsilon_{1h} - u^\varepsilon_{2h} \|_{H^{\frac{1}{2}}_{00}(\Gamma)}.$$

- **Error indicators related to the finite element discretization**

For each element K of $\mathcal{T}_h$ contained in Ω_i, $i = 1, 2$,

$$\eta_K = h_K \, \| f_h + \Delta u^\varepsilon_{ih} \|_{L^2(K)} + \frac{1}{2} \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \, \| [\partial_{\mathbf{n}} \mathbf{u}^\varepsilon_{ih}] \|_{L^2(e)},$$

where f_h is a piecewise polynomial approximation of f.

A posteriori error estimate

$$h_m = \min_{K\cap\Gamma\neq\emptyset} h_K.$$

Proposition : Assume that each space X_{ih} contains

$$Y_{ih} = \Big\{ \mathbf{v}_h \in H^1_*(\Omega_i);\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_{h\,|K} \in \mathcal{P}_1(K) \Big\}.$$

The following a posteriori error estimate holds

$$\sum_{i=1}^{2} |u - \mathbf{u}^{\varepsilon}_{ih}|_{H^1(\Omega_I)} \le c\left(\eta_\varepsilon + \Big(\sum_{K\in\mathcal{T}_h} \mu_K^2 (\eta_K^2 + \|f - f_h\|^2_{L^2(K)})\Big)^{\frac{1}{2}}\right),$$

where μ_K is equal to 1 if $K \cap \Gamma = \emptyset$, à $h_m^{-\frac{1}{2}}$ otherwise.

The loss of optimality is local. It is due to the nonconformity of the discretization.

An upper bound for the indicators

Proposition : The following estimate holds

$$\eta_\varepsilon \leq c \sum_{i=1}^{2} |u - u_{ih}^\varepsilon|_{H^1(\omega_i)},$$

and, for any element K of $\mathcal{T}_h$,

$$\eta_K \leq c \Big(|u - u_h^\varepsilon|_{H^1(\omega_K)} + h_K \, \|f - f_h\|_{L^2(\omega_K)} \Big),$$

where ω_K is the union of elements of $\mathcal{T}_{nh}$thta share at least and edge ($d = 2$) or a face ($d = 3$) with K.

Key numerical difficulty :

Computing the scalar product $[[\cdot,\cdot]]_\Gamma$ is very expensive, even when using an extension of

E. Casas, J.-P. Raymond

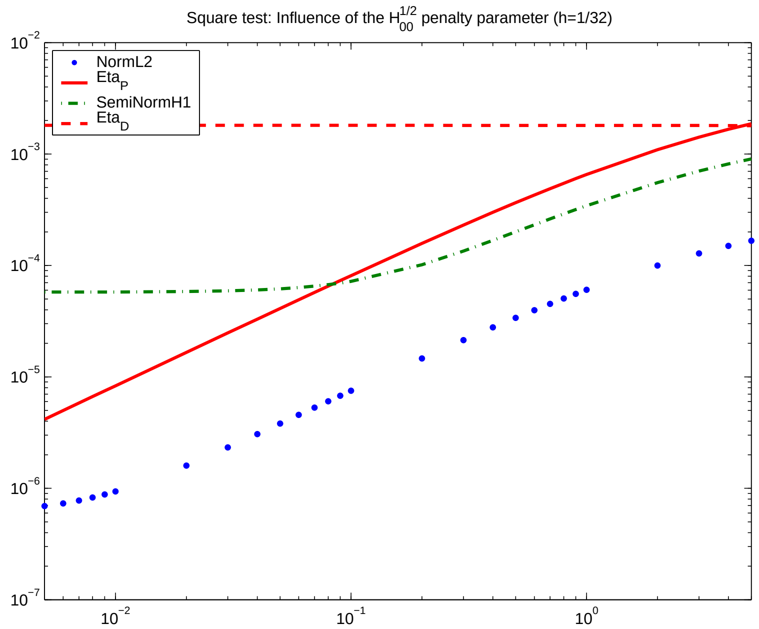
So the method is not so efficient as it could be thought frm the estimates.

