Equilibrated error estimator for contact problems

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Workshop on:
A posteriori estimates for adaptive mesh refinement and error control

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Outline

1. **Equilibration techniques for error control**
   - Equilibrated fluxes for Laplace operator
   - One-sided obstacle problem
   - Two-sided obstacle problem

2. **A posteriori error estimates for contact problems**
   - $H(\text{div})$-conforming aproximations for symmetric tensors
   - Local definition of the estimator
   - Efficiency and reliability

3. **AFEM strategy for one-body problems**
   - Modified error estimator
   - Edge residuals
   - (Energy based error decay)

Prager–Synge theorem (Laplace operator)

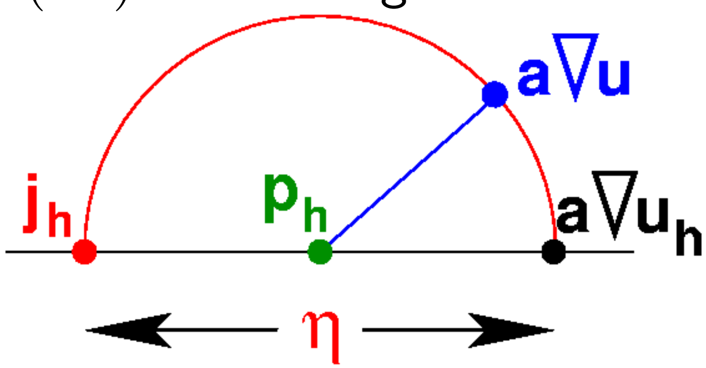
Prager, Synge: Approximations in elasticity based on the concept of function space. Quart. Appl. Math. 5, (1947). 241–269.

Let u_h be a conforming finite element solution then

$$\|\nabla u - \nabla u_h\|_0 \le \|\nabla u_h - \mathbf{j}\|_0 + C\|\mathrm{div}\mathbf{j} - f\|_{-1}$$

for all $H(\mathrm{div})$-conforming vector fields $\mathbf{j}$

Idea: Construct a suitable $H(\mathrm{div})$-conforming finite element approximation j_h



Remark: This result can also be regarded as a hypercycle method
$\Longrightarrow$ asymptotically exact for postprocessed solution $p_h := \frac{1}{2}(j_h + a\nabla u_h)$

How to construct suitable $H(\text{div})$-approximations?

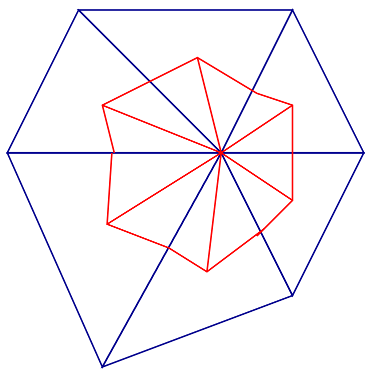
Idea: Use standard mixed finite elements, e.g., RT or BDM, such that

$$\text{div} j_h = P_h f,$$

where P_h is locally defined and reproduces constants

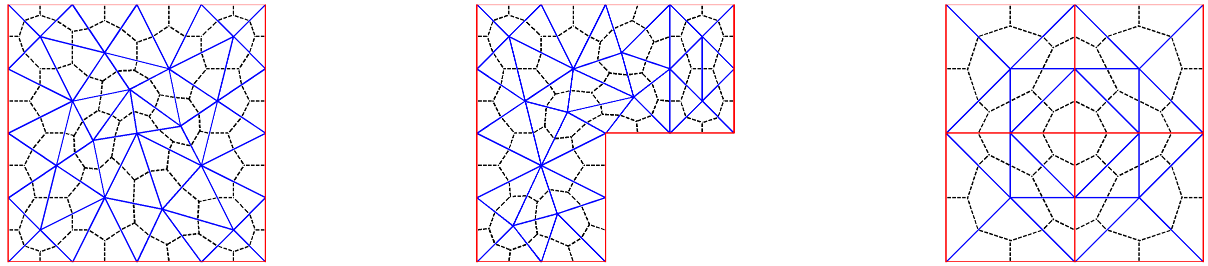
But: Solution of a global mixed finite element problem to **expensive**
Need to recover j_h **locally** from the conforming fe solution u_h

Possibility one: Define j_h on a dual finite volume mesh and use a macro-element based Raviart–Thomas space of lowest order (jww Robert Luce,04)



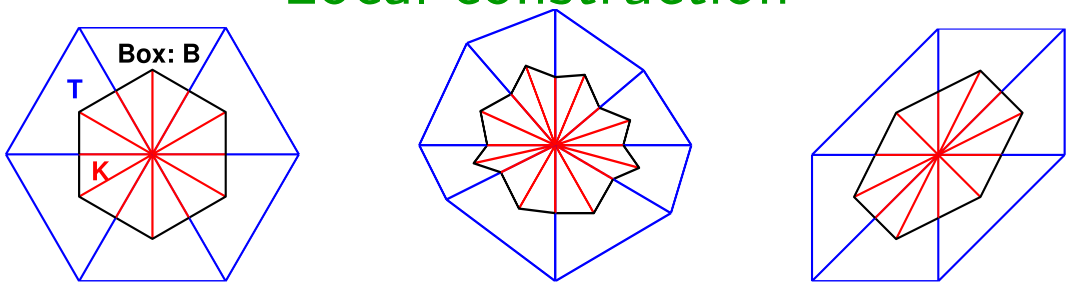
One macro-element associated with each vertex

Simplicial triangulation and finite volume boxes



$\mathcal{B}_h$: Finite volume boxes on Ω and $\mathcal{T}_h \prec \mathcal{K}_h$: Simplicial triangulations on Ω

Local construction

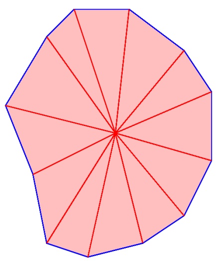


Definition of the Raviart-Thomas space S_h:

$$S_h := \{ j \in H(\text{div};\Omega); \quad j_{|_K} \in RT_0(K); K \in \mathcal{K}_h \;, \quad \text{div} j_{|_B} \in P_0(B); B \in \mathcal{B}_h \} \subset RT_h$$

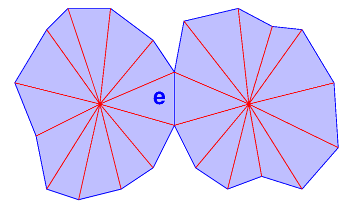
Local basis of S_h: $S_h = \sum_{\substack{e \in \mathcal{E}_h \\ e \subset \partial B}} \text{span}\ \{w_e\} \oplus \sum_{\substack{B \in \mathcal{B}_h \\ \text{meas}(\partial\Omega \cap \partial B) \neq 0}} \text{span}\ \{w_B\}$

Flux approximation in S_h



Definition of w_B

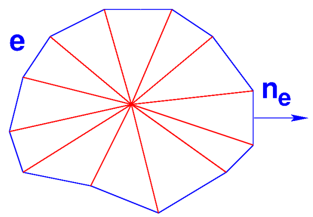
$$w_B := \beta_B \text{curl } \phi_B,$$
$$\beta_B^{-2} := (\text{curl } \phi_B, \text{curl}\phi_B)_0$$



Definition of w_e

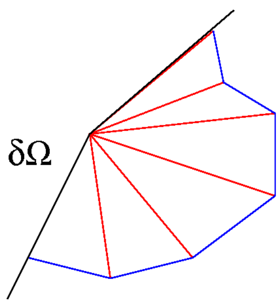
$$w_e n_{\hat{e}|_{\hat{e}}} := \frac{1}{h_e}\delta_{e\hat{e}},$$
$$(w_e, w_B)_0 = 0$$

Case I: B is interior box



$$\begin{aligned} j_h &:= \sum_{e\in\mathcal{E}_B} \alpha_e w_e + \alpha_B w_B \\ \alpha_e &:= \int_e a\nabla u_h n_e \, d\sigma \\ \alpha_B &:= \int_B a\nabla u_h \, w_B \, dx \end{aligned}$$

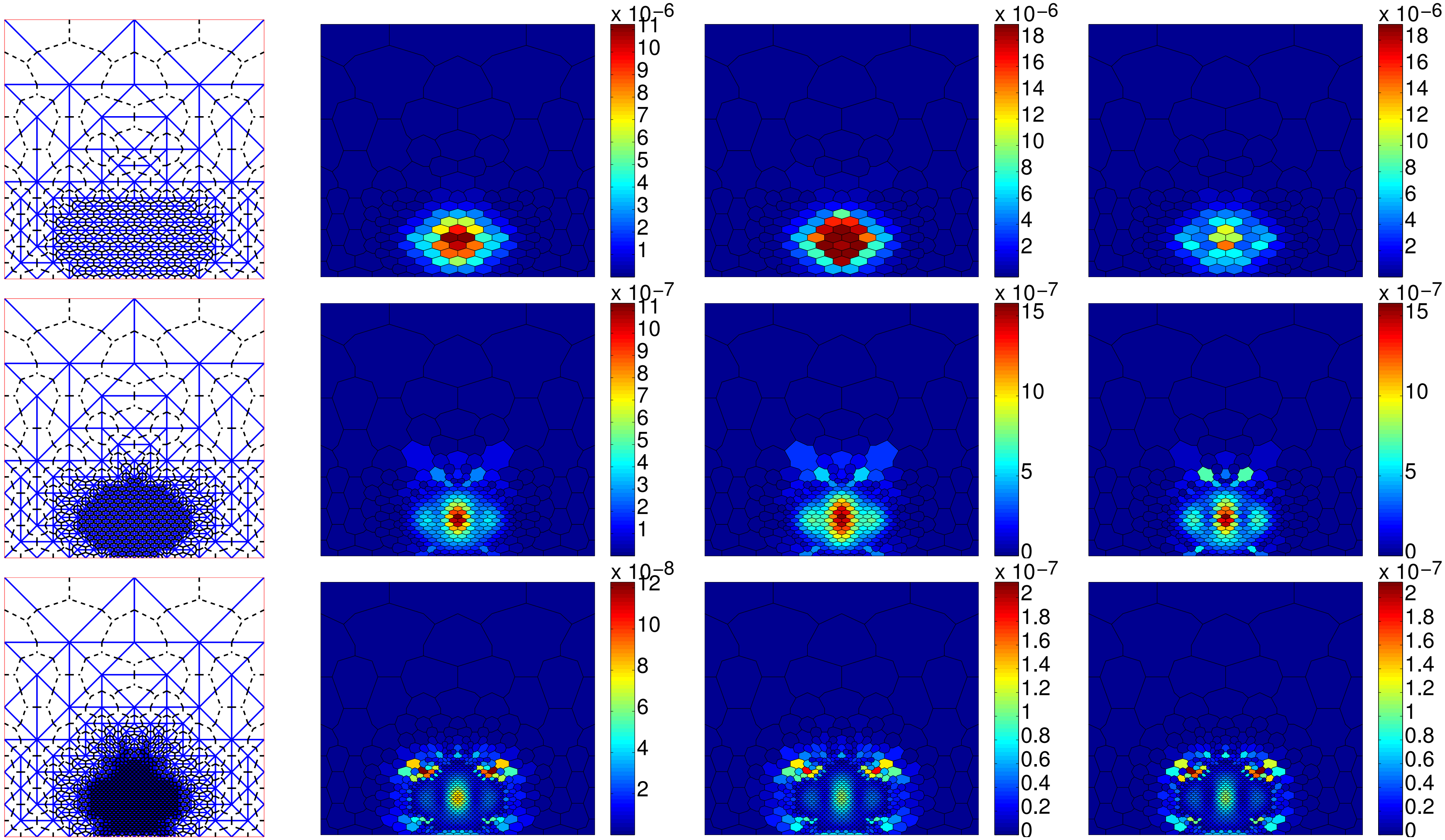
Case II: B is boundary box



$$\begin{aligned} j_h &:= \sum_{\substack{e\in\mathcal{E}_B \\ e\subset\Omega}} \alpha_e w_e + \sum_{\substack{e\in\mathcal{E}_B \\ e\subset\partial\Omega}} \alpha_e w_e \\ \alpha_e &:= \int_e a\nabla u_h n_e \, d\sigma \quad e \text{ in } \Omega \\ \alpha_e &:= \alpha_e + \frac{1}{2}\left(\int_\Omega -f\phi_B \, dx - \int_{\partial B} a\nabla u_h n \, d\sigma\right) \end{aligned}$$

Lemma: $\text{div} j_{h|_B} = \frac{1}{|B|}\int_\Omega f\phi_B \, dx =: P_Q f_{|B}$

Local error contributions (Laplace operator)

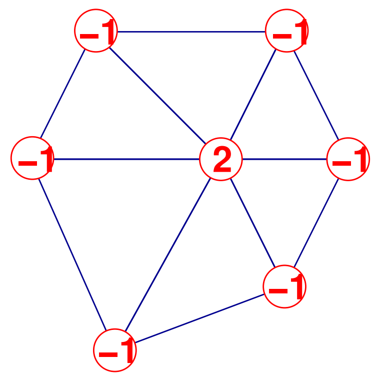


Triangulation, error in u_h, estimator and error for postprocessed solution (same scale)

