A posteriori error estimates for efficiency and error control in numerical simulations

Martin Vohralík

Outline

Outline

Heat equation

The problem

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega \end{aligned}$$

Model setting

- exact solution $u = e^{x+y+t-3}$ on square domain $\Omega = (0,3) \times (0,3)$, $T = 1.5$ or $T = 3$
- square meshes: 10×10, 30×30, 90×90
- time steps: 0.3, 0.1, 0.3333
- vertex-centered finite volumes
- additional quadrature/mass lumping estimator

Heat equation

The problem

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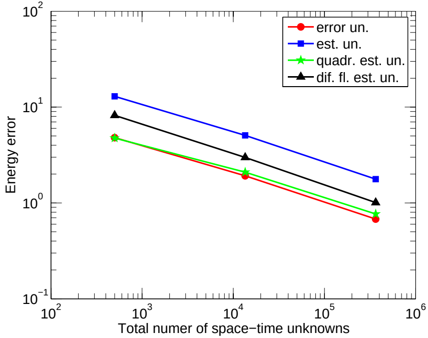
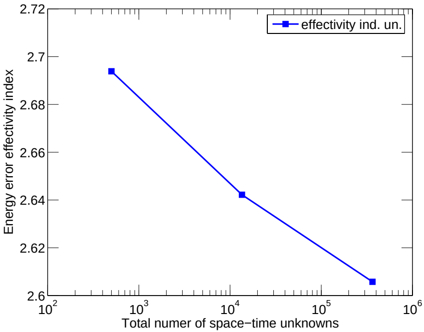
Energy norm results, $T = 1.5$

