

Equivalence of local- and global-best approximations (a posteriori tools in a priori analysis)

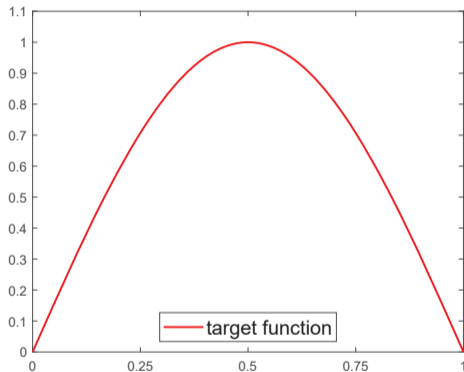
Alexandre Ern, Thirupathi Gudi, Iain Smears, **Martin Vohralík**

Inria Paris & Ecole des Ponts

Amiens, 8 juin 2021

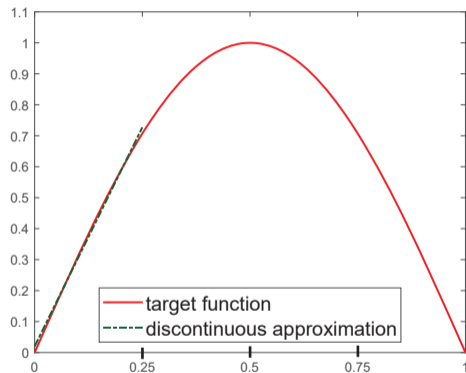


Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



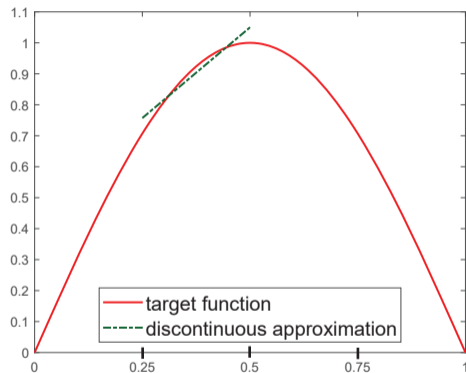
Target function

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



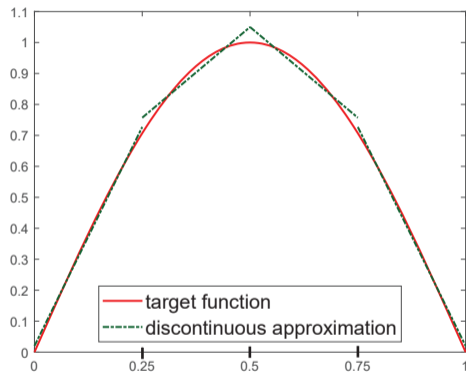
Approximation by **discontinuous**
piecewise polynomials

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



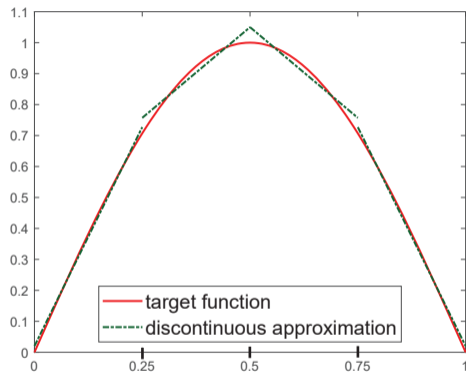
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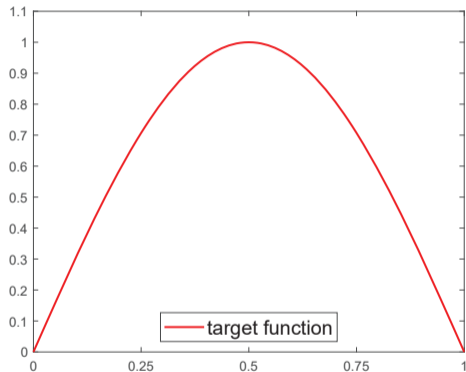


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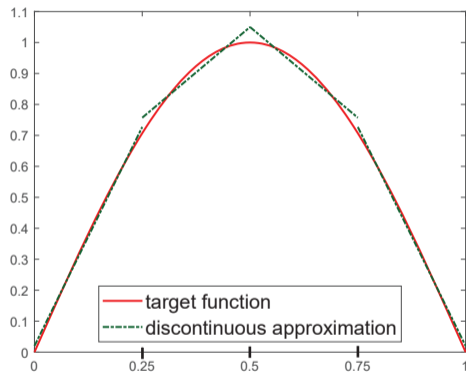


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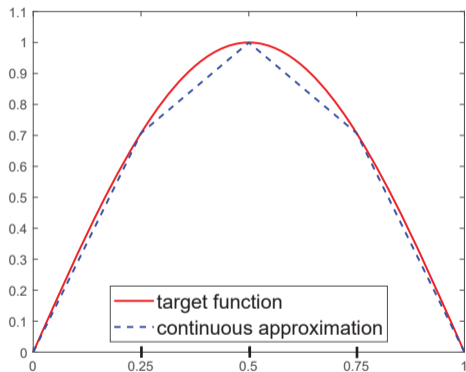


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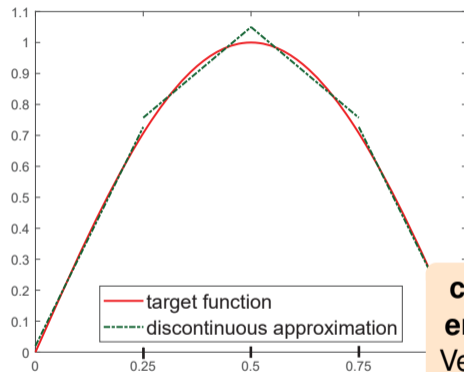