Characteristic properties of the construction

- Easy and simply construction for low order elements (✓)
- Decoupling of the global problem by “inner” boundaries (✓)
- Generalization to high order elements not straightforward (✗)
- Generalization to symmetric tensors not straightforward (✗)

Observation: Each edge of $\mathcal{T}_h$ is decomposed into two subedges with constant flux
$\Longrightarrow$ this motivates alternative approach in terms of **equilibrated fluxes**



Linear equilibrated fluxes per edge are decoupled by biorthogonality

Obstacle problem

- **Discrete primal formulation:** Find $u_h \in \mathcal{K}_h$ such that

$$a(u_h, v - u_h) \leq f(v - u_h), \quad v \in \mathcal{K}_h,$$

where $\mathcal{K}_h$ is the discrete set of admissible elements, i.e.,

$$\mathcal{K}_h := \{v \in X_h, \int_\Omega v\mu_p \, dx \geq \int_\Omega \psi\mu_p \, dx\}$$

and $\{\mu_p\}_p$ forms a set of biorthogonal basis functions wrt $\{\phi_p\}_p$

- **Discrete hybrid formulation:** $(u_h, \lambda_h) \in (X_h, M_h^+)$, $M_h^+ := \{\sum_p \alpha_p\mu_p, \alpha_p \geq 0\}$

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v_h) &= f(v_h), & v_h \in X_h, \\ b(\mu_h - \lambda_h, u_h) &\leq \langle \psi, \mu_h - \lambda_h \rangle, & \mu_h \in M_h^+. \end{aligned}$$

$a(\cdot,\cdot)$ bilinear form, $b(\cdot,\cdot) := \langle \cdot,\cdot \rangle$ duality pairing between H^{-1} and H_0^1

λ_h can be seen as an additional source term for the a posteriori analysis

Sinus-shaped obstacle

Problem Setting:

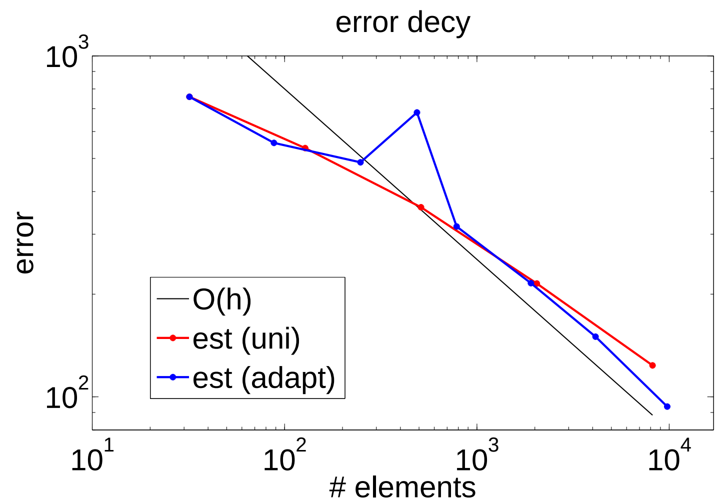
Obstacle:

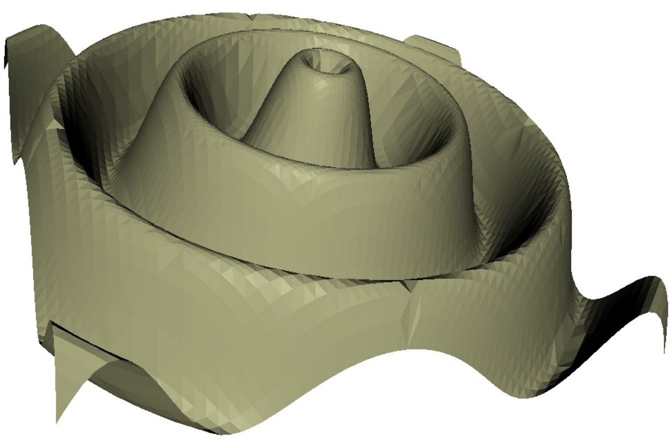
$$\psi = 3\|x-(0.5,0.5)\| - \sin(10\pi\|x-(0.5,0.5)\|)$$

Rhs:

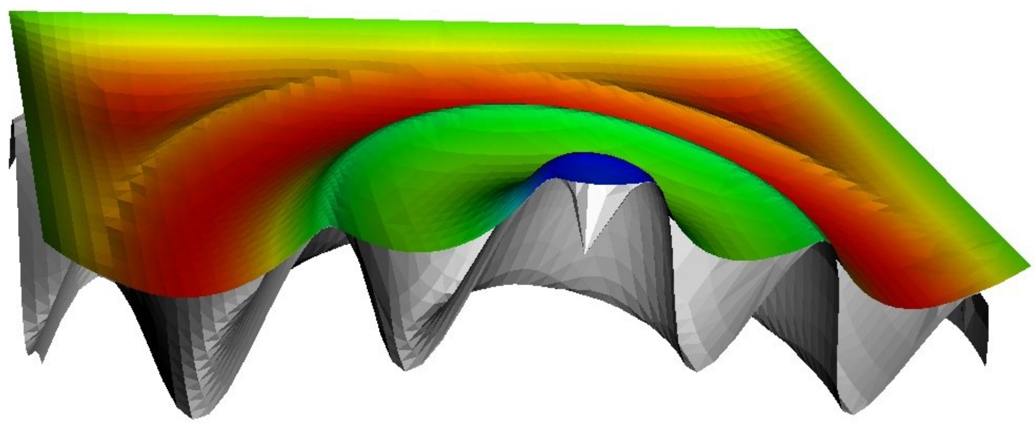
$$f = 0$$

Zero Dirichlet boundary conditions



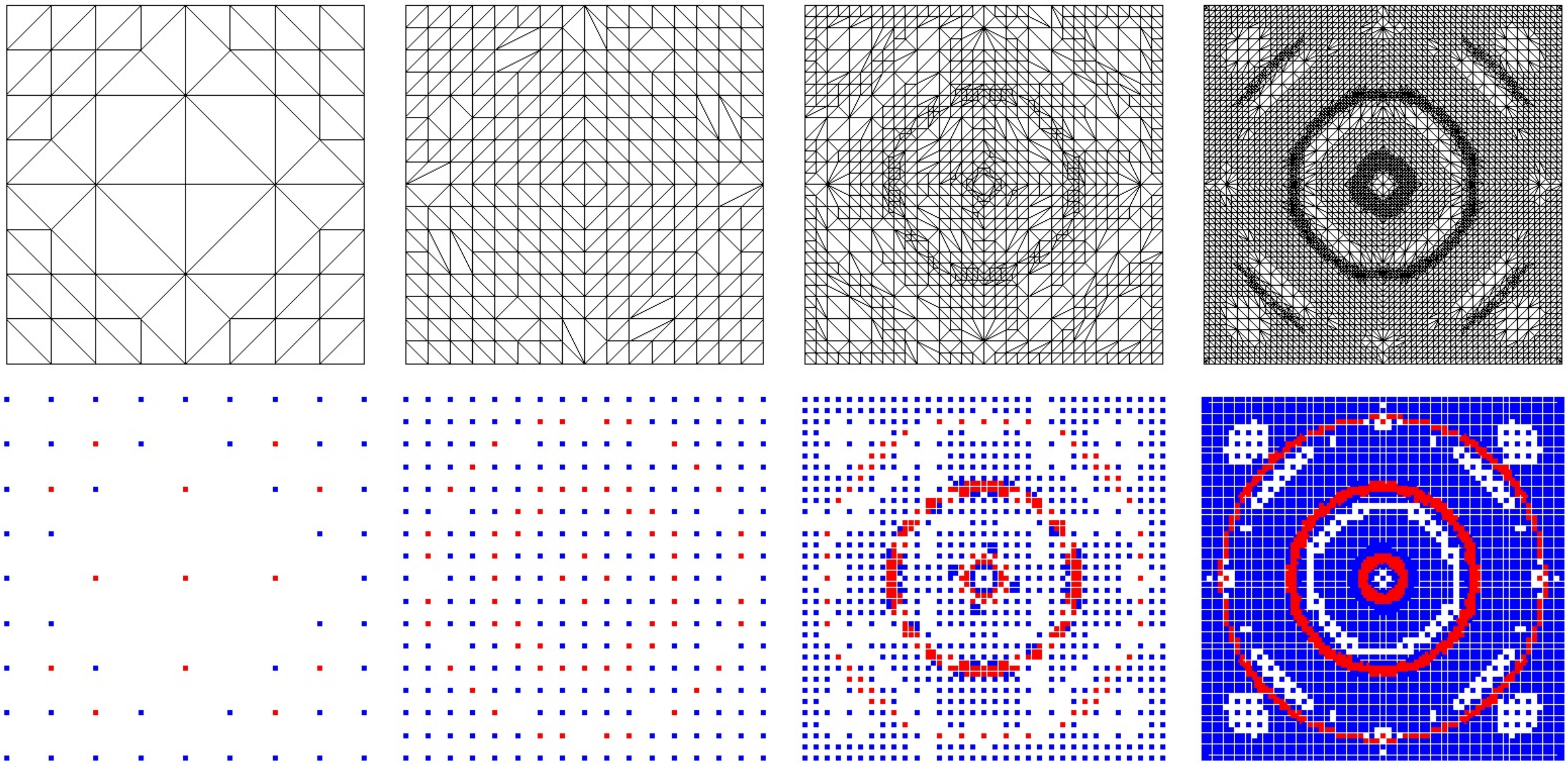


Obstacle



Solution of contact problem (cut)

Grid and active set on different refinement levels



Level 1 Level 3 Level 5 Level 7

Non-smooth obstacle

Problem Setting:

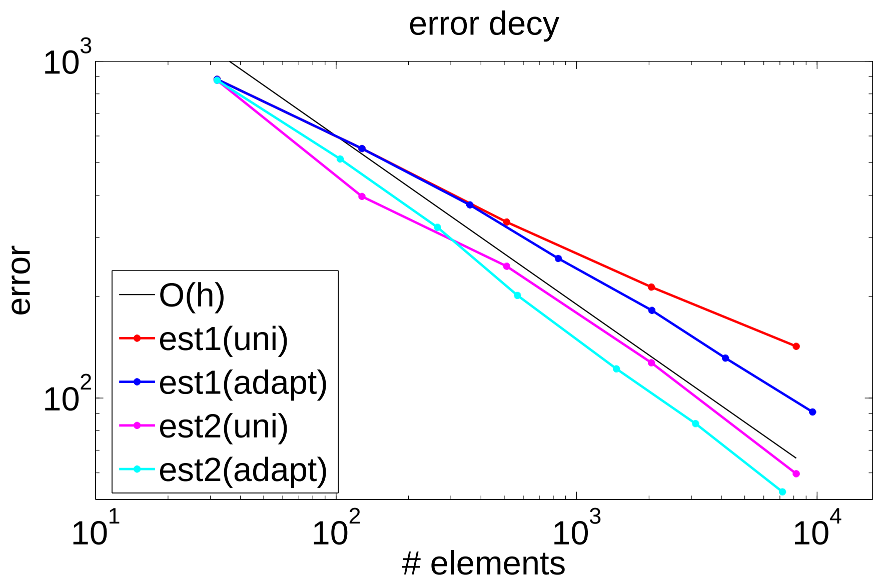
Obstacle:

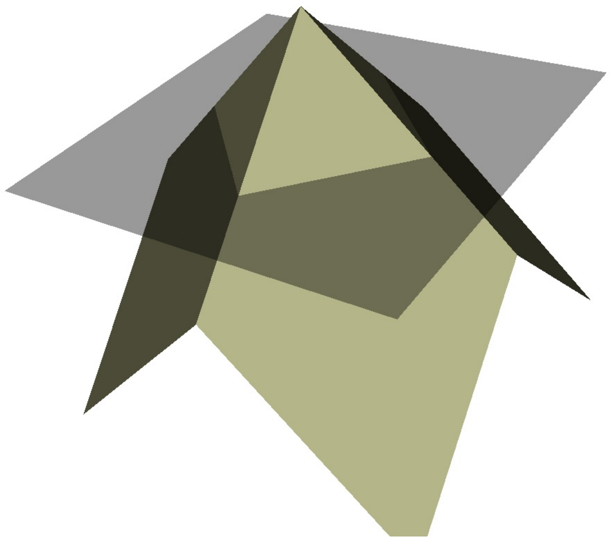
$\psi = \|x - (0.5, 0.5)\|_1 - 0.3$

Rhs:

