Numerical methods, a priori and a posteriori error estimates, and *hp* finite elements

Martin Vohralík

Inria Paris & Ecole des Ponts

ENSTA, March 8, 2024

Outline

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Numerical approximations of PDEs

Numerical methods

- mathematically-based algorithms evaluated by **computers**
- deliver **approximate solutions**
- conception: more effort $\Rightarrow$ closer to the unknown solution
- example: elastic string

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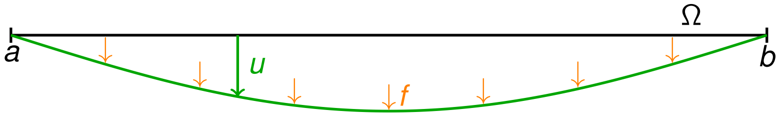

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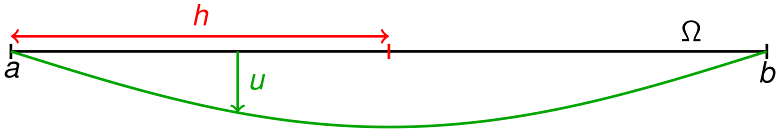

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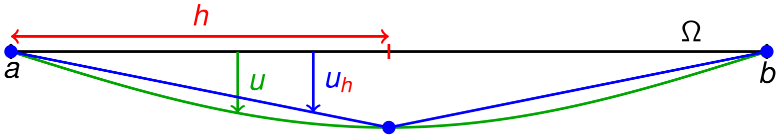

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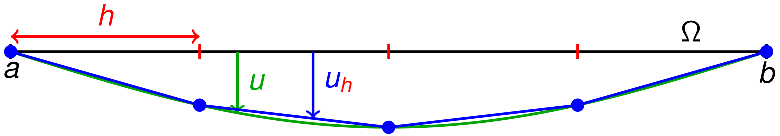

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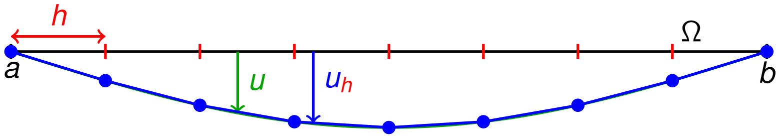

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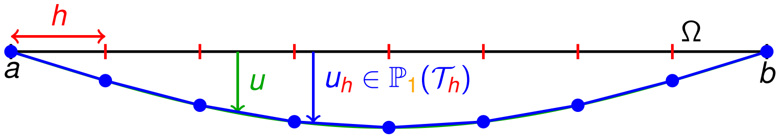

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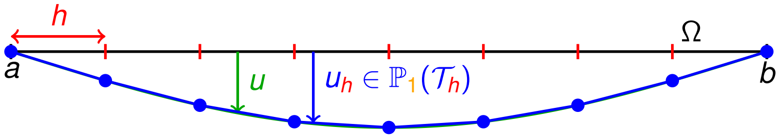

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Error

$$\|\nabla(u-u_h)\| = \left\{\int_a^b |(u-u_h)'|^2\right\}^{\frac{1}{2}}$$

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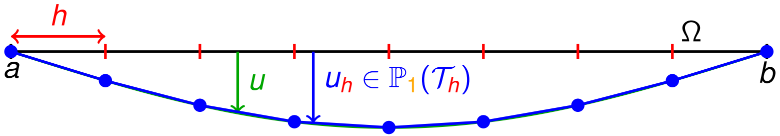

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Polynomial degree p

$$u_h \in \mathbb{P}_p(\mathcal{T}_h)$$

Outline

A priori & a posteriori error estimates

Crucial questions

1. Does the method **converge**?
   $\|\nabla(u - u_h)\| \to 0$? For $h \searrow 0$? For $p \nearrow \infty$?
2. At which **speed**? $\|\nabla(u - u_h)\| \leq ?$
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3. **Adaptivity**, focusing, h & p refined **non uniformly**.