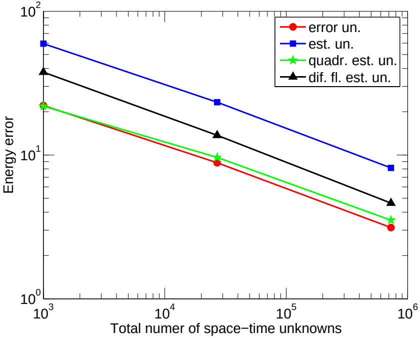


Energy error and estimators

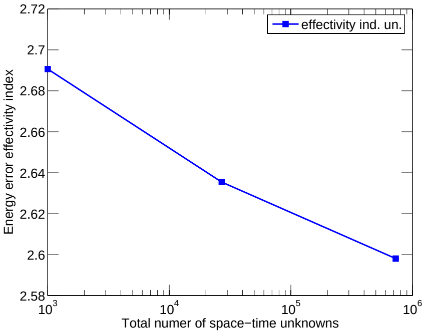


Effectivity index

Energy norm results, $T = 3$

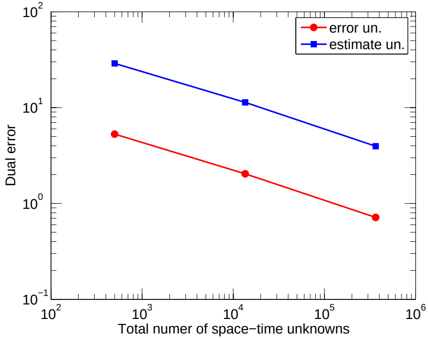


Energy error and estimators

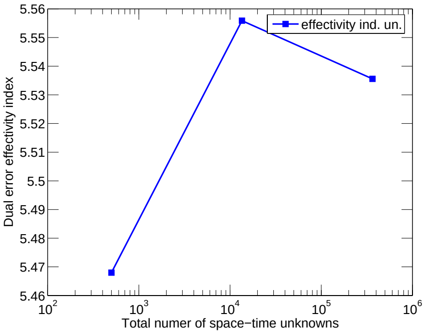


Effectivity index

Dual norm results, $T = 1.5$

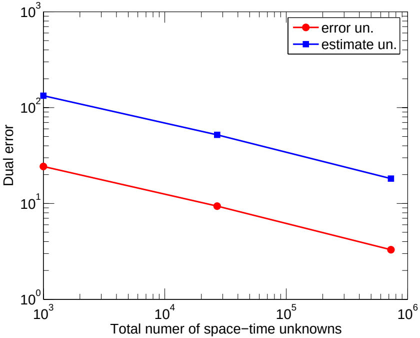


Dual error and estimators

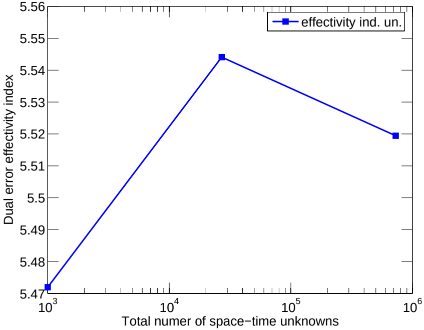


Effectivity index

Dual norm results, $T = 3$



Dual error and estimators



Effectivity index

Outline

Unsteady advection–diffusion–reaction equation

The problem

$$
\begin{aligned}
u_t - \nabla\cdot(\underline{\mathbf{K}}\nabla u) + \nabla\cdot(u\mathbf{w}) + ru &= f && \text{in } \Omega\times(0,T),\\
u(\cdot,0) &= u_0 && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega\times(0,T)
\end{aligned}
$$

Model setting

- $\underline{\mathbf{K}} = \nu Id$, ν is a parameter
- $\mathbf{w} = (0.8, 0.4)$
- $r = 0$, $f = 0$

Exact solution

$$
u(x,y,t) = \frac{1}{200\nu(t+t_0)+1} e^{-50\frac{(x-x_0-v_1(t+t_0))^2+(y-y_0-v_2(t+t_0))^2}{200\nu(t+t_0)+1}}
$$

Unsteady advection–diffusion–reaction equation

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u_t - \nabla\cdot(\underline{\mathbf{K}}\nabla u) + \nabla\cdot(u\mathbf{w}) + ru = f \qquad & \text{in } \Omega\times(0,T),\\
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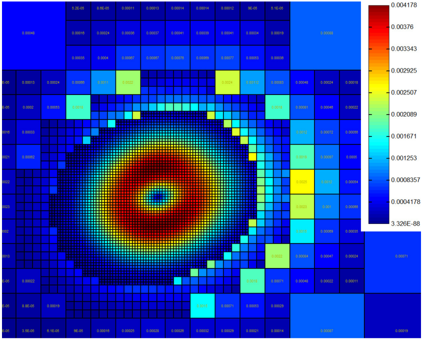
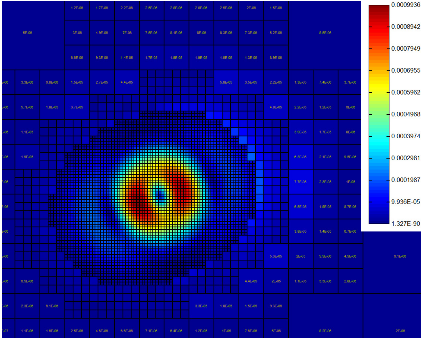
Error distributions