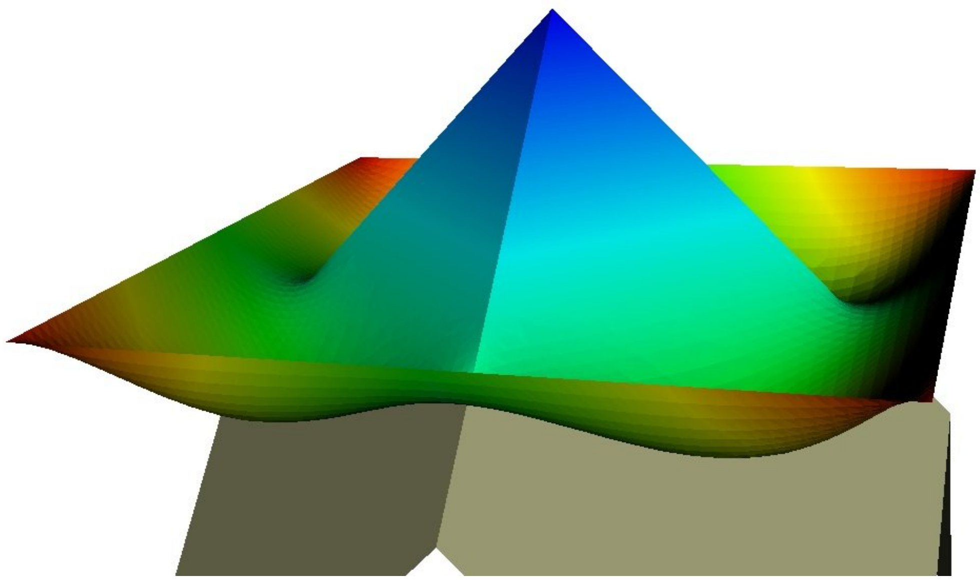
$f = 0$

Zero Dirichlet boundary conditions

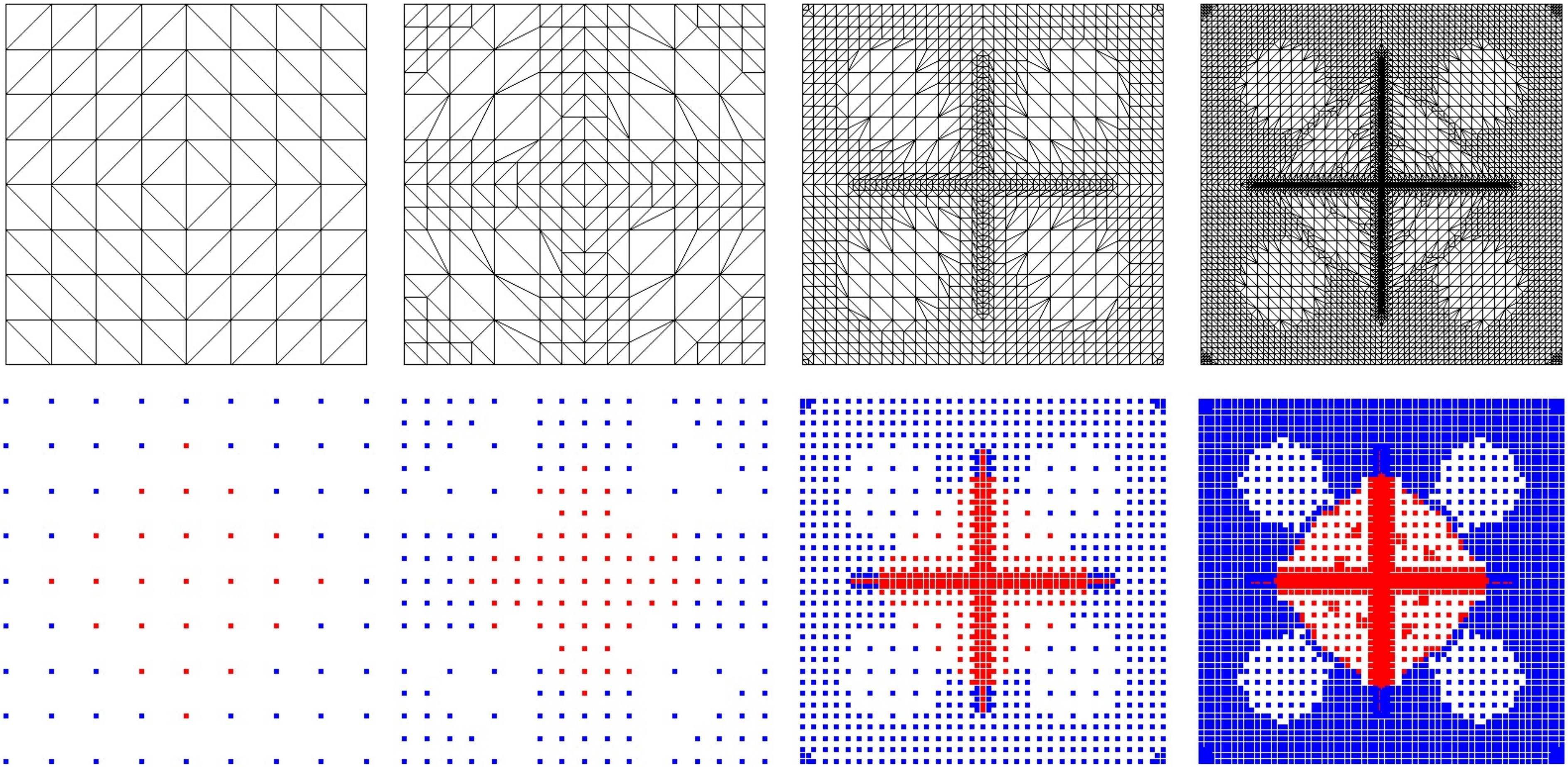




Obstacle



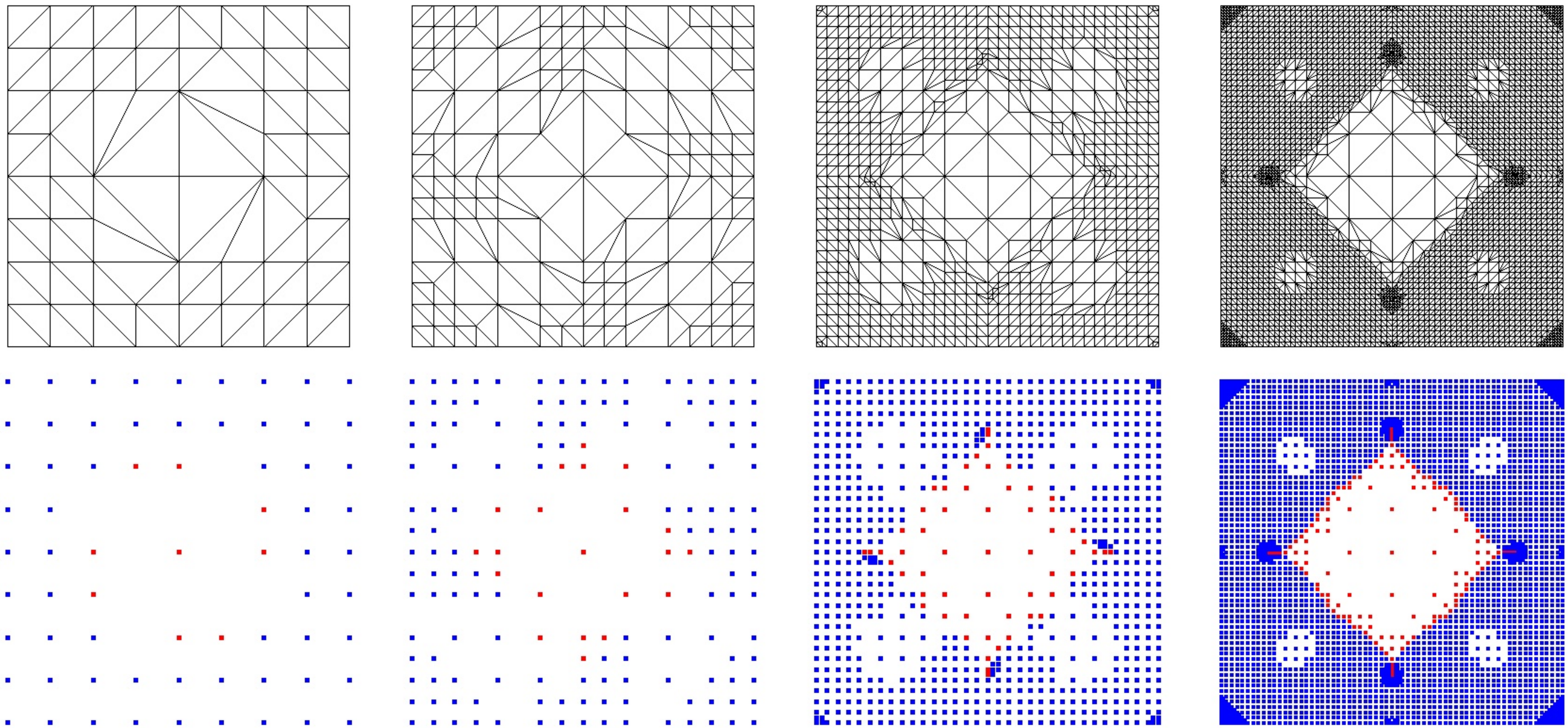
Adaptive meshes and active sets



Level 1 Level 2 Level 4 Level 6

Solution is not in $H(\mathrm{div}) \Longrightarrow$ Overestimation and no correct asymptotic

Adaptive meshes and active sets



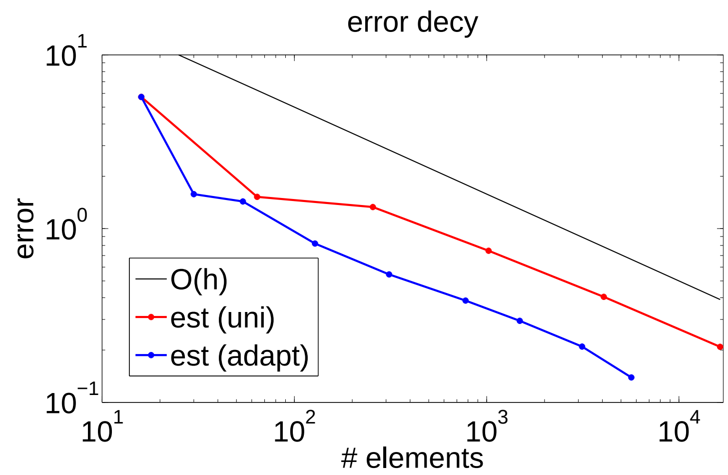
Level 1 Level 2 Level 4 Level 6

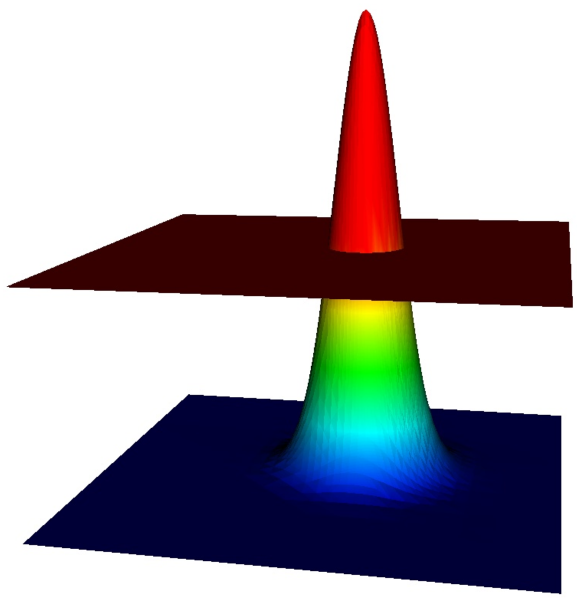
Regularity of j_h has to be weakened $\Longrightarrow$ $[j_h n]$ not necessarily non-zero

Obstacle problem between two membranes

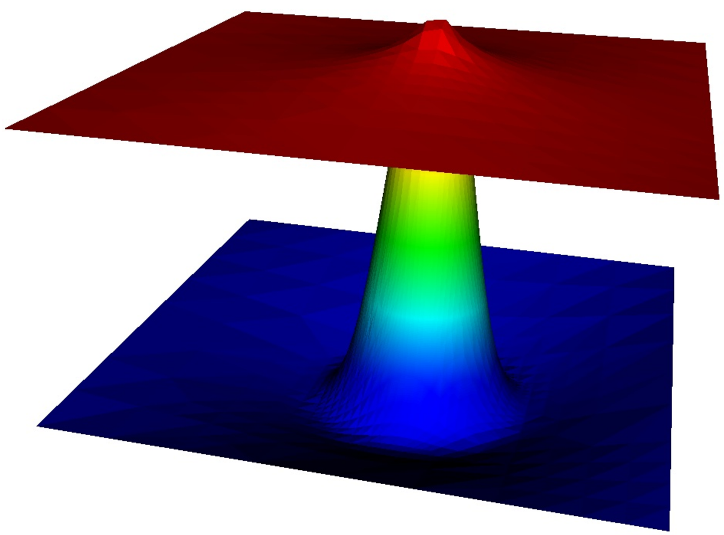
Problem Setting (unconstrained):

$u_m = 0.5$
$u_s = e^{-100\|x-(0.45,0.57)\|}$
$K_m = 3Id, K_s = Id$
Dirichlet boundary conditions