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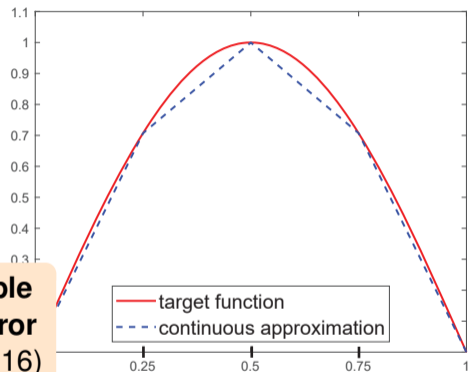


Approximation by **continuous** piecewise polynomials

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



comparable energy error
Veeseer (2016)



Approximation by **discontinuous** piecewise polynomials

Approximation by **continuous** piecewise polynomials

Outline

- 1 Introduction
- 2 Potential reconstruction
- 3 Flux reconstruction
- 4 A priori estimates
 - Global-best – local-best equivalence in H^1
 - Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
 - Stable commuting local projector in $\mathbf{H}(\text{div})$
- 5 A posteriori estimates
 - Guaranteed upper bound
 - Polynomial-degree-robust local efficiency
 - Numerical illustration
- 6 Tools (*hp*-optimality, *p*-robustness)
- 7 Conclusions and outlook

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw polynomial ▶ potential reconstruction

- a posteriori analysis of mixed and nonconforming FEs:

estimate error

- a priori analysis of conforming FEs:

global-best–local-best equivalence in approximation

approximation continuous pw pols \approx_p discontinuous pw pols

flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace

estimate error \rightarrow continuous normal trace

analysis of mixed FEs

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Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ flux reconstruction

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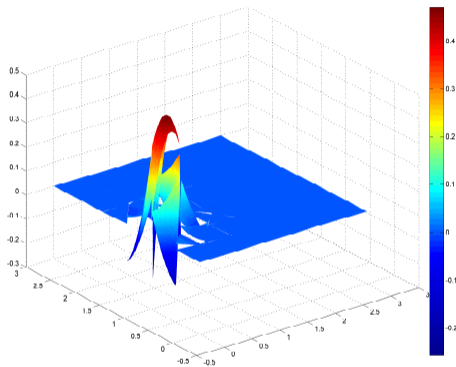
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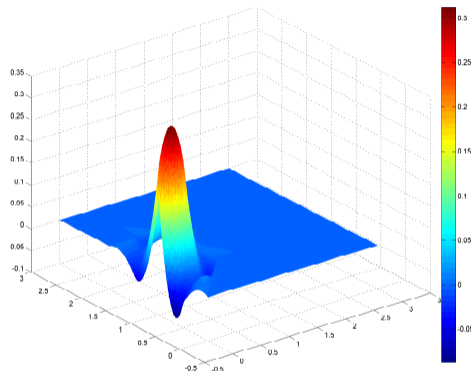
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Potential reconstruction



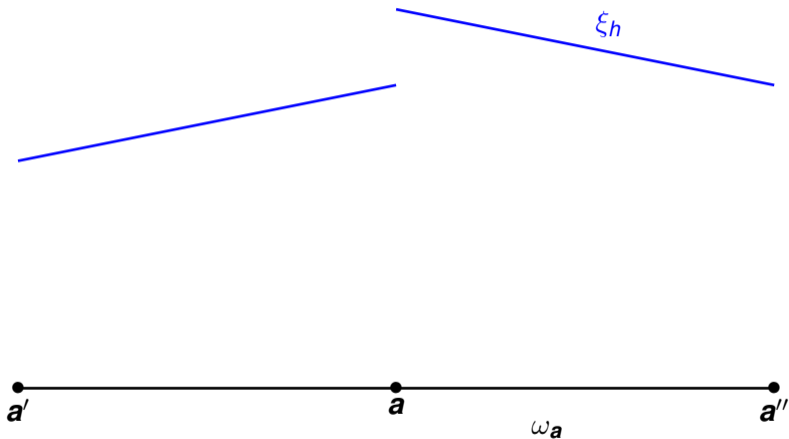
Potential ξ_h



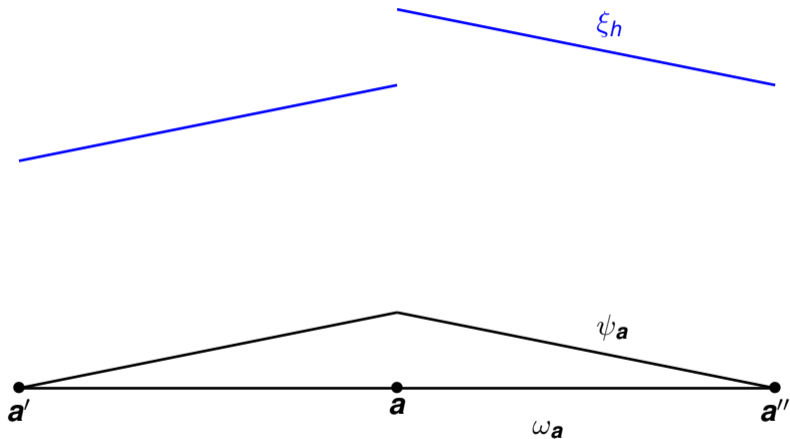
Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \rightarrow s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