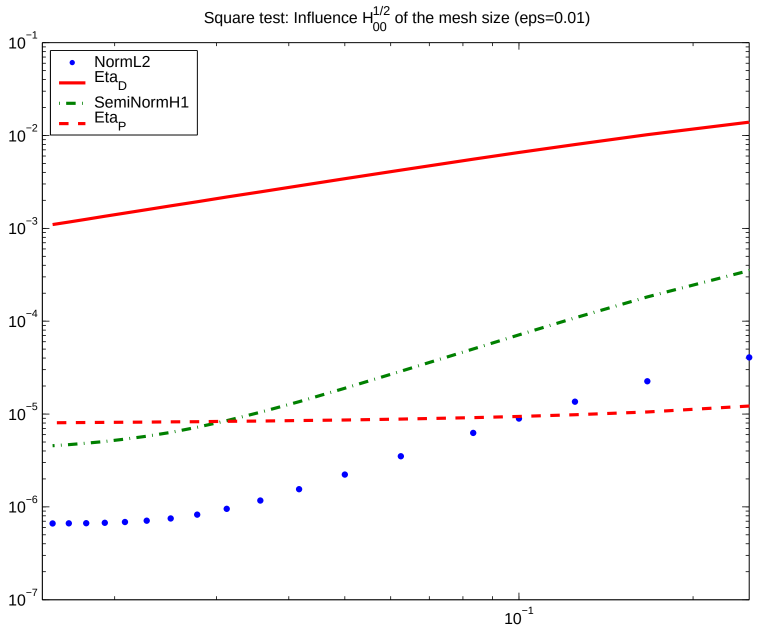
Some numerical experiments

Smooth solution in a square divided into two rectangles.

Influence of the penaty parameter (left part), of the mesh (right part)