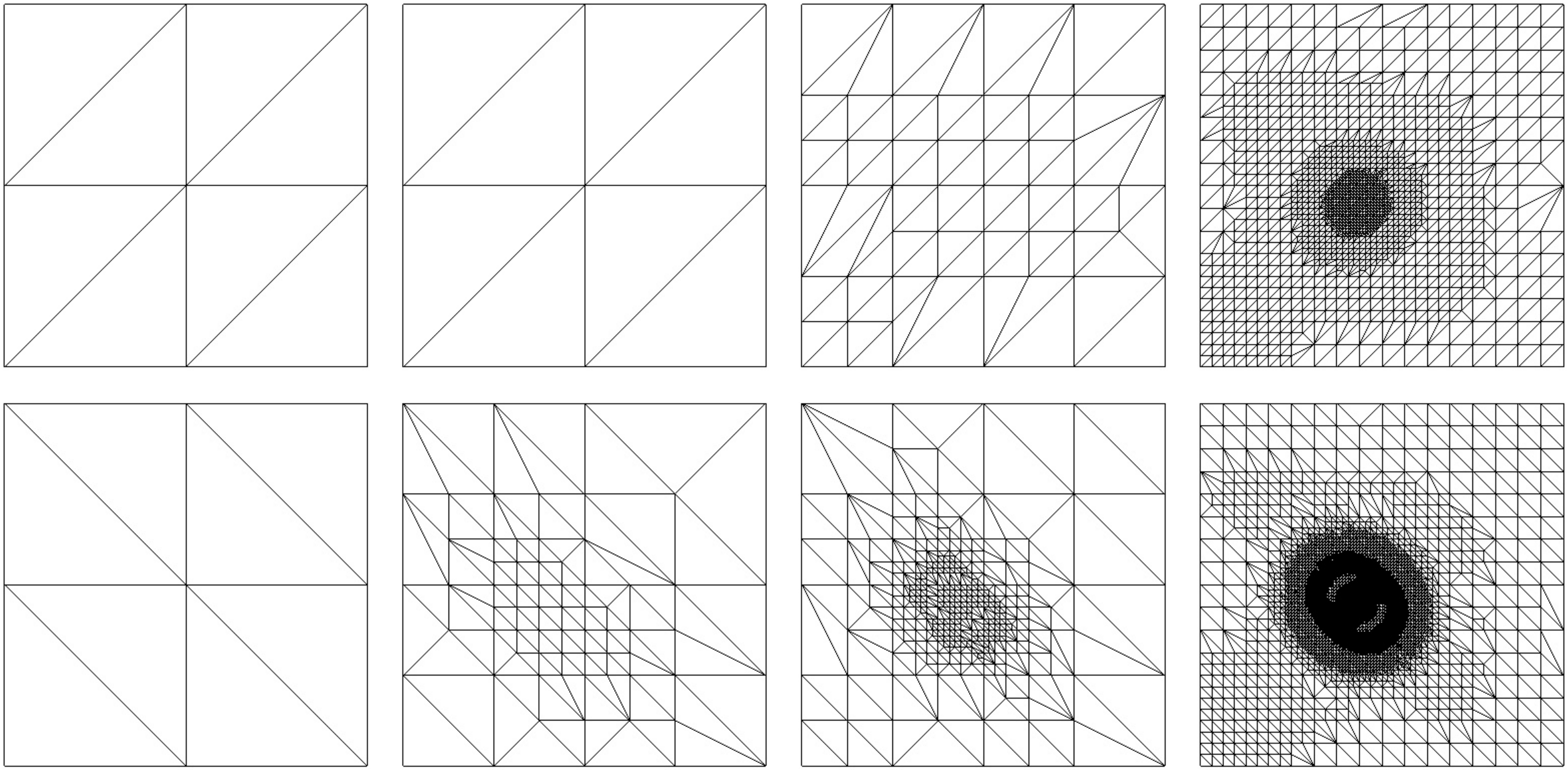


Solution without restriction



Solution of contact problem

Non-matching adaptive meshes



Level 0 Level 3 Level 5 Level 8

Obstacle problem between two membranes

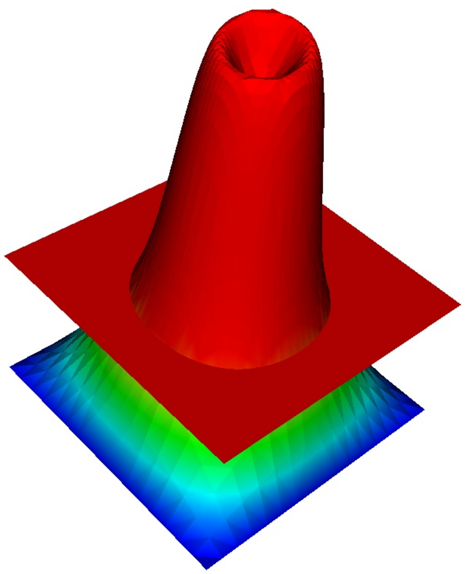
Problem Setting (unconstrained):

$u_m = 0.5$

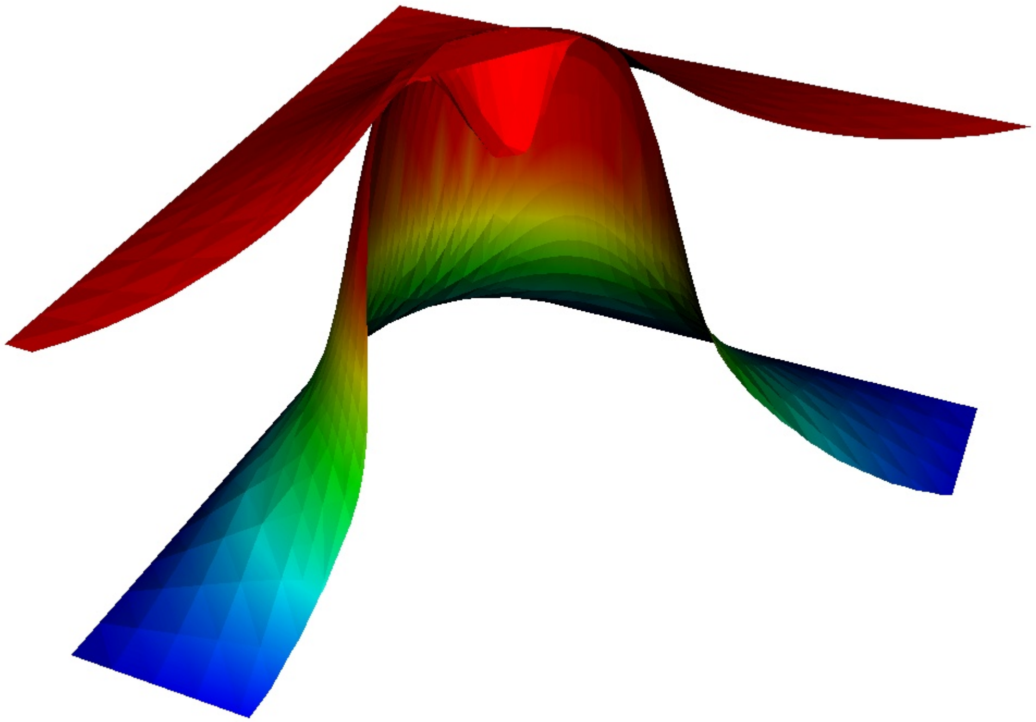
$u_s = e^{-1000(\|x-(0.45,0.57)\|^2-0.1^2)^2}$

$K_m = 3Id, K_s = Id$

Dirichlet boundary conditions

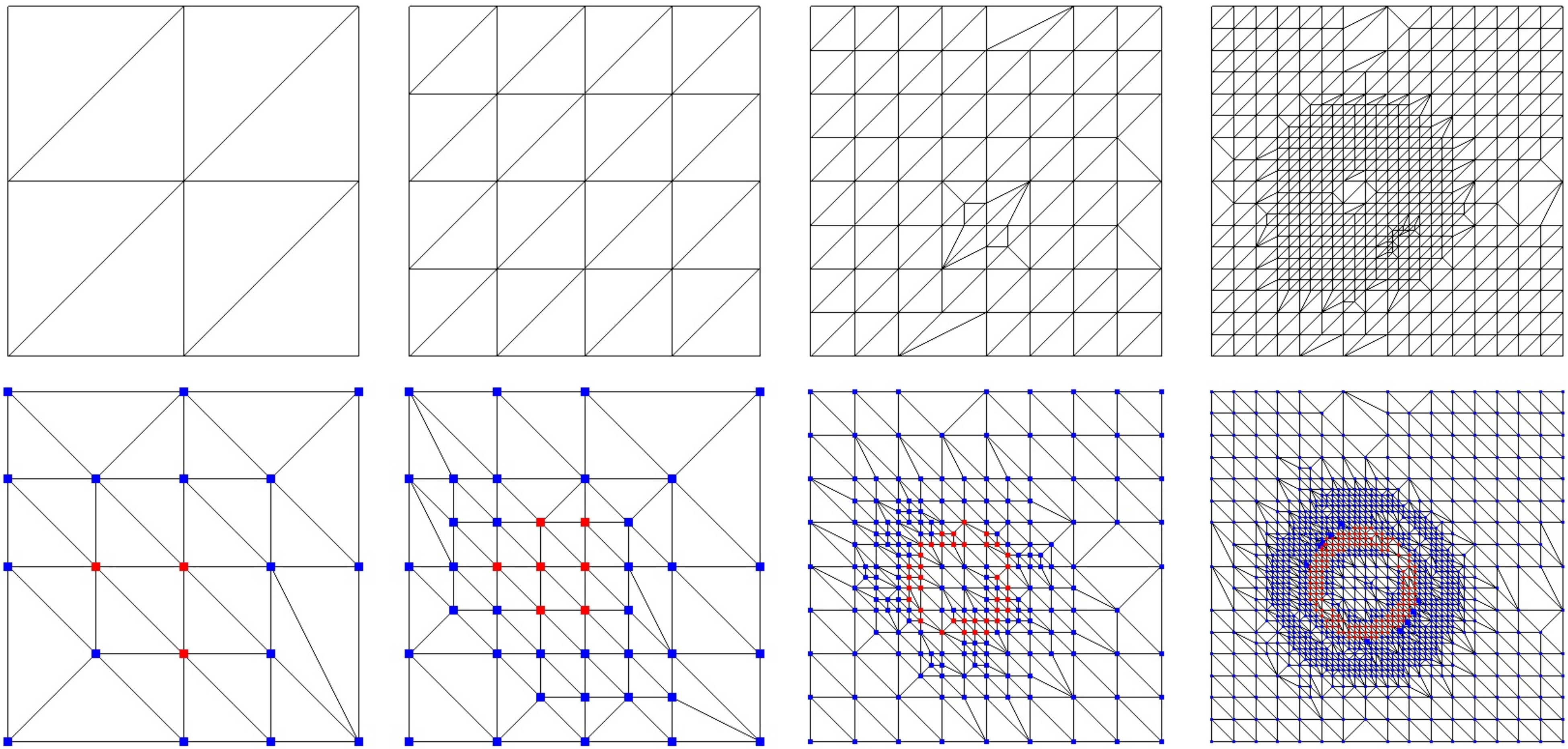


Solution without restriction



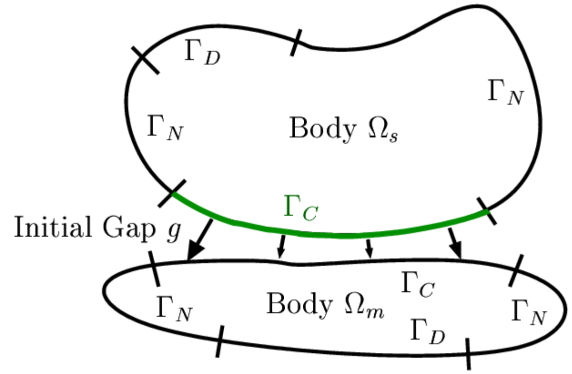
Solution of contact problem (cut)

Non-matching meshes and active sets



Level 1 Level 2 Level 4 Level 6

Contact problem with Coulomb friction



Linear Elasticity:

$$-\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{p} \quad \text{on } \Gamma_N$$

Non-penetration:

$$[\mathbf{u}]_n - g \leq 0,$$

$$\sigma_{\mathbf{n}} := \sigma_{\mathbf{n}}(\mathbf{u}_m) = \sigma_{\mathbf{n}}(\mathbf{u}_s) \leq 0,$$

$$\sigma_{\mathbf{n}}([\mathbf{u}]_n - g) = 0$$

jump: $[\mathbf{u}] := (\mathbf{u}_s - P^s_m \mathbf{u}_m)$

$$[\mathbf{u}]_n := [\mathbf{u}] \cdot \mathbf{n}, \quad [\mathbf{u}]_{\mathbf{t}} := [\mathbf{u}] - [\mathbf{u}]_n \mathbf{n}$$

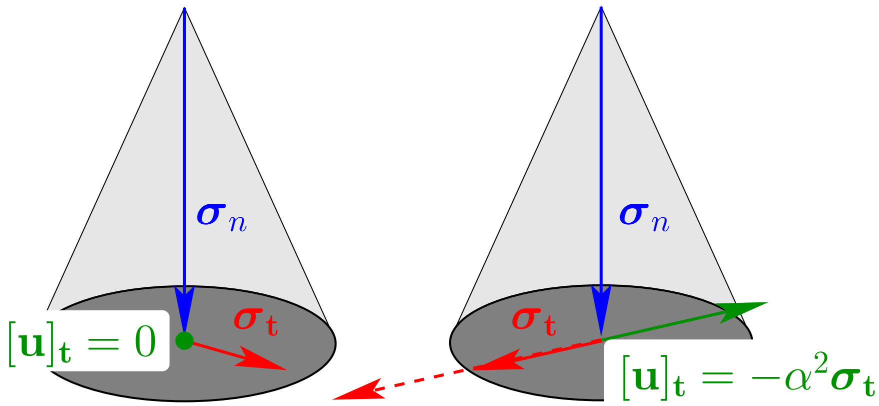
stress: $\sigma_{\mathbf{n}} := \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}, \quad \boldsymbol{\sigma}_{\mathbf{t}} := \boldsymbol{\sigma}\mathbf{n} - \sigma_n \mathbf{n}$