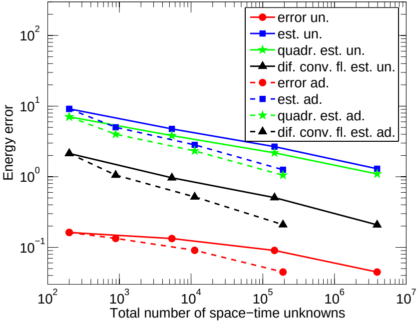


Estimated error distribution,
$\nu = 0.001$, $T = 0.6$

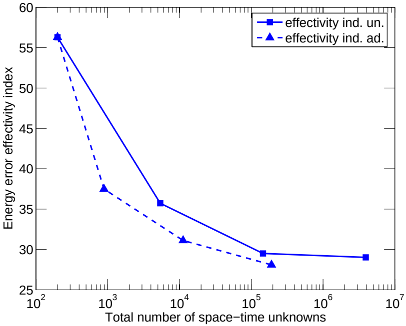


Exact error distribution,
$\nu = 0.001$, $T = 0.6$

Estimated and actual energy errors

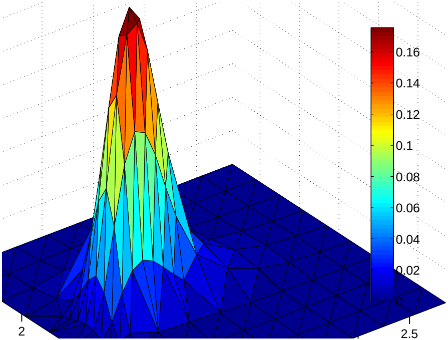


Estimated and actual errors,
$\nu = 0.001$, $T = 0.6$

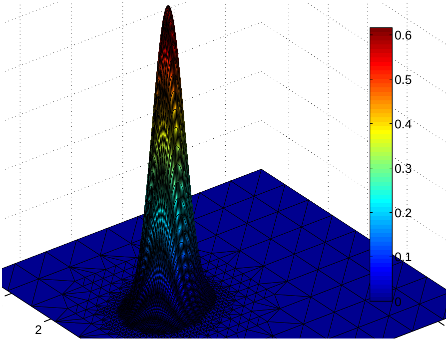


Effectivity indices,
$\nu = 0.001$, $T = 0.6$

Adaptive refinement approximate solutions

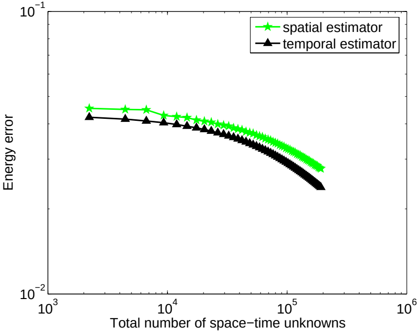


Approximate solutions, $\nu = 0.001$, $T = 0.6$, two levels refinement



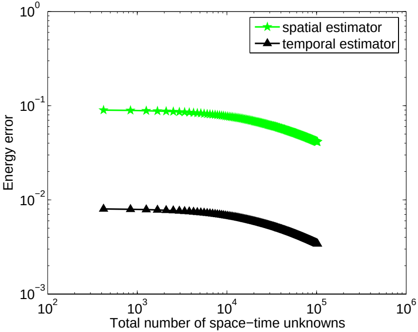
Approximate solutions, $\nu = 0.001$, $T = 0.6$, four levels refinement

Spatial and temporal estimators equilibrated

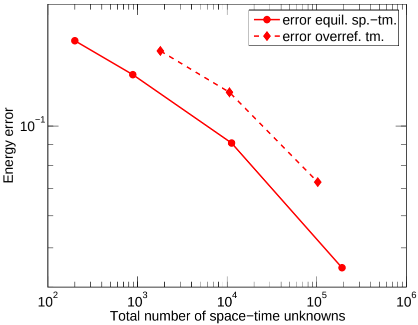


Spatial estimators η_{sp}^{n} and temporal estimators η_{tm}^{n} equilibrated, $\nu = 0.001$, $T = 0.6$

Overrefinement in time

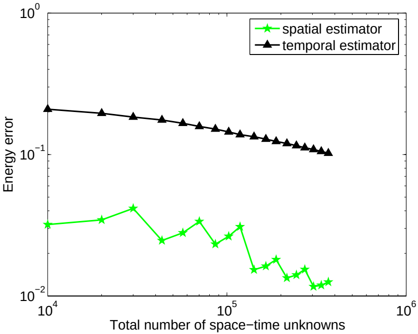


Spatial estimators η_{sp}^n and temporal estimators η_{tm}^n

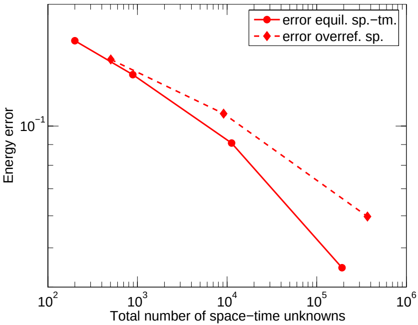


Comparison with the equilibrated case

Overrefinement in space



Spatial estimators $\eta_{\rm sp}^n$ and temporal estimators $\eta_{\rm tm}^n$



Comparison with the equilibrated case

Outline

1. Heat equation
2. Unsteady advection–diffusion–reaction equation
3. Nonlinear Laplace equation

Numerical experiment I

Model problem

- p-Laplacian

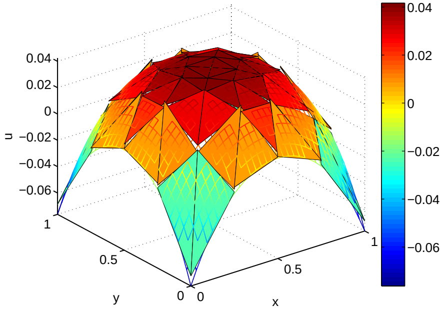
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad & \text{in } \Omega, \\ u = u_0 \qquad & \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

