Examination
May, 21, 2018

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal (polyhedral) domain (open, bounded, and connected set), $\mathcal{T}_h$ its simplicial mesh, $\mathcal{E}_h$ the set of all its faces, and $\mathcal{E}_h^{\mathrm{int}}$ the set of its interior faces.

Question 1. (Guaranteed a posteriori error estimate for the Laplace equation) The Poisson problem for the Laplace equation reads: for $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{1b}$$

1) Recall the weak formulation of (1a)–(1b).

2) Let u be the weak solution of (1a)–(1b). Let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. State and prove rigorously an a posteriori error estimate of the form

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K})^2 + \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2. \tag{2}$$

Specify in particular the quantities $\eta_{\mathrm{F},K}$, $\eta_{\mathrm{R},K}$, and $\eta_{\mathrm{NC},K}$.

Question 2. (Application to the finite element method)

1) Define the finite element method (of arbitrary polynomial degree) for the problem (1a)–(1b).

2) Show in details how to apply the estimate (2) to this method (define all the quantities, give formulas how to compute them, and verify all assumptions necessary).

Question 3. (Guaranteed a posteriori error estimate for the Stokes equation) Consider the Stokes problem: for $\mathbf{f} \in [L^2(\Omega)]^d$, find $\mathbf{u}$ and p such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{3a}$$
$$\nabla{\cdot}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{3b}$$
$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{3c}$$

Let $(\mathbf{u}, p)$ be the weak solution of (3a)–(3c). Let $(\mathbf{u}_h, p_h) \in [H^1(\mathcal{T}_h)]^d \times L_0^2(\Omega)$ be arbitrary. Let $\beta > 0$ be the constant from the inf–sup condition

$$\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{v}\in[H_0^1(\Omega)]^d} \frac{(q, \nabla{\cdot}\mathbf{v})}{\|\nabla\mathbf{v}\|\,\|q\|} = \beta.$$

State and prove a posteriori error estimates on the errors $\|\nabla(\mathbf{u} - \mathbf{u}_h)\|$ and $\|p - p_h\|$.

Question 4. (Equivalence augmented norm–dual norm of the residual for the heat equation)

Let $T > 0$ be a final simulation time. The heat equation reads: for $f \in L^2(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$, find u such that

$$\partial_t u - \Delta u = f \quad \text{in } \Omega \times (0,T), \tag{4a}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{4b}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega. \tag{4c}$$

1) Consider the spaces $X := L^2(0,T;H_0^1(\Omega))$ and $Y := \{v \in X; \partial_t v \in X'\}$. Recall the variational formulation of (4a)–(4c), where we look for $u \in Y$.

2) On the space X, consider the energy norm $\|v\|_X := \left\{\int_0^T \|\nabla v\|^2(t)\,\mathrm{d}t\right\}^{1/2}$. Equip the space Y with the norm

$$\|v\|_Y^2 := \|v\|_X^2 + \|\partial_t v\|_{X'}^2 + v(\cdot,T)^2,$$

where $\|\partial_t v\|_{X'} := \left\{\int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2(t)\,\mathrm{d}t\right\}^{1/2}$. For $v \in Y$, show the link between $\|v\|_Y$, $\|v(\cdot,0)\|$, and $\sup_{\varphi\in X,\,\|\varphi\|_X=1} \int_0^T \{\langle \partial_t v, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\nabla v, \nabla\varphi)\}(t)\,\mathrm{d}t$.

3) Consider an arbitrary $v_{h\tau} \in Y$. Define the residual $\mathcal{R}(v_{h\tau}) \in X'$ of $v_{h\tau}$ such that for all $\varphi \in X$,

$$\langle \mathcal{R}(v_{h\tau}), \varphi\rangle_{X',X} := \int_0^T \{(f,\varphi) - \langle \partial_t v_{h\tau}, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - (\nabla v_{h\tau}, \nabla\varphi)\}(t)\,\mathrm{d}t.$$

Use the link from the previous question to give the link between $\|u - v_{h\tau}\|_Y$ (with u the weak solution of the heat equation) to the dual norm of the residual and to the initial error.

Question 5. (Stopping criteria, balancing error components, adaptive strategies)

1) Explain the principle of optimal stopping criteria for linear and nonlinear solvers.

2) On the example of the heat equation, explain the concept of identification of different error components and of their balancing.

3) Describe the principles of adaptive strategies for efficient numerical approximation of partial differential equations.

This is a conceptual question, no proofs are to be given here.