Coulomb friction:

$$|\boldsymbol{\sigma}_{\mathbf{t}}| - \mathfrak{F}|\sigma_{\mathbf{n}}| \leq 0,$$

$$[\mathbf{u}]_{\mathbf{t}} + \alpha^2 \boldsymbol{\sigma}_{\mathbf{t}} = 0,$$

$$[\mathbf{u}]_{\mathbf{t}}(|\boldsymbol{\sigma}_{\mathbf{t}}| - \mathfrak{F}|\sigma_{\mathbf{n}}|) = 0$$



$\Longrightarrow$ Discretization in terms of mortar finite elements and dual Lagrange multipliers

Discretization on non-matching meshes

- Constraints are **weakly** satisfied in terms of Lagrange multipliers:
  Displacement $\mathbf{u}$: primal variable
  Contact stress $\boldsymbol{\lambda} := -\boldsymbol{\sigma}\mathbf{n}$: dual variable

- **Discrete hybrid formulation:** $(\mathbf{u}_h, \boldsymbol{\lambda}_h) \in (X_h, M_h(\boldsymbol{\lambda}_h))$

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda}_h, \mathbf{v}_h) &= f(\mathbf{v}_h), & \mathbf{v}_h \in X_h, \\ b(\boldsymbol{\mu}_h - \boldsymbol{\lambda}_h, \mathbf{u}_h) &\leq \langle g, (\boldsymbol{\mu}_h - \boldsymbol{\lambda}_h)_n \rangle, & \boldsymbol{\mu}_h \in M_h(\boldsymbol{\lambda}_h). \end{aligned}$$

  $a(\mathbf{u}_h, \cdot)$ elasticity linear form, $b(\cdot,\cdot) := \langle [\cdot], \cdot \rangle$ contact bilinear form
  (X_h, M_h) stable pair of mortar finite elements, W_h trace space of X_h
  $M_h(\boldsymbol{\lambda}_h) := \{\boldsymbol{\mu} \in M_h; \langle \mathbf{v}, \boldsymbol{\mu} \rangle \leq \langle |\mathbf{v}_\mathbf{t}|_h, \mathfrak{F}(\boldsymbol{\lambda}_h)_n \rangle, \ \mathbf{v} \in W_h, \mathbf{v}_n \leq 0\}$
  Local static elimination of $\boldsymbol{\lambda}_h$ possible due to biorthogonality

- **Coulomb friction:** **quasi** variational inequality
  No friction ($\mathfrak{F} = 0$)/Tresca friction ($|(\boldsymbol{\lambda}_h)_n|_h \rightarrow g_f$): variational inequality

A posteriori error estimator for contact

Observation: Discrete displacement satisfies a variational equality for given Lagrange multiplier $\boldsymbol{\lambda}_h$

Idea: Find a $H(\text{div})$-conforming approximation $\boldsymbol{\sigma}_h$ for the stress such that

- the divergence satisfies

$$\textbf{(CD)} \qquad \text{div}\boldsymbol{\sigma}_h = -\boldsymbol{\Pi}_1 \mathbf{f},$$

  where $\boldsymbol{\Pi}_1$ is the L^2-projection onto piecewise affine functions.

- the surface traction satisfies

$$\textbf{(CS)} \qquad (\boldsymbol{\sigma}_h \mathbf{n}^l)_{|_{\Gamma_N^l}} = \mathbf{0}, \quad \text{and} \quad (\boldsymbol{\sigma}_h \mathbf{n}^l)_{|_{\Gamma_C^l}} = -\mathbf{n}^l \cdot \mathbf{n}^s \boldsymbol{\Pi}_l^* \boldsymbol{\lambda}_h, \quad l \in \{m, s\}$$

  where $\boldsymbol{\Pi}_l^*$ is the dual mortar projection onto the Lagrange multiplier space.

Definition of the error estimator

$$\eta^2 := \sum_T \eta_T^2 , \qquad \eta_T^2 := \|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}(\mathbf{u}_h))\|_{0;T}^2 .$$

Remark: The error estimator is elementwise defined.

How to obtain a suitable σ_h?

Idea: A posteriori error estimator based on equilibrated fluxes [Ainsworth-Oden 99, Ladeveze/Leguillon 83, Stein et al 97-01]

Let $\mathbf{u}_h$ be the mortar finite element solution of the variational inequality, i.e.,

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{0;\Omega_i} - b(\boldsymbol{\lambda}_h, \mathbf{v}_h), \quad \mathbf{v}_h \in X_h^i, \quad i \in \{s, m\}$$

$\Longrightarrow$ $\boldsymbol{\lambda}_h$ plays role of Neumann boundary condition

Equilibrated fluxes

Then there exists a $\mathbf{g}_e \in [P_1(e)]^2$ such that $\mathbf{g}_e = -\mathbf{n}^l \cdot \mathbf{n}^s \boldsymbol{\Pi}_1 \boldsymbol{\Pi}_l^* \boldsymbol{\lambda}_h$ on Γ_C^l

$$\int_{\partial T} (\mathbf{n}_T \cdot \mathbf{n}_e)\mathbf{g}_e \cdot \mathbf{v}\, ds = \Delta_T(\mathbf{v}) := a_T(\mathbf{u}_h, \mathbf{v}) + b_T(\boldsymbol{\lambda}_h, \mathbf{v}) - (\mathbf{f}, \mathbf{v})_{0;T}, \quad \mathbf{v} \in [P_1(T)]^2$$

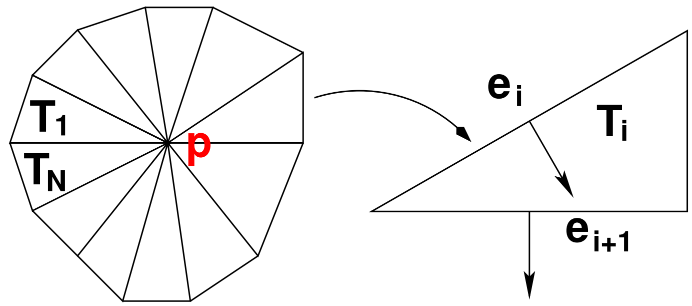
Moreover, $\mathbf{g}_e$ can be locally computed by rewriting

$$\mathbf{g}_e = \boldsymbol{\mu}_{e,p_1}\psi_{p_1} + \boldsymbol{\mu}_{e,p_2}\psi_{p_2},$$

where $\int_e \psi_{p_j}\phi_{p_i}\, ds = \delta_{ij}$. Then the moments $\boldsymbol{\mu}_{e,p_i}$ are given by $\boldsymbol{\mu}_{e,p_i} := \int_e \mathbf{g}_e \phi_{p_i}\, ds$.

Local postprocess to compute the moments

For each vertex p a local system has to be solved for the moments



Singular system (interior vertex), but compatible rhs: $\sum_i \Delta_{T_i}(\phi_p) = \mathbf{0}$

$$\begin{pmatrix} -\mathrm{Id} & \mathrm{Id} & & & \\ & \ddots & \ddots & & \\ & & -\mathrm{Id} & \mathrm{Id} & \\ \mathrm{Id} & & & -\mathrm{Id} & \end{pmatrix} \begin{pmatrix} \boldsymbol{\mu}_{e_1,p} \\ \boldsymbol{\mu}_{e_2,p} \\ \vdots \\ \boldsymbol{\mu}_{e_N,p} \end{pmatrix} = \begin{pmatrix} \Delta_{T_1}(\phi_p) \\ \Delta_{T_2}(\phi_p) \\ \vdots \\ \Delta_{T_N}(\phi_p) \end{pmatrix}$$

Set e.g. $\boldsymbol{\mu}_{e_1,p} = \mathbf{0} \Longrightarrow$ **lower tridiagonal matrix**

$\mathbf{g}_e$ is conservative, i.e., $\int_{\partial T}(\mathbf{n}_T \cdot \mathbf{n}_e)\mathbf{g}_e \, ds = \int_T \mathbf{f} \, dx$

Alternative choice

Better approximation of the flux:

$$\min_{\mathbf{g}_e} \sum_e h_e \|\mathbf{g}_e - \{\boldsymbol{\sigma}(\mathbf{u}_h)\}\mathbf{n}_e\|^2_{0;e}$$

$\Longrightarrow$ **minimization problem** (Lagrange multipliers on elements)

New local system (interior vertex):

$$\begin{pmatrix} \mathrm{Id} & -\frac{1}{2}\mathrm{Id} & 0 & \cdots & 0 & -\frac{1}{2}\mathrm{Id} \\ -\frac{1}{2}\mathrm{Id} & \mathrm{Id} & -\frac{1}{2}\mathrm{Id} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\frac{1}{2}\mathrm{Id} & \mathrm{Id} & -\frac{1}{2}\mathrm{Id} \\ -\frac{1}{2}\mathrm{Id} & 0 & \cdots & 0 & -\frac{1}{2}\mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\mu}}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \tilde{\boldsymbol{\mu}}_N \end{pmatrix} = \begin{pmatrix} \widetilde{\Delta_1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \widetilde{\Delta_N} \end{pmatrix}$$

Similar systems for boundary nodes depending on boundary cond. (D-D, D-N, N-N)

$\mathbf{g}_e$ is uniquely defined [Ainsworth-Oden, Ladeveze, Stein et al]

Arnold–Winther [02] elements

The Arnold–Winther element is locally defined on each T by the 24-dimensional space

$$X_T := \left\{\boldsymbol{\tau}_h \in [P_3(T)]^{2\times 2},\ (\boldsymbol{\tau}_h)_{12} = (\boldsymbol{\tau}_h)_{21},\ \operatorname{div}\boldsymbol{\tau}_h \in [P_1(T)]^2\right\} ,$$

and a global finite element space $X_h := X_s \times X_m$ which is on each body $H(\text{div})$-conforming can be obtained using

- the **nodal values** (3 dof) at each node p,
- the **zero** and **first order moments** of $\boldsymbol{\tau}_h n_e$ (4 dof) on each edge e,
- the **mean value** (3 dof) on each element T

as degrees of freedom on each of the two bodies.

Norm equivalence

$$\|\boldsymbol{\tau}\|_0^2 \equiv \underbrace{\sum_p |T|\|\boldsymbol{\tau}(p)\|^2}_{=:m_p(\boldsymbol{\tau})} + \underbrace{\sum_e \|\int_e \boldsymbol{\tau}\mathbf{n}_e\,ds\|^2 + \|\int_e \boldsymbol{\tau}\mathbf{n}_e\phi_e\,ds\|^2}_{=:m_e(\boldsymbol{\tau})} + \underbrace{\frac{1}{|T|}\|\int_T \boldsymbol{\tau}\,dx\|^2}_{=:m_i(\boldsymbol{\tau})}$$

Definition of $\boldsymbol{\sigma}_h$

$$\boldsymbol{\sigma}_h(p) := \frac{1}{N_T^p} \sum_{T \in \mathcal{T}_p} \boldsymbol{\sigma}(\mathbf{u}_h)_{|T}(p) + \boldsymbol{\alpha}(p), \qquad (1)$$

$$\int_e \boldsymbol{\sigma}_h \mathbf{n}_e \cdot q ds := \int_e \mathbf{g}_e \cdot \mathbf{q} ds, \qquad \mathbf{q} \in [P_1(e)]^2 , \qquad (2)$$

$$\int_T \boldsymbol{\sigma}_h : \nabla \mathbf{v} ds := a_T(\mathbf{u}_h, \mathbf{v}), \qquad \mathbf{v} \in [P_1(T)]^2 . \qquad (3)$$

$\boldsymbol{\alpha}(p)$ depends on the type of the node, e.g. $\boldsymbol{\alpha}(p) = \mathbf{0}$ if p is an interior node

Lemma:

i) Let $\boldsymbol{\sigma}_h \in X_h$ be defined such that (2) and (3) hold. Then,

$$\text{div}\boldsymbol{\sigma}_h = -\boldsymbol{\Pi}_1 \mathbf{f}.$$

ii) Let $\boldsymbol{\sigma}_h \in X_h$ be defined such that (1) and (2) hold. Then,

$$(\boldsymbol{\sigma}_h \mathbf{n}^l)_{|\Gamma_N^l} = 0, \quad \text{and} \quad (\boldsymbol{\sigma}_h \mathbf{n}^l)_{|\Gamma_C^l} = -\mathbf{n}^l \cdot \mathbf{n}^s \boldsymbol{\Pi}_l^* \boldsymbol{\lambda}_h.$$

Reliability of the error estimator

Theorem: Under suitable regularity assumptions the error estimator η yields a global upper bound for the discretization error

$$a(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h)^{\frac{1}{2}} \leq \eta + \mathcal{O}(h^{\frac{3}{2}})$$

The definition of the error estimator yields

$$\|\mathbf{u}-\mathbf{u}_h\|_a^2 \leq \eta\|\mathbf{u}-\mathbf{u}_h\|_a + \underbrace{\sum_{l\in\{m,s\}}\int_{\Omega^l}(\boldsymbol{\sigma}(\mathbf{u})-\boldsymbol{\sigma}_h)) : (\boldsymbol{\epsilon}(\mathbf{u})-\boldsymbol{\epsilon}(\mathbf{u}_h))\,dx}_{=:I}$$