$h = 3.10^{-2}$ $\qquad\qquad$ $\varepsilon = 10^{-2}$

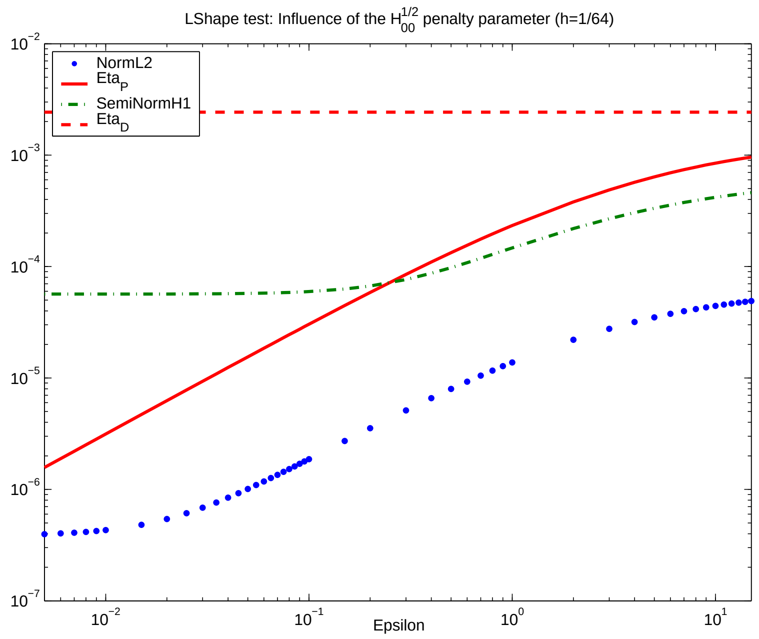


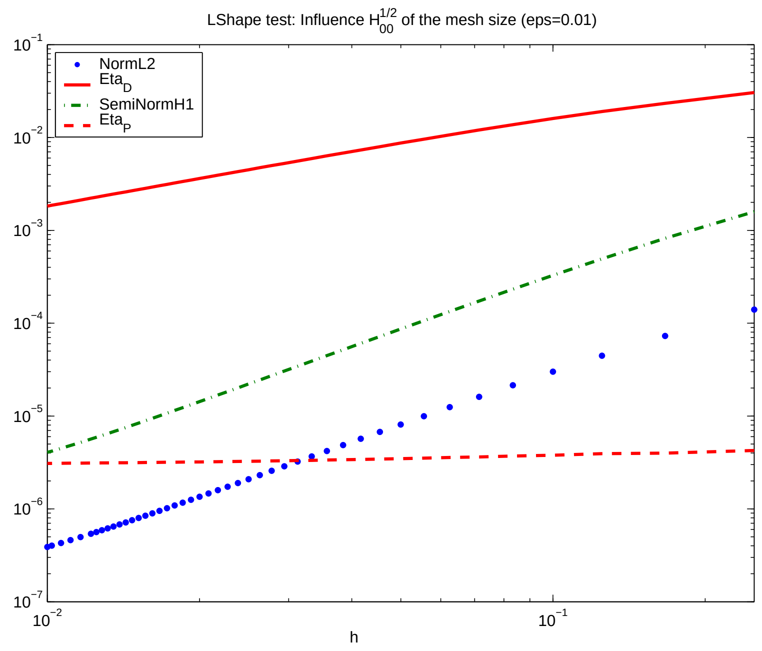


Smooth solution in a L-shaped domain divided into two trapezia.

Influence of the penaty parameter (left part), of the mesh (right part)

$h = 3.10^{-2}$ $\varepsilon = 10^{-2}$





The adaptivity algorithm in the L-shaped domain

Comparison of the methods with $H_{00}^{\frac{1}{2}}(\Gamma)$- and $L^2(\Gamma)$-matching

	ε_{opt}	**Iterations**	**CPU time**
$H_{00}^{1/2}$	**0.0503**	**165**	**126.3s**
L^2	**0.0029**	**157**	**99.0s**

$h = 1/64$

- **Finite element discretization of a shell model**

with A. Blouza, F. Hecht, and H. Le Dret

The Naghdi model and its variational formulations

ω : polygon in $\mathbb{R}^2$.
The midsurface of the shell is given by $S = \varphi(\bar{\omega})$ where φ is a one-to-one mapping in $W^{2,\infty}(\omega)^3$ such that the two vectors

$$\mathrm{a}_\alpha(\mathrm{x}) = (\partial_\alpha \varphi)(\mathrm{x})$$

are linearly independent at each point x of $\overline{\omega}$. The thickness of the shell is denoted by e.

Unit normal vector to S at point $\varphi(\mathrm{x})$: $\mathrm{a}_3(\mathrm{x}) = \frac{\mathrm{a}_1(\mathrm{x}) \wedge \mathrm{a}_2(\mathrm{x})}{|\mathrm{a}_1(\mathrm{x}) \wedge \mathrm{a}_2(\mathrm{x})|}$.
Local contravariant basis : $\mathrm{a}_i(\mathrm{x}) \cdot \mathrm{a}^j(\mathrm{x}) = \delta_i^j$.
Area element of the midsurface : $\sqrt{a}(\mathrm{x}) = |\mathrm{a}_1(\mathrm{x}) \wedge \mathrm{a}_2(\mathrm{x})|$.
First fundamental form of the surface : $a^{\alpha\beta} = \mathrm{a}^\alpha \cdot \mathrm{a}^\beta$.

In the case of homogeneous, isotropic material with Young modulus $E > 0$ and Poisson coefficient ν, $0 \leq \nu < \frac{1}{2}$, the coefficients of the elasticity tensor are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}.$$

This tensor is symmetric and uniformly strictly positive.

Covariant components of the change of metric tensor :

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \mathbf{a}_\beta + \partial_\beta \mathbf{u} \cdot \mathbf{a}_\alpha).$$

Covariant components of the change of transverse shear tensor :

$$\delta_{\alpha 3}(\mathbf{u}, \mathbf{r}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \mathbf{a}_3 + \mathbf{r} \cdot \mathbf{a}_\alpha).$$

Covariant components of the change of curvature tensor :

$$\chi_{\alpha\beta}(\mathbf{u}, \mathbf{r}) = \frac{1}{2}(\partial_\alpha \mathbf{u} \cdot \partial_\beta \mathbf{a}_3 + \partial_\beta \mathbf{u} \cdot \partial_\alpha \mathbf{a}_3 + \partial_\alpha \mathbf{r} \cdot \mathbf{a}_\beta + \partial_\beta \mathbf{r} \cdot \mathbf{a}_\alpha).$$

Boundary conditions :