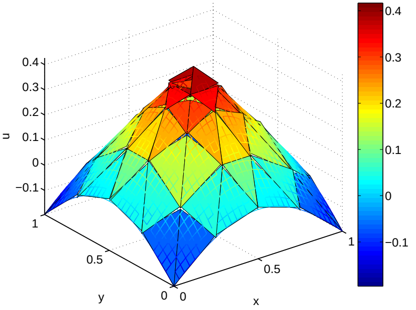
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and $p = 10$
- nonconforming finite elements

Analytical and approximate solutions

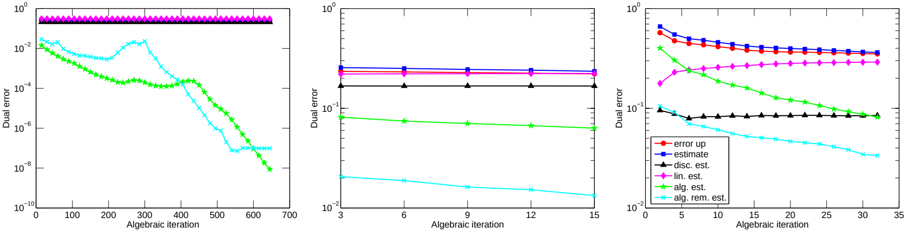


Case $p = 1.5$



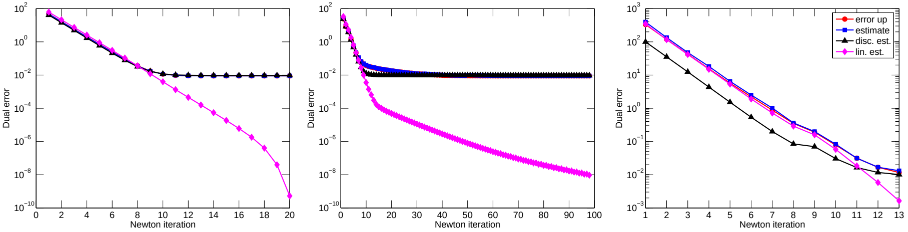
Case $p = 10$

Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



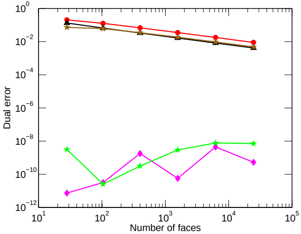
Newton inexact Newton ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh

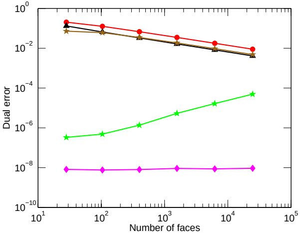


Newton inexact Newton ad. inexact Newton

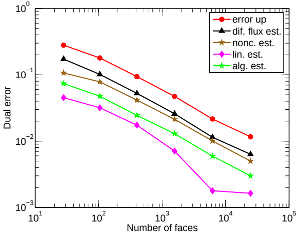
Error and estimators, $p = 10$



Newton

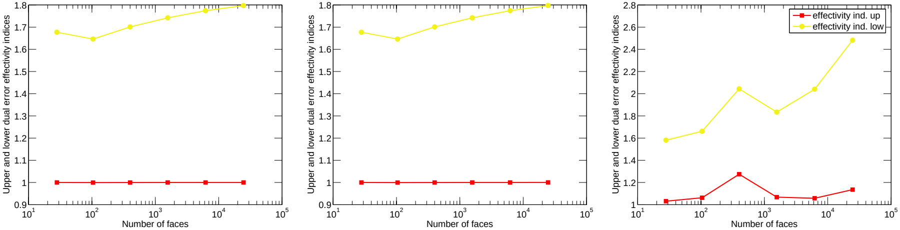


inexact Newton



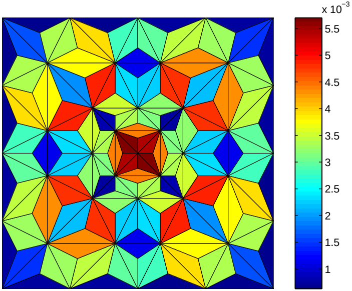
ad. inexact Newton

Effectivity indices, $p = 10$

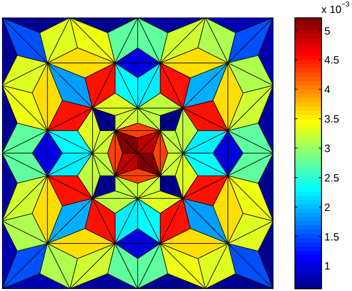


Newton inexact Newton ad. inexact Newton

Error distribution, $p = 10$

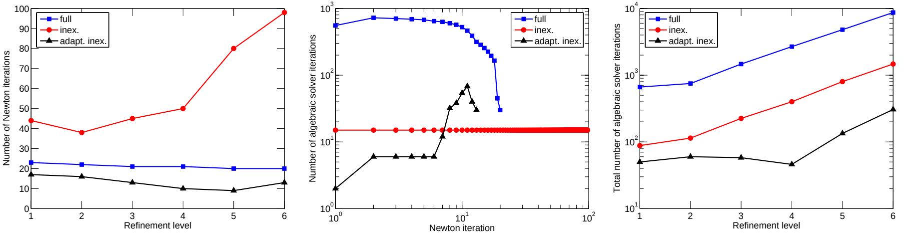


Estimated error distribution



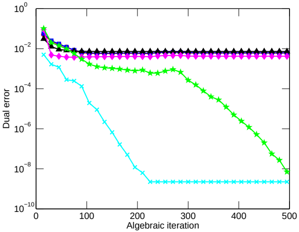
Exact error distribution

Newton and algebraic iterations, $p = 10$

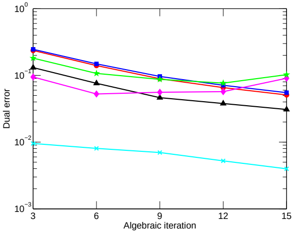


Newton it. / refinement alg. it. / Newton step alg. it. / refinement

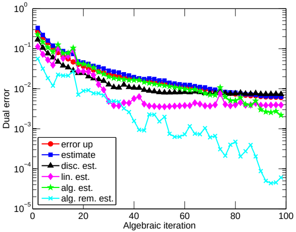
Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.



Newton

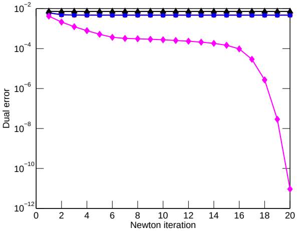


inexact Newton

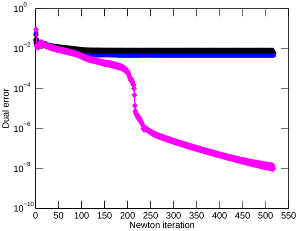


ad. inexact Newton

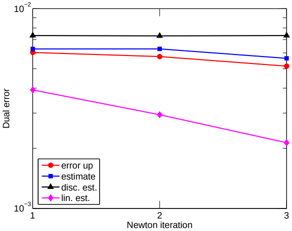
Error and estimators as a function of Newton iterations, $p = 1.5$, 6th level mesh