To bound I one has to exploit:

- the properties (CD) and (CS) of $\boldsymbol{\sigma}_h$
- the a priori results for $b(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\lambda}_h-\boldsymbol{\lambda})$
- the approximation property of $\mathbf{\Pi}_l^*$, i.e., $\int_{\Gamma_C^m}(\boldsymbol{\lambda}_h-\mathbf{\Pi}_m^*\boldsymbol{\lambda}_h)\cdot(\mathbf{u}^m-\mathbf{u}_h^m)ds$

Remark: There is no constant in the upper bound
No additional terms enter due to the contact

Efficiency of the error estimator

Theorem: Under suitable regularity assumptions the error estimator η_T yields a local lower bound for the discretization error

$$\eta_T \;\leq\; C a_{\omega_T}(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h)^{\frac{1}{2}} + \mathcal{O}(h^{\frac{3}{2}})$$

Proof is based on the norm equivalence and $\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}(\mathbf{u}_h)) \in X_T$:

- $m_i(\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}(\mathbf{u}_h))) = 0$
- $m_e(\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}(\mathbf{u}_h))) \leq C \sum_e h_e(\|\mathbf{g}_e - \boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_e\|_e^2 + \|\mathbf{\Pi}_l^* \lambda_h - \boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_m\|_e^2)$
- $m_p(\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}(\mathbf{u}_h))) \leq C \sum_e h_e \|[\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_e]\|_e^2$

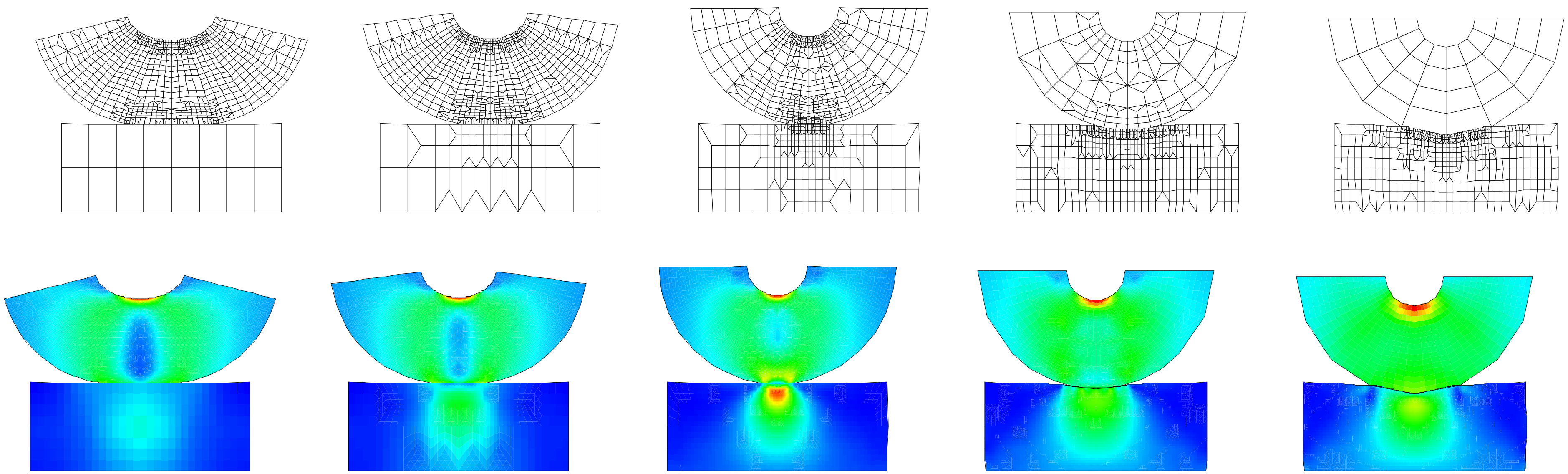
These terms can be found in the analysis of

- the residual based error estimator for a variational equality
- the equilibrated error estimator for a variational equality
- the a priori estimates for the variational inequality

Influence of the material parameters E_i

Poisson number: $\nu_1 = \nu_2 = 0.3$, Coulomb friction coefficient: $\mathcal{F} = 0.4$;

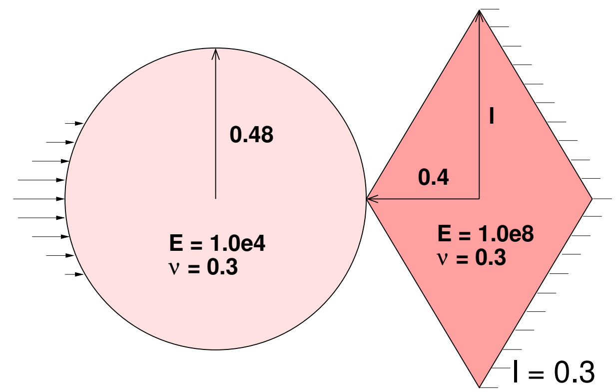
$E_1 = 500$	$E_1 = 10^3$	$E_1 = 10^4$	$E_1 = 10^5$	$E_1 = 10^6$
$E_2 = 10^6$	$E_2 = 10^5$	$E_2 = 10^4$	$E_2 = 10^3$	$E_2 = 500$



Solution after 4 refinement steps; top: deformed mesh, bottom: effective stress

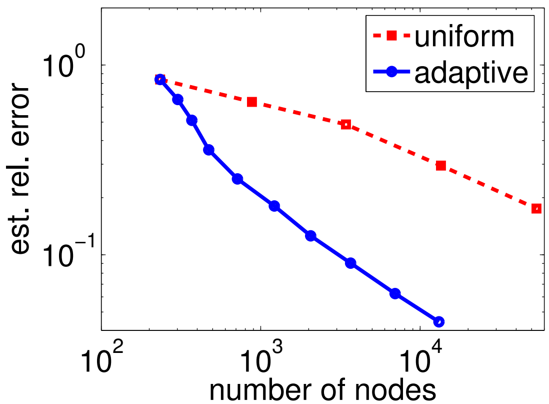
Adaptivity preserves optimality

Low-regularity problem:
soft ball on hard diamond

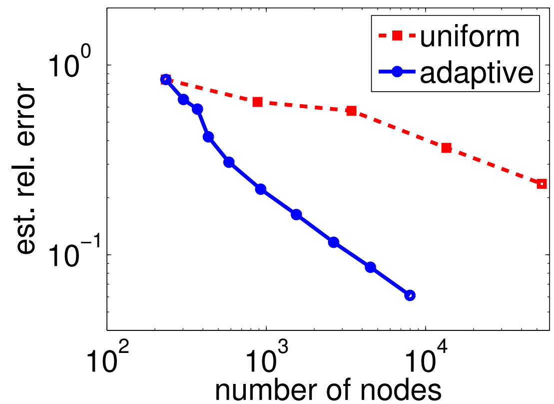


height l determines regularity of problem

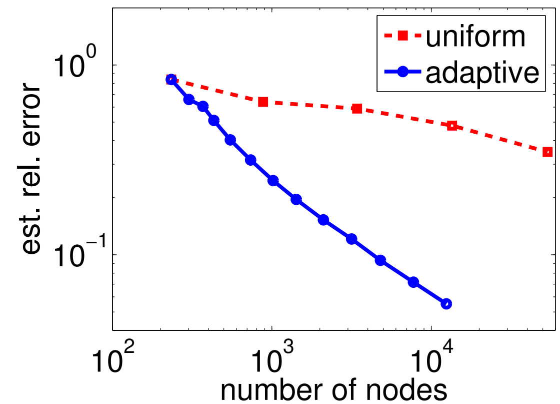
l = 0.7



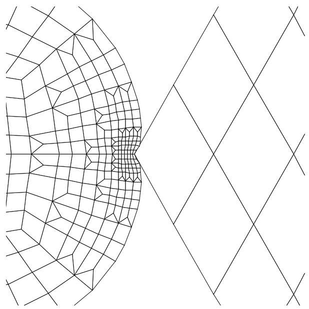
l = 0.5



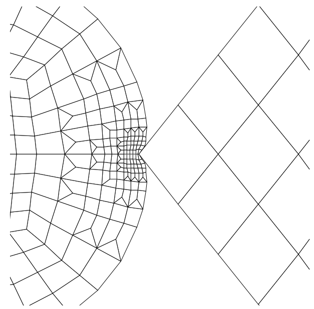
l = 0.3



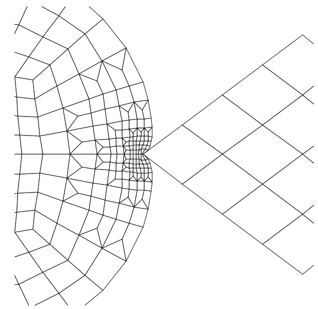
0.7
1.1



0.6
1.1



0.5
1.1

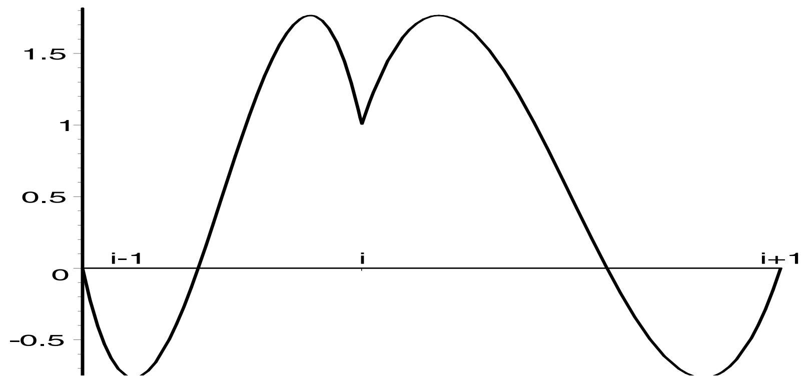


Why is the high order term non standard?

Observation: Higer order term $\mathcal{O}(h^{\frac{3}{2}})$ depends not only on given data **but** also on unknown solution

- **Primal nonconformity:** Non-matching meshes
  $\Longrightarrow$ weak but no strong non-penetration

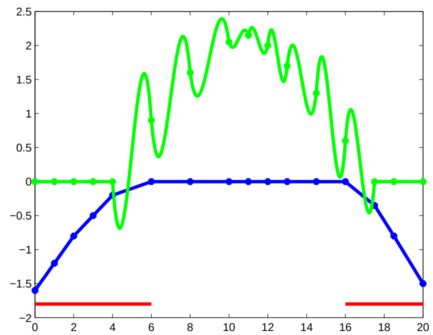
- **Dual nonconformity:** Biorthogonality
  $\Longrightarrow$ LM is weakly but not in a strong sense non-negative

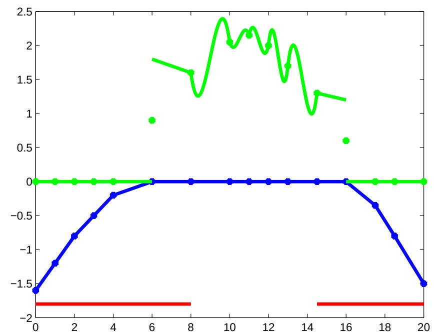


Remedy: Postprocessing of the discrete LM λ_h

Operator on the Lagrange multiplier

Observation: Discrete Lagrange multiplier $\boldsymbol{\lambda}_h$ is not non-negative





Orthogonality between the normal components of $\mathbf{u}_h$ and $\boldsymbol{\lambda}_h$ (left) and $P_{\mathbf{u}_h}\boldsymbol{\lambda}_h$ (right)

$$P_{\mathbf{u}_h}\boldsymbol{\mu}_h := \begin{cases} \mathbf{0} & e \in \mathcal{E}_h^s \\ \boldsymbol{\mu}_h & e \in \mathcal{E}_h^i \text{ and if } (\boldsymbol{\mu}_h)_n \geq 0 \text{ on } e \\ (\alpha_e^1\phi_e^1 + \alpha_e^2\phi_e^2)\mathbf{n} & e \in \mathcal{E}_h^i \text{ and otherwise} \\ (\alpha_e^1 w_e^1\phi_e^1 + \alpha_e^2 w_e^2\phi_e^2)\mathbf{n} & e \in \mathcal{E}_h^b \end{cases},$$

where ϕ_e^1, ϕ_e^2 are the local nodal Lagrange basis functions, and

$$w_e^i := \begin{cases} \frac{\text{meas}(\text{supp}\psi_{p_{ge(i)}})}{\text{meas}(e)} & \text{if supp } \psi_{p_{ge(i)}} \subset \text{supp}_h \mathbf{u}_h, \\ 1 & \text{otherwise} \end{cases}$$

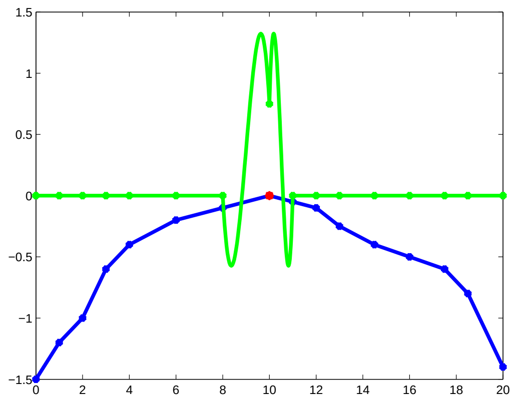
Modified error estimator

In addition to η, we define the quantity

$$\eta_C^2 := \sum_{e \in \mathcal{E}_h^C} \eta_e^2, \quad \eta_e^2 := \frac{h_e}{\sqrt{2\mu}} \|\boldsymbol{\lambda}_h - P_{\mathbf{u}_h} \boldsymbol{\lambda}_h\|_{0;e}^2 .$$

Assumption: For each edge $e \subset \operatorname{supp} \boldsymbol{\lambda}_h \cap \operatorname{supp} \mathbf{u}_h$, we assume that there exists an adjacent edge $\hat{e}$ such that $\hat{e} \subset \Gamma_C \setminus \operatorname{supp} \mathbf{u}_h$.