The boundary $\partial\omega$ is divided into two parts : γ_0 on which the shell is clamped and its complementary part $\gamma_1 = \partial\omega \setminus \gamma_0$

$$H^1_{\gamma_0}(\omega) = \Big\{\mu \in H^1(\omega);\ \mu = 0 \textbf{ on } \gamma_0\Big\}.$$

The unknowns are the midsurface displacement $\mathbf{u}$ of the shell and its rotation $\mathbf{r}$ (so that $\mathbf{r}$ is tangential to the midsurface).

$$\mathbb{V}(\omega) = \Big\{(\mathbf{v}, \mathbf{s}) \in H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3;\ \mathbf{s} \cdot \mathbf{a}_3 = 0 \textbf{ in } \omega\Big\}.$$

First variational problem

Find (u, r) in $\mathbb{V}(\omega)$ such that

$$\forall (\mathbf{v}, \mathbf{s}) \in \mathbb{V}(\omega), \quad a\Big((\mathbf{u}, \mathbf{r}); (\mathbf{v}, \mathbf{s})\Big) = \mathcal{L}((\mathbf{v}, \mathbf{s})),$$

where the bilinear form $a(\cdot;\cdot)$ is defined by

$$a\Big((\mathbf{u}, \mathbf{r}); (\mathbf{v}, \mathbf{s})\Big) = \int_\omega \Big\{ e a^{\alpha\beta\rho\sigma} \Big[\gamma_{\alpha\beta}(\mathbf{u}) \gamma_{\rho\sigma}(\mathbf{v}) + \frac{e^2}{12} \chi_{\alpha\beta}(\mathbf{u}, \mathbf{r}) \chi_{\rho\sigma}(\mathbf{v}, \mathbf{s}) \Big] + 2e \frac{E}{1+\nu} a^{\alpha\beta} \delta_{\alpha 3}(\mathbf{u}, \mathbf{r}) \delta_{\beta 3}(\mathbf{v}, \mathbf{s}) \Big\} \sqrt{a}\, d\mathbf{x},$$

and the linear form $\mathcal{L}(\cdot)$ is given by

$$\mathcal{L}((\mathbf{v}, \mathbf{s})) = \int_\omega \mathbf{f} \cdot \mathbf{v} \sqrt{a}\, d\mathbf{x} + \int_{\gamma_1} (\mathbf{M} \cdot \mathbf{v} + \mathbf{N} \cdot \mathbf{s})\, d\tau.$$

The data f, M and N represent a given resultant force density, an applied moment density and an applied traction density, respectively.

A. Blouza, H. Le Dret

For any data $(\mathrm{f}, \mathrm{M}, \mathrm{N})$ in $H^1_{\gamma_0}(\omega)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime} \times H^{\frac{1}{2}}_{00}(\gamma_1)^{3\prime}$, this problem admits a unique solution (u, r) in $\mathbb{V}(\omega)$.

But, in view of the discretization, a Lagrange multiplier must be introduced to handle the tangency constraint $\mathrm{r} \cdot \mathrm{a}_3 = 0$.

Second (mixed) variational problem

A. Blouza, H. Le Dret

$$\mathbb{X}(\omega) = H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3, \qquad \mathbb{M}(\omega) = H^1_{\gamma_0}(\omega).$$

From now on, we set : $U = (\mathrm{u}, \mathrm{r}), V = (\mathrm{v}, s)$.

Find (U,ψ) in $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$ such that

$$\forall V\in\mathbb{X}(\omega),\quad a(U;V)+b(V;\psi)=\mathcal{L}(V),$$

$$\forall \chi\in\mathbb{M}(\omega),\quad b(U;\chi)=0,$$

where the new bilinear form $b(\cdot;\cdot)$ is defined by

$$b(V;\chi)=\int_\omega \partial_\alpha(\mathbf{s}\cdot\mathbf{a}_3)\partial_\alpha\chi\,d\mathbf{x}.$$

The penalized problem

Find (U,ψ) in $\mathbb{X}(\omega)\times\mathbb{M}(\omega)$ such that

$$\forall V\in\mathbb{X}(\omega),\quad a(U;V)+b(V;\psi)=\mathcal{L}(V),$$

$$\forall \chi\in\mathbb{M}(\omega),\quad b(U;\chi)=\varepsilon\, c(\psi,\chi),$$

where the new bilinear form $c(\cdot;\cdot)$ is defined by

$$c(\psi;\chi)=\int_\omega \partial_\alpha\psi\partial_\alpha\chi\,d\mathbf{x}.$$

All these problems are well-posed.