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Case Studies in Engineering Failure Analysis 3 (2015) 88–95

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A posteriori error estimates: **certify** the error

Poisson equation

$$
\begin{aligned}
-\Delta u &= f \quad \text{in} \quad \Omega,\\
u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Guaranteed error upper bound (reliability)

$$
\underbrace{\|\nabla(u-u_h)\|}_{\text{unknown error}} \le \underbrace{\eta(u_h)}_{\text{computable estimator}}
$$

Global error lower bound (global efficiency; mathematical equivalence of the error and estimator)

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\eta(u_h) \le C\|\nabla(u-u_h)\|
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Local error lower bound (local efficiency; if the estimator predicts error in an element K, then it is in K and around)

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Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\|\nabla(u - u_h)\|^2 = \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v} = f}} \|\nabla u_h + \boldsymbol{v}\|^2 + \min_{v \in H_0^1(\Omega)} \|\nabla(u_h - v)\|^2.$$

Comments

- It is now enough to choose suitable $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $s_h \in H_0^1(\Omega)$.
- A simple choice for nonconforming finite elements given in the lecture notes.

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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	1.25	28%	1.07	24%	1.17
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
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$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	**1.06**
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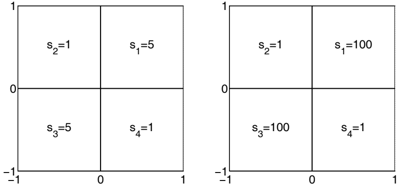
Outline

Problem with singular solution

- consider the pure diffusion equation

$$-\nabla\cdot\boldsymbol{S}\nabla u = 0 \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

- discontinuous and inhomogeneous $\boldsymbol{S}$, two cases:

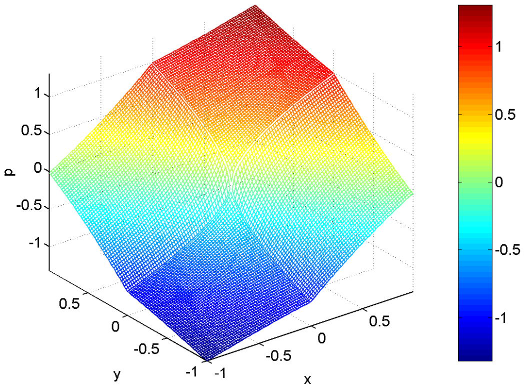


- analytical solution: **singularity** at the origin

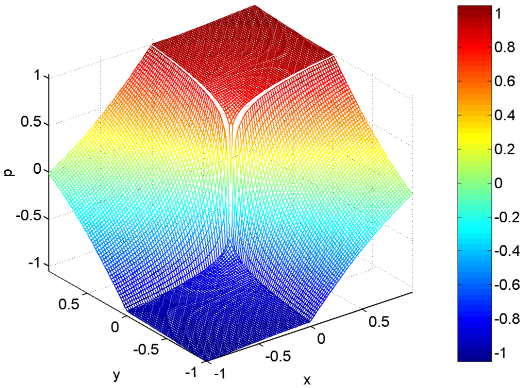
$$u(r,\theta) = r^\alpha(a_i\sin(\alpha\theta) + b_i\cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution, $u \in H^{1+\alpha}(\Omega)$

Analytical solutions

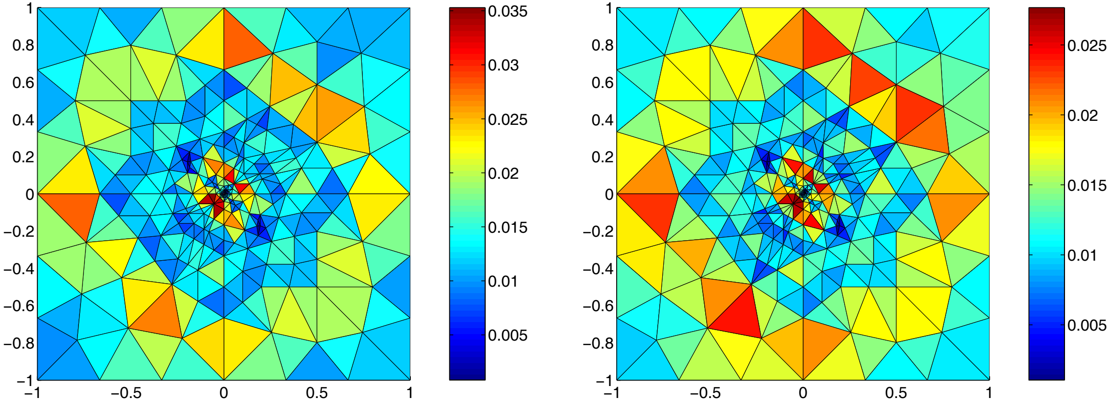


Case 1 ($\alpha \approx 0.54$)



Case 2 ($\alpha \approx 0.13$)

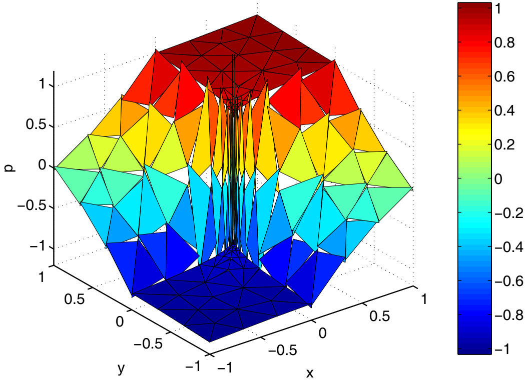
Where is the error **localized**?