Newton

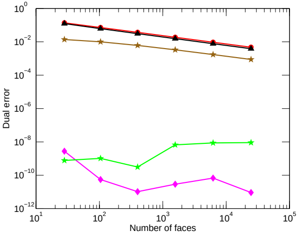


inexact Newton

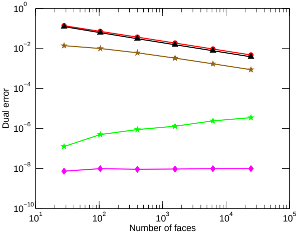


ad. inexact Newton

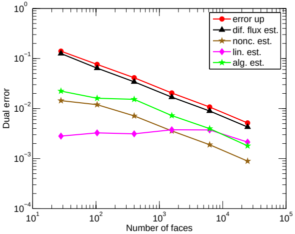
Error and estimators, $p = 1.5$



Newton

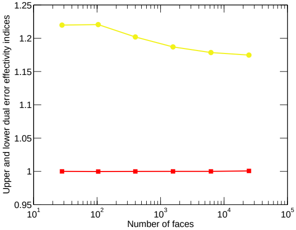


inexact Newton

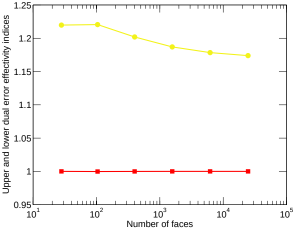


ad. inexact Newton

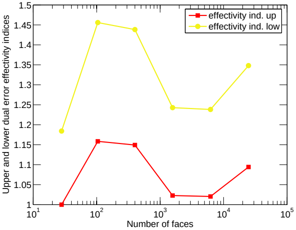
Effectivity indices, $p = 1.5$



Newton

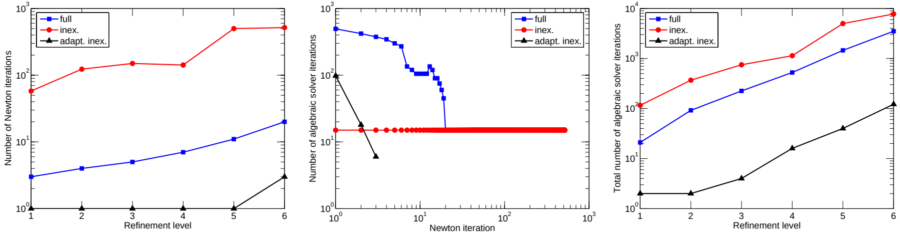


inexact Newton



ad. inexact Newton

Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement   alg. it. / Newton step   alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

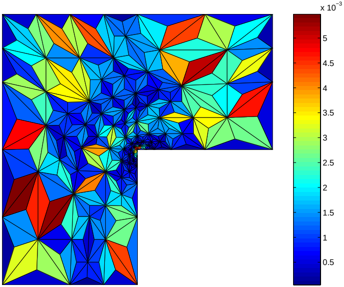
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

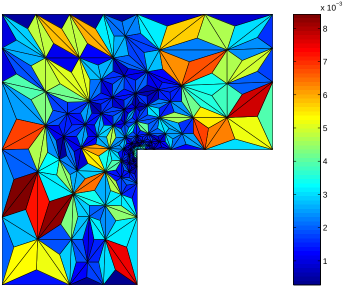
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

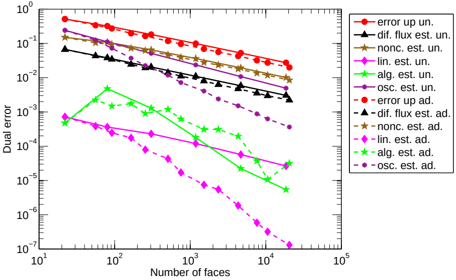


Estimated error distribution

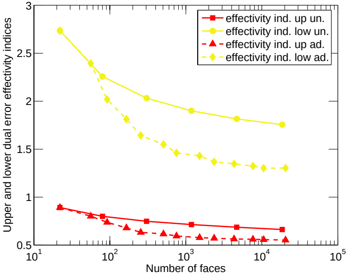


Exact error distribution

Estimated and actual errors and the effectivity index

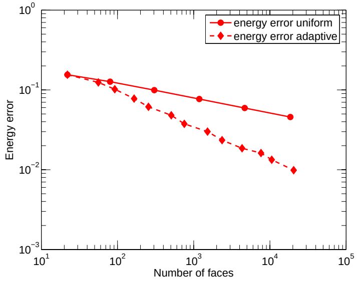


Estimated and actual errors

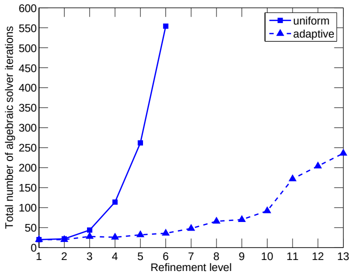


Effectivity index

Energy error and overall performance



Energy error



Overall performance