The discrete problem

$(\mathcal{T}_h)_h$: regular family of trianngulations of ω by triangles such that γ_0 is the union of whole edges of elements of $\mathcal{T}_h$.

We thus define the basic discrete space

$$\mathbb{M}_h = \left\{\chi_h \in H^1(\omega);\ \forall K \in \mathcal{T}_h,\ \chi_{h|K} \in \mathcal{P}_1(K)\right\},$$

next the spaces that are involved in the discrete problem

$$\mathbb{M}^{\gamma_0} = \mathbb{M}_h \cap H^1_{\gamma_0}(\omega), \qquad \mathbb{X}_h = \left(\mathbb{M}_h^{\gamma_0}\right)^3 \times \left(\mathbb{M}_h^{\gamma_0}\right)^3.$$

Find (U_h, ψ_h) in $\mathbb{X}_h \times \mathbb{M}_h^{\gamma_0}$ such that

$$\forall V_h \in \mathbb{X}_h, \quad a(U_h, V_h) + b(V_h, \psi_h) = \mathcal{L}(V_h),$$

$$\forall \chi_h \in \mathbb{M}_h^{\gamma_0}, \quad b(U_h, \chi_h) = \varepsilon\, c(\psi_h, \chi_h).$$

This problem has a unique solution.

Approximation of the data : We consider an approximation f_h of f in $\mathbb{Z}_h$ and approximations N_h and M_h of N and M in $\mathbb{Z}_h^1$, where the spaces $\mathbb{Z}_h$ and $\mathbb{Z}_h^1$ are defined by

$$\mathbb{Z}_h = \left\{ \mathbf{g}_h \in L^2(\omega)^3;\ \forall K \in \mathcal{T}_h,\ \mathbf{g}_{h|K} \in \mathcal{P}_0(K)^3 \right\},$$

$$\mathbb{Z}_h^1 = \left\{ \mathbf{P}_h \in L^2(\gamma_1)^3;\ \forall e \in \mathcal{E}_h^1,\ \mathbf{P}_{h|e} \in \mathcal{P}_0(e)^3 \right\}.$$

$\mathcal{E}_h^1$; set of edges of elements of $\mathcal{T}_h$ which are contained in $\overline{\gamma}_1$.

Approximation of the coefficients : We introduce approximations $a_h^{\alpha\beta}$, $a_h^{\alpha\beta\rho\sigma}$, $(\sqrt{a})_h$ and ℓ_h of the scalar coefficients $a^{\alpha\beta}$, $a^{\alpha\beta\rho\sigma}$, $\sqrt{a}$ and ℓ in the space $\mathbb{M}_h$. Similarly, we consider approximations a_k^h of the vectors a_k and d_α^h of the $\partial_\alpha \mathrm{a}_3$ in the space $\left(\mathbb{M}_h\right)^3$.

We also agree to denote by $\gamma^h_{\alpha\beta}(\cdot)$, $\delta^h_{\alpha 3}(\cdot)$ and $\chi^h_{\alpha\beta}(\cdot)$ the coefficients of the tensors introduced above where all coefficients are replaced by their approximations.

Some new notation : We define the contravariant components of the stress resultant

$$n_h^{\rho\sigma}(\mathbf{u}) = e a_h^{\alpha\beta\rho\sigma}\, \gamma^h_{\alpha\beta}(\mathbf{u}),$$

of the stress couple

$$m^{\rho\sigma}(U) = \frac{e^3}{12}\mathbf{a}^{\alpha\beta\rho\sigma}\, \chi_{\alpha\beta}(U),$$

and of the transverse shear force

$$t^{\beta}(U) = e\frac{E}{1+\nu}a^{\alpha\beta}\, \delta_{\alpha 3}(U).$$

We also observe that

$$\chi_{\rho\sigma}(V) = \theta_{\rho\sigma}(\mathbf{v}) + \gamma_{\rho\sigma}(\mathbf{s}), \qquad \text{with} \qquad \theta_{\rho\sigma}(\mathbf{v}) = \frac{1}{2}(\partial_\rho \mathbf{v}\cdot\partial_\sigma \mathbf{a}_3 + \partial_\sigma \mathbf{v}\cdot\partial_\rho \mathbf{a}_3).$$