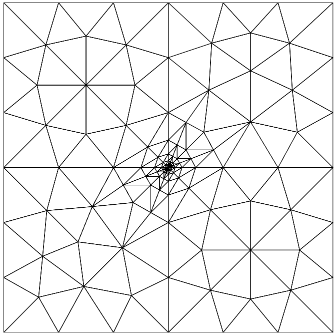
Estimated error distribution $\eta_K(u_h)$ Exact error distribution $\|\nabla(u-u_h)\|_K$

M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Can we adapt the mesh to better approximate the solution?



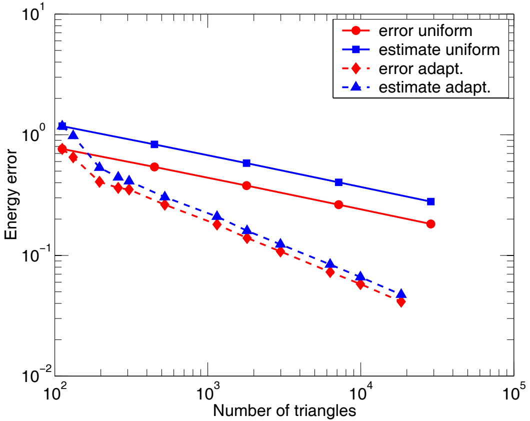
Nonconforming finite elements



Adaptively refined mesh

M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Does this lead to a better error decrease?



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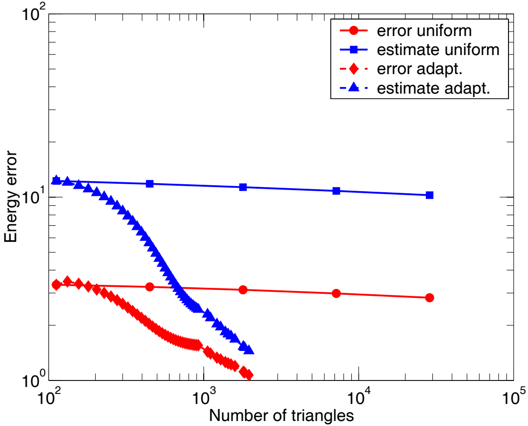

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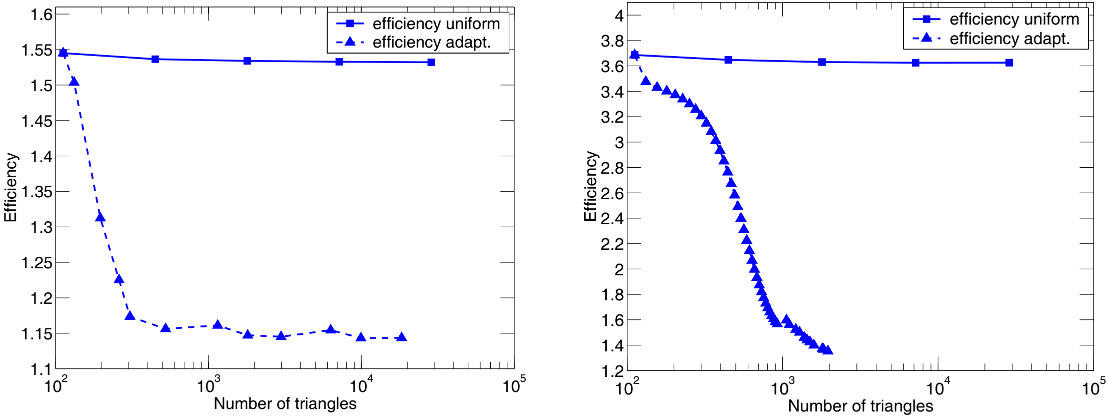
Quality of the estimates for a singular solution