This assumption excludes isolated points such as



A posteriori error estimator for a one-body problem

As it is standard for a posteriori estimates, we define a higher order term which only depends on the given data

$$\xi^2 := \sum_{T \in \mathcal{T}_h} \xi_T^2, \quad \xi_T^2 := \frac{h_T^2}{2\mu} \|\mathbf{f} - \mathbf{\Pi}_1 \mathbf{f}\|_{0;T}^2 .$$

Theorem: Under the Assumption, there exist constants $C_1, C_2 < \infty$ independent of the mesh-size such that

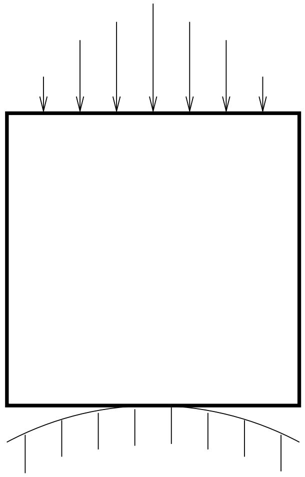
$$\|\mathbf{u} - \mathbf{u}_h\|_a \leq \eta + C_1 \eta_C + C_2 \xi .$$

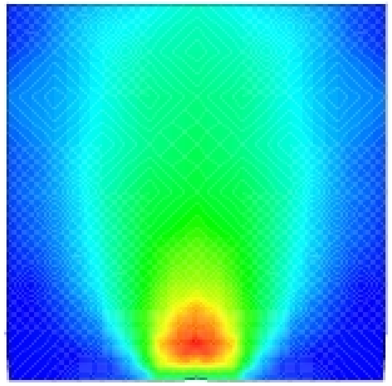
Theorem: Under the Assumption, there exists a constant $C < \infty$ independent of the mesh-size such that $\beta(h) \leq C$ and

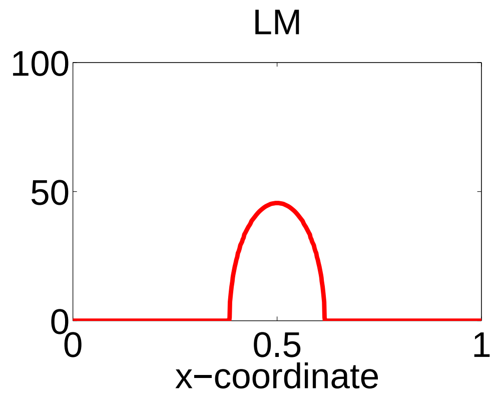
$$\|\mathbf{u} - \mathbf{u}_h\|_a \leq (1 + C_1 \beta(h))\eta + C_2 \xi) .$$

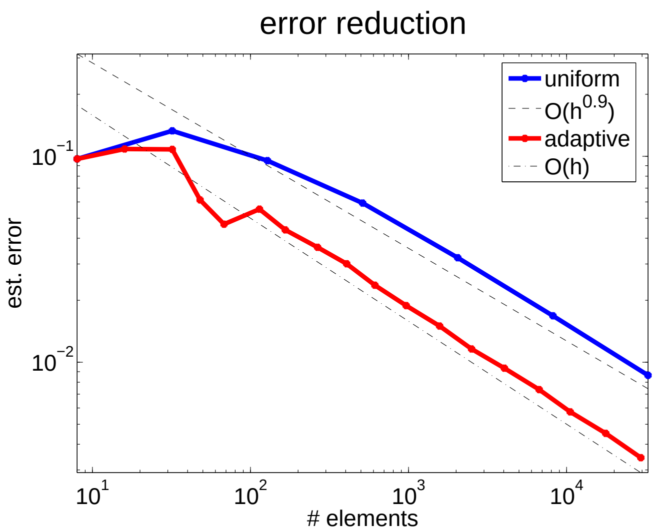
Remark: The numerical results show that $\beta(h)$ tends asymptotically to zero and the upper bound $1 + C_1 \beta(h)$ tends to one.

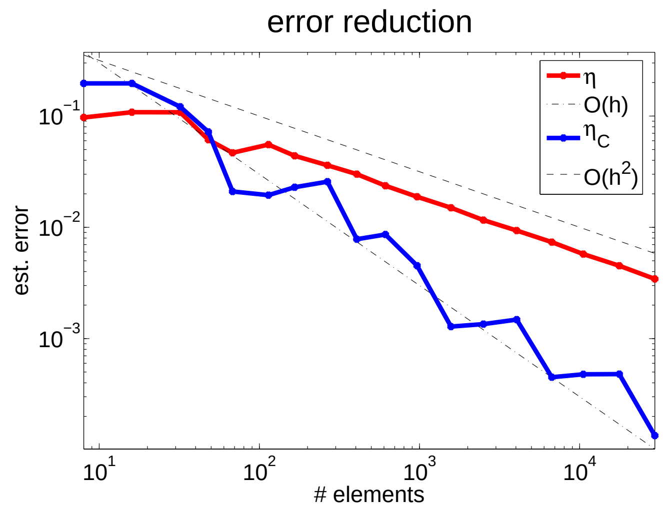
Hertz-problem

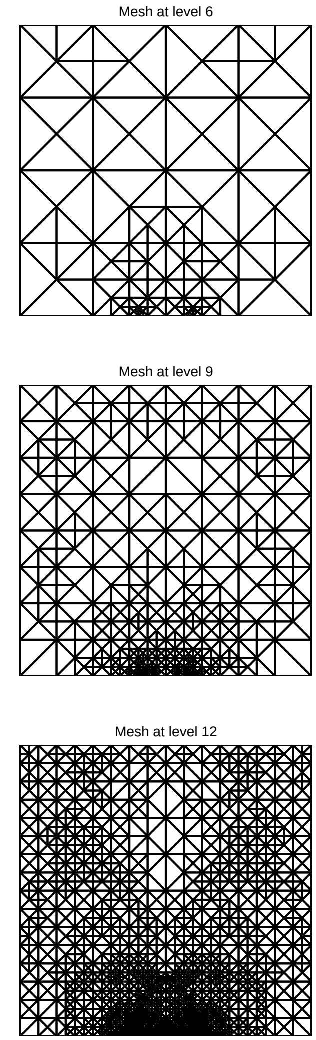




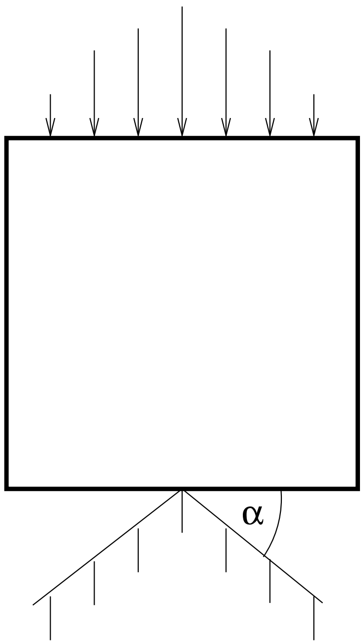


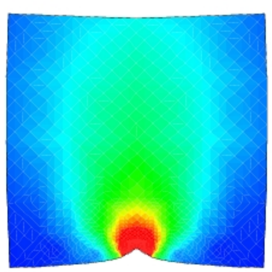




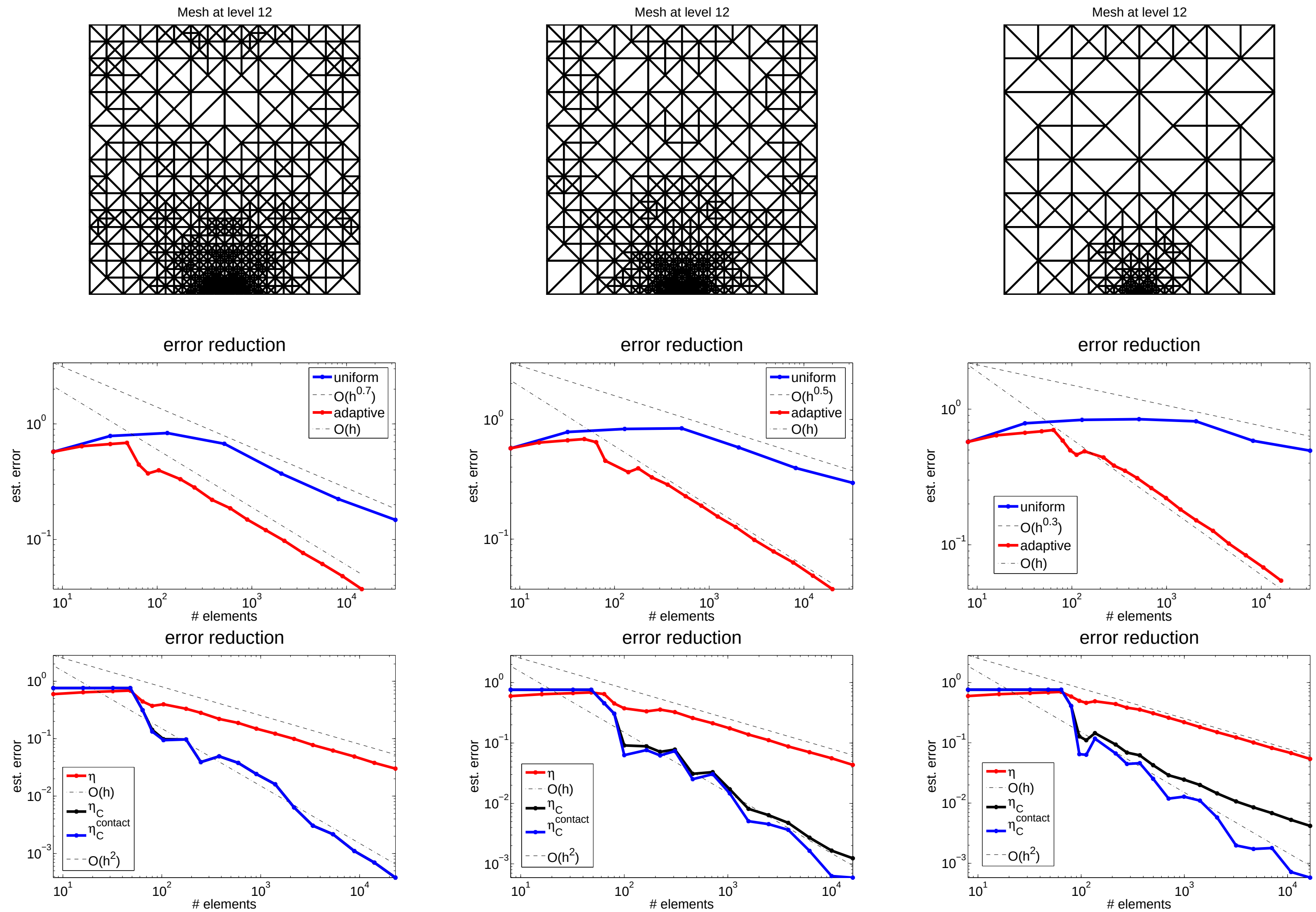


Square on triangle





$\alpha = \pi/6$ $\alpha = \pi/4$ $\alpha = \pi/3$



A posteriori estimator for the Lagrange multiplier

Standard a priori estimate:

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2};\Gamma_C} \leq C\Big(\underbrace{\inf_{\boldsymbol{\mu}_h \in \mathbf{M}_h} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{-\frac{1}{2};\Gamma_C}}_{\mathcal{O}(h^{\frac{3}{2}})} + \|\mathbf{u} - \mathbf{u}_h\|_a\Big) .$$

But: $\boldsymbol{\lambda}$ is not a given data

Data oscillation term:

$$\tilde{\xi}^2 := \sum_{T \in \mathcal{T}_h} \tilde{\xi}_T^2, \quad \tilde{\xi}_T^2 := \frac{h_T^2}{2\mu} \|\mathbf{f} - \mathbf{Q}^* \mathbf{f}\|_{0;T}^2$$

$\tilde{\xi} = 0$ if $\mathbf{f}$ is constant on Ω, $\mathbf{Q}^*$ Scott–Zhang type operator

Theorem: There exists a constant $C < \infty$ independent of the mesh-size such that

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{-\frac{1}{2};\Gamma_C} \leq C\left(\|\mathbf{u} - \mathbf{u}_h\|_a + \tilde{\xi}\right) .$$

Conclusion

- Equlibration techniques can be generalized to elasticity
- Error bound for the LM in terms of the primal bound
- Variational inequality does not bring in extra terms
- Sign controlling terms are of higher order
- Non-matching meshes are problematic (theory)

AFEM based strategies

AFEM for standard fe estimates: guaranteed error decay

W. Dörfler, SIAM J. Numer. Anal., 33 (1996), pp. 1106–1124 .

P. Binev, W. Dahmen, R. DeVore, Numer. Math., 97 (2004), pp. 219–268

AFEM for obstacle problems:

No Galerkin orthogonality but minimization property on convex set

D. Braess, C. Carstensen, R. Hoppe, Numer. Math., 107 (2007), pp. 455–471

These results can be applied for one-body contact problems

In the case of a two-body contact problem on non-matching meshes:

- Convex sets are non-nested, i.e., $K_l \not\subset K_{l+1} \not\subset K$
- Higher order term cannot be controlled by given data
- No classical inverse estimate for the discrete trace, i.e.,

$$\|[v_h]\|^2_{\frac{1}{2};\Gamma_C} \not\leq C\frac{1}{h}\|[v_h]\|^2_{0;\Gamma_C} \quad \textbf{BUT} \quad \|[v_h]\|^2_{\frac{1}{2};\Gamma_C} \leq C\frac{|\ln\epsilon|+1}{h}\|[v_h]\|^2_{0;\Gamma_C}$$

ϵ minimal relative shift

Strong monotonicity in the energy

Corollary: There exists a constant independent of the mesh-size such that

$$J(\mathbf{u}_h) - J(\mathbf{u}) \le C(\eta^2 + \xi^2) \, ,$$

where the energy $J(\mathbf{v})$ is given by $J(\mathbf{v}) := \frac{1}{2}a(\mathbf{v},\mathbf{v}) - f(\mathbf{v})$.