Two families of error indicators

- **Error indicator related to the penalty term**

$$\eta_\varepsilon = \varepsilon \, |\psi_h^p|_{H^1(\omega)}$$

- **Error indicators related to the finite element discretization**

For each element K of $\mathcal{T}_h$,

$$\eta_K = \eta_{K1} + \eta_{K2} + \eta_{K3},$$

with

$$\begin{aligned}\eta_{K1} = {} & h_K \, \|\mathbf{f}_h(\sqrt{a})_h + \partial_\rho\Big((n_h^{\rho\sigma}(\mathbf{u}_h)\mathbf{a}_\sigma^h + m_h^{\rho\sigma}(U_h)\mathbf{d}_\sigma^h + t_h^\rho(U_h)\mathbf{a}_3^h)(\sqrt{a})_h\Big)\|_{L^2(K)^3} \\ & + \sum_{e\in\mathcal{E}_K\backslash\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|\Big[\nu_\rho(n_h^{\rho\sigma}(\mathbf{u}_h)\mathbf{a}_\sigma^h + m_h^{\rho\sigma}(U_h)\mathbf{d}_\sigma^h + t_h^\rho(U_h)\mathbf{a}_3^h)(\sqrt{a})_h\Big]_e\|_{L^2(e)^3} \\ & + \sum_{e\in\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|\mathbf{N}_h \, \ell_h - \nu_\rho(n_h^{\rho\sigma}(\mathbf{u}_h)\mathbf{a}_\sigma^h + m_h^{\rho\sigma}(U_h)\mathbf{d}_\sigma^h + t_h^\rho(U_h)\mathbf{a}_3^h)(\sqrt{a})_h\|_{L^2(e)^3},\end{aligned}$$

$$\eta_{K2} = h_K \, \|\partial_\rho\Big(m_h^{\rho\sigma}(U_h)\mathbf{a}_\sigma^h(\sqrt{a})_h\Big) - t_h^\beta(U_h)\mathbf{a}_\beta^h\,(\sqrt{a})_h + \partial_\rho\Big(\mathbf{a}_3^h\partial_\rho\psi_h\Big) - \mathbf{d}_\rho^h\partial_\rho\psi_h\|_{L^2(K)^3}$$

$$+ \sum_{e\in\mathcal{E}_K\setminus\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|\Big[\nu_\rho m_h^{\rho\sigma}(U_h)\mathbf{a}_\sigma^h(\sqrt{a})_h \; + \nu_\rho\partial_\rho\psi_h\mathbf{a}_3^h\Big]_e\|_{L^2(e)^3}$$

$$+ \sum_{e\in\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|\mathbf{M}_h\,\ell_h - \nu_\rho m_h^{\rho\sigma}(U_h)\mathbf{a}_\sigma^h(\sqrt{a})_h \; - \nu_\rho\partial_\rho\psi_h\mathbf{a}_3^h\|_{L^2(e)^3},$$

and

$$\eta_{K3} = h_K \, \|\partial_\alpha(\partial_{\mathbf{a}}\mathbf{r}_h^p \cdot \mathbf{a}_{3h} + \mathbf{r}_h^p \cdot \mathbf{d}_{\mathbf{a}}^h - \varepsilon_p \, \partial_\alpha\psi_h^p)\|_{L^2(K)}$$

$$+ \sum_{e\in\mathcal{E}_K\setminus\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|[\partial_\nu(\mathbf{r}_h \cdot \mathbf{a}_3^h - \varepsilon_p \, \psi_h^p)]_e\|_{L^2(e)} + \sum_{e\in\mathcal{E}_K^1} h_e^{\frac{1}{2}} \, \|\partial_\nu(\mathbf{r}_h \cdot \mathbf{a}_3^h - \varepsilon_p \, \psi_h^p)\|_{L^2(e)}.$$

$\mathcal{E}_K$ **: set of edges of** K **not contained in** γ_0**,**
$\mathcal{E}_K^1$ **: set of edges of** K **contained in** γ_1**.**