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Adaptive mesh refinement

Mesh adaptivity

- Dörfler marking: subset $\mathcal{M}_\ell$ containing θ-fraction of the estimates

$$\sum_{K \in \mathcal{M}_\ell} \eta_K(u_\ell)^2 \geq \theta^2 \sum_{K \in \mathcal{T}_\ell} \eta_K(u_\ell)^2$$

- refine the elements in $\mathcal{M}_\ell$

Convergence on a sequence of adaptively refined meshes $\mathcal{T}_\ell$

- $$\|\nabla(u - u_\ell)\| \to 0 \text{ for } \ell \to \infty$$
- some mesh elements may not be refined at all: ~~$h \searrow 0$ **uniformly**~~

Optimal error decay rate wrt degrees of freedom

- $$\|\nabla(u - u_\ell)\| \leq C|\mathrm{DoF}_\ell|^{-p/d} \qquad \text{(replaces } h^p\text{)}$$
- same for smooth & singular solutions: ~~higher-order only pays-off for sm. sol.~~
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h vs. *hp* a priori analysis

Theorem (Deny–Lions/Bramble–Hilbert)

For all $K \in \mathcal{T}_h$ *and* $v \in H^{p+1}(K)$,

$$\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K \leq \sqrt{(p+1)!} \left(\frac{h_K}{\pi}\right)^p |v|_{H^{p+1}(K)}.$$

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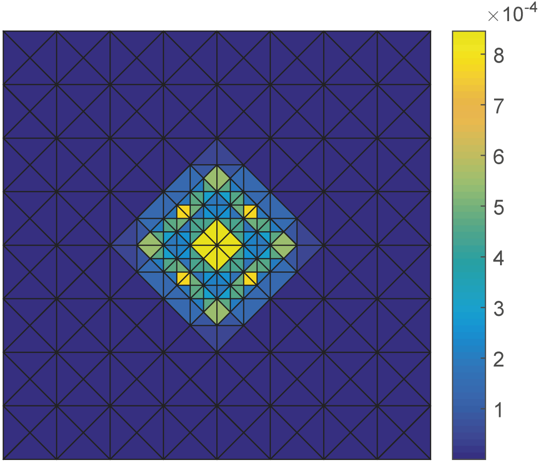

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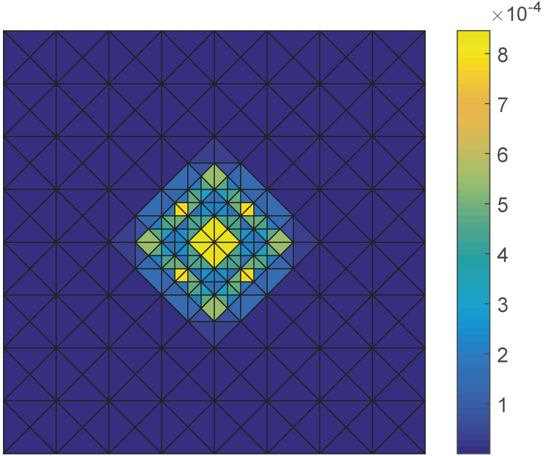

Exact error distribution $\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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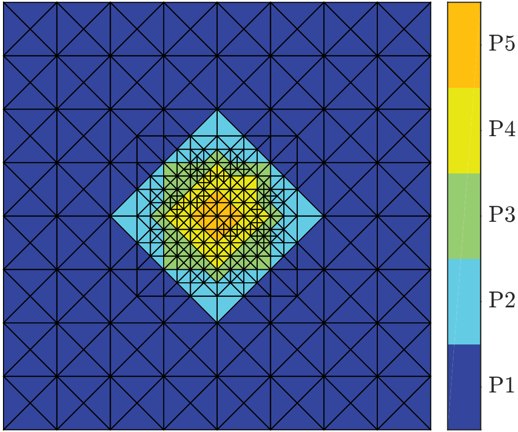

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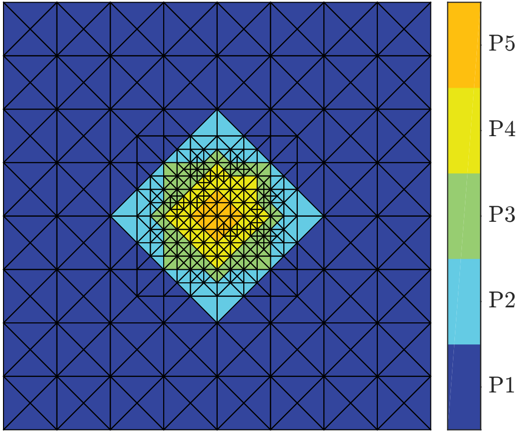