The variational inequality is equivalent to a constrained minimization problem, i.e., $J(\mathbf{u}) \le J(\mathbf{v})$, $\mathbf{v} \in K$, and in terms of $K_l \subset K_{l+1}$, we have

$$0 \le \delta_{l+1} \le \delta_l := J(\mathbf{u}_l) - J(\mathbf{u}) \, .$$

Theorem: There exist constants $\rho_1, \rho_2 < 1$ and $c_\xi, C_\xi < \infty$ such that

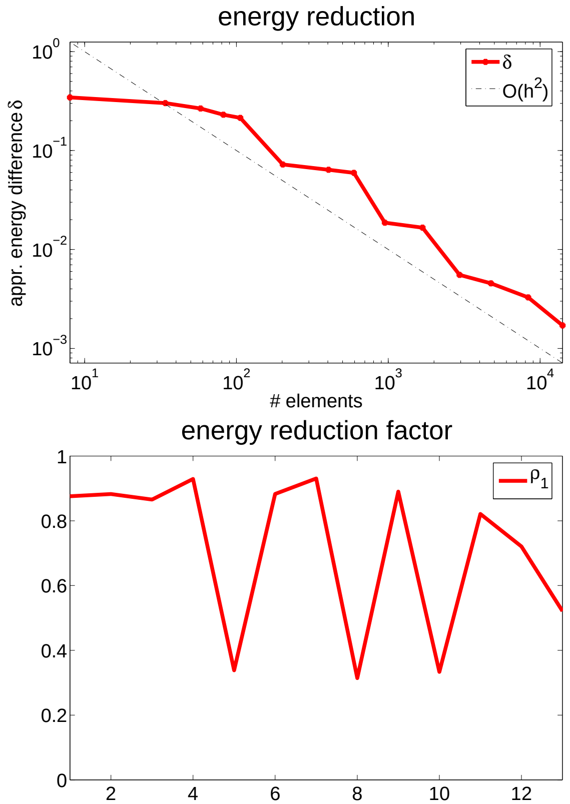
$$\begin{aligned} \delta_{l+1} &\le \rho_1 \delta_l + c_\xi \hat{\xi}_l^2 \, , \\ \delta_{l+1} + C_\xi \hat{\xi}_{l+1}^2 &\le \rho_2 (\delta_l + C_\xi \hat{\xi}_l^2) \, . \end{aligned}$$

Remark: We observe that $\hat{\xi}_l = 0$ for a constant $\mathbf{f}$. In that case, the energy term δ_l is a strictly decreasing function with respect to the refinement level l.

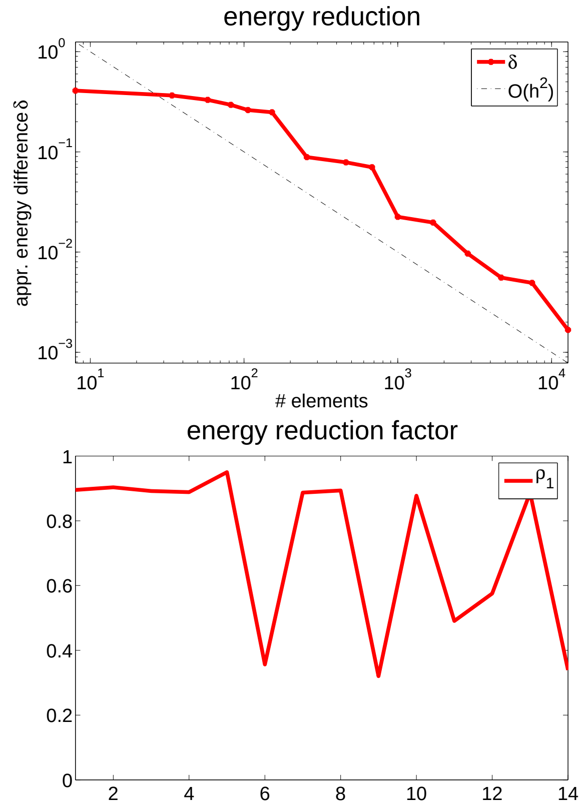
AFEM strategy for example 3

Energy difference $\delta_l := J(\mathbf{u}_l) - J(\mathbf{u})$

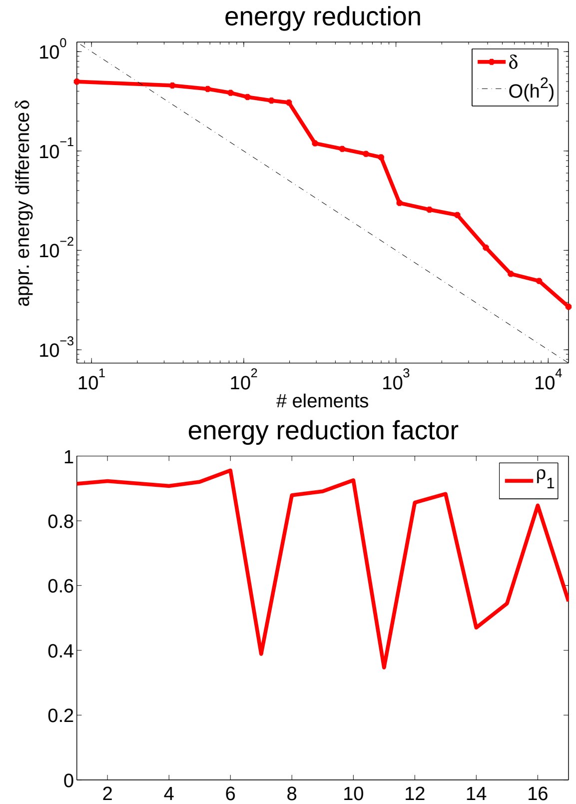
$\alpha = \pi/6$



$\alpha = \pi/4$

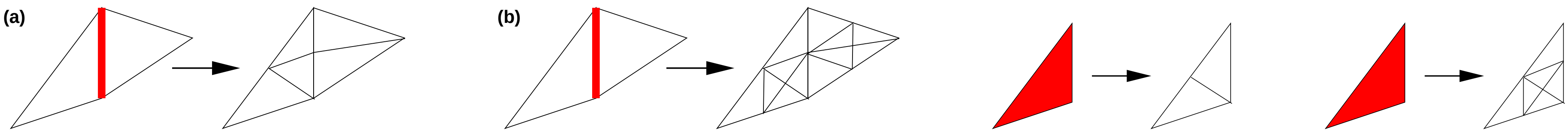


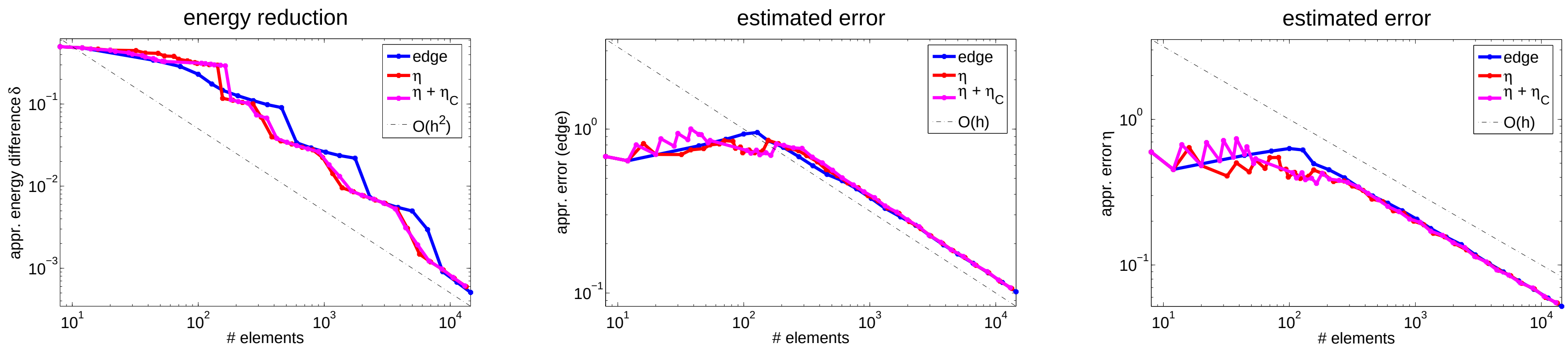
$\alpha = \pi/3$



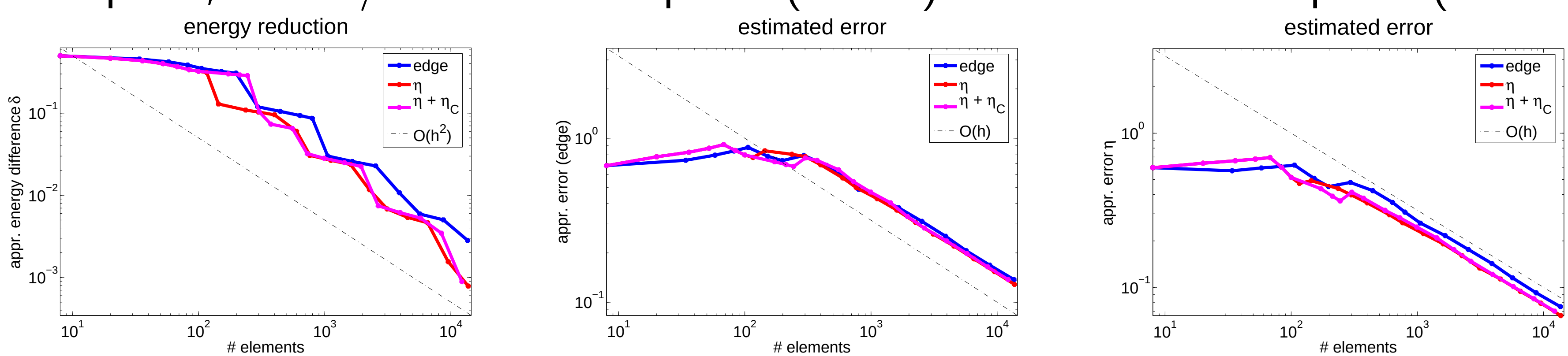
Comparison of different refinement strategies

edge | edge (ip) | η | η (ip)



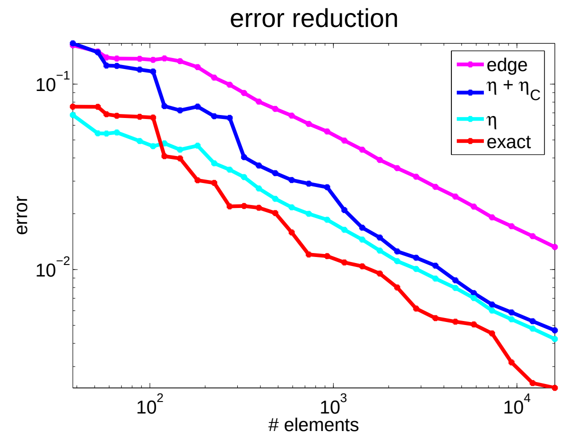


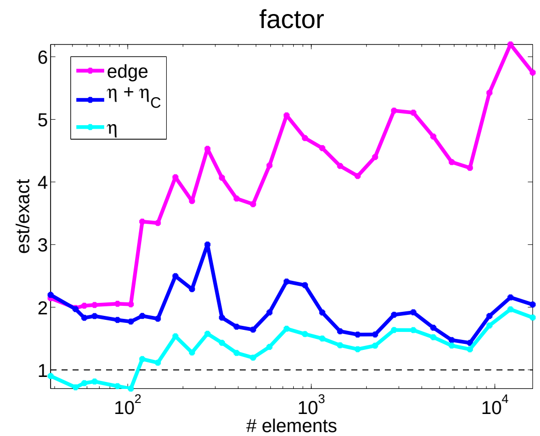
Example 3, $\alpha = \pi/3$: no interior point (above) and with interior point (below)



Comparison of different error estimators

Example 2:





Example 3: ($\alpha = \pi/3$)