A posteriori error estimates

Despite the slightly alarming aspect of the indicators, all a posteriori error estimates and upper bounds for these indicators are fully optimal,

up to the terms involving the data

$$\varepsilon_K^{(d)} = h_K \, \|\mathbf{f} - \mathbf{f}_h\|_{L^2(K)^3} + \sum_{e \in \mathcal{E}_K^1} h_e^{\frac{1}{2}} \left(\|\mathbf{N} - \mathbf{N}_h\|_{L^2(e)^3} + \|\mathbf{M} - \mathbf{M}_h\|_{L^2(e)^3} \right),$$

and the coefficients

$$\varepsilon_h^{(c)} = \|\sqrt{a} - (\sqrt{a})_h\|_{L^\infty(\omega)} + h^{\frac{1}{2}} \, \|\ell - \ell_h\|_{L^\infty(\gamma_1)} + \ldots$$

Key numerical difficulty :

- The main part of the solution $(\mathrm{u}_h, \mathrm{r}_h - (\mathrm{r}_h \cdot \mathrm{a}_3)\mathrm{a}_3)$ sems fully independent of ε.
- The error indicator η_ε is much smaller than the η_K.

So, no adaptivity procedure with respect to ε seems possible. We only choose an initial ε such that

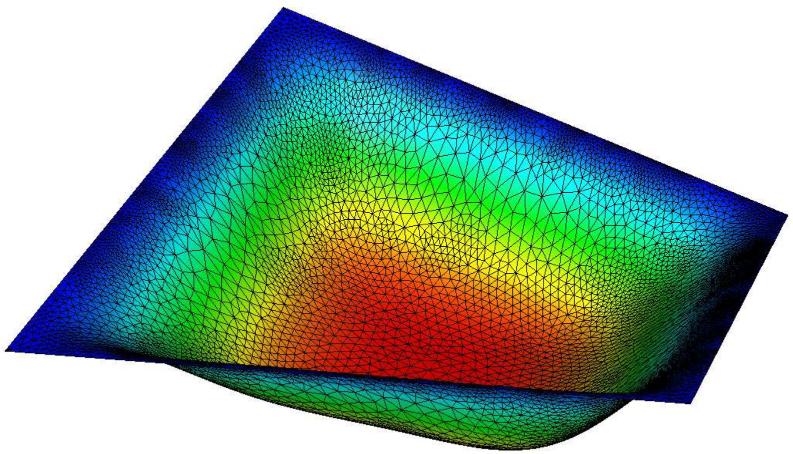
$$\eta_\varepsilon \leq 10^{-3} \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{\frac{1}{2}}.$$

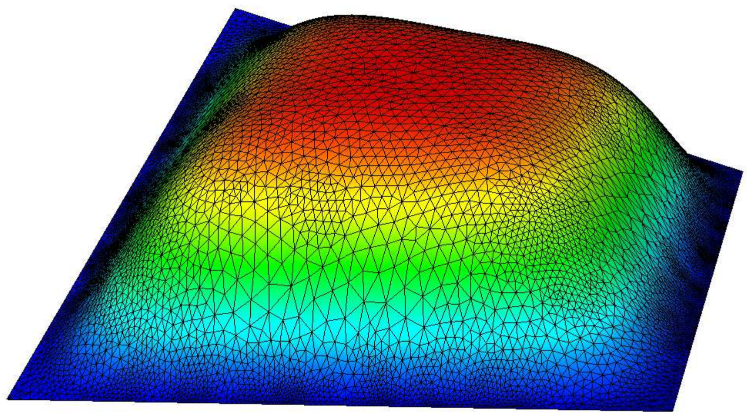
Case of a hyperbolic paraboloid shell clamped on the whole boundary and subjected to a uniform pressure.

$$\omega = \left\{(x,y); |x| + |y| \leq b\sqrt{2}\right\}, \qquad \boldsymbol{\varphi}(x,y) = \left(x, y, \frac{c}{2b^2}(x^2 - y^2)\right)^T,$$

with $b = 50\,\mathrm{cm}$**,** $c = 10\,\mathrm{cm}$**, and** $e = 0.8$ **cm.**

The "over-deformed" shell $\boldsymbol{\varphi}(\mathbf{x}) + 1000\,\mathbf{u}(\mathbf{x})$





Thank you for your attention