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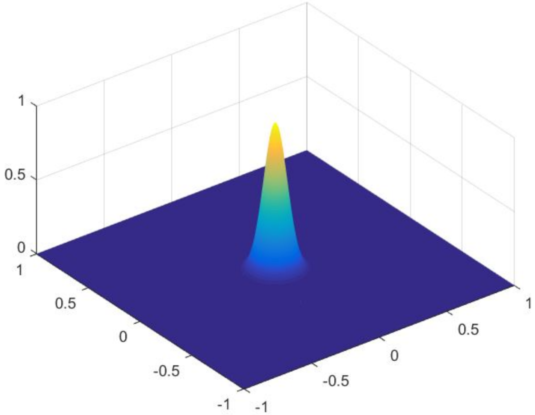
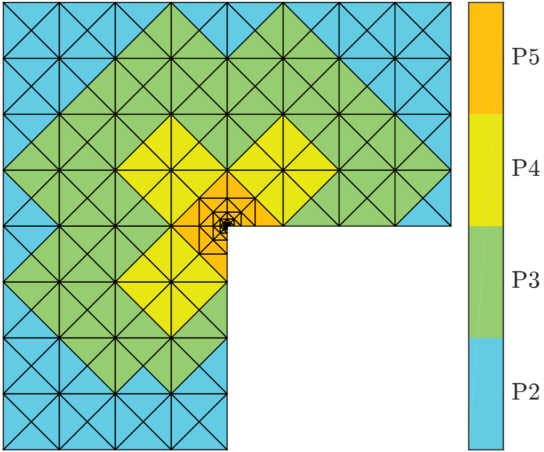


Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**singular** solution)



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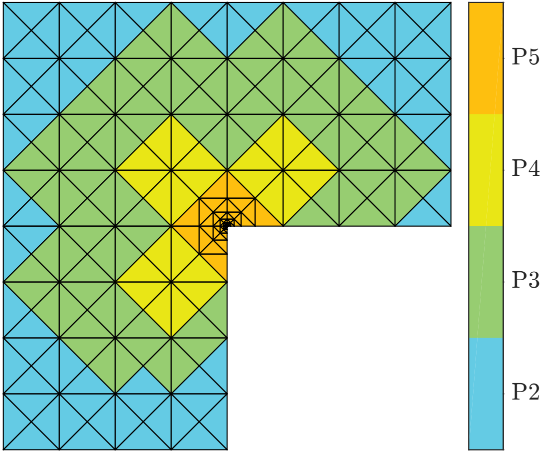

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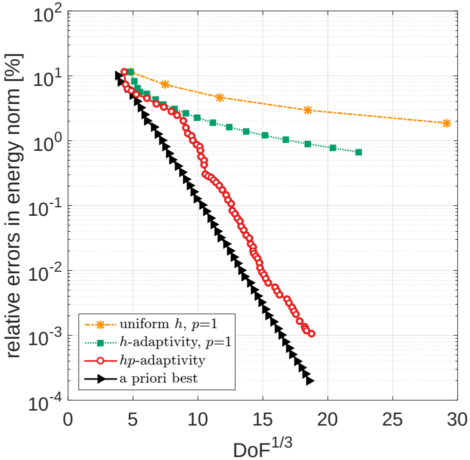

Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Adaptive mesh & polynomial degree refinement

Mesh & polynomial degree adaptivity

- **decision** between *h* or *p* refinement needs to be done
- much harder than just *h*-adaptivity

Convergence on a sequence of adaptively refined *hp* spaces V_ℓ

- $$\|\nabla(u - u_\ell)\| \to 0 \text{ for } \ell \to \infty$$
- ~~$h \searrow 0$ **uniformly**~~, ~~$p \nearrow 0$ **uniformly**~~

Optimal error decay rate wrt degrees of freedom

- for $d = 2$, *hp* refinement gives

$$\|\nabla(u - u_\ell)\| \le C_1 \frac{1}{e^{C_2 \mathrm{DoF}^{1/3}}}$$

- **exponential** convergence **rate**
- for $d = 2$ and $p = 1$ fixed, adaptive mesh *h* refinement only gives

$$\|\nabla(u - u_\ell)\| \le C \frac{1}{\mathrm{DoF}^{1/2}}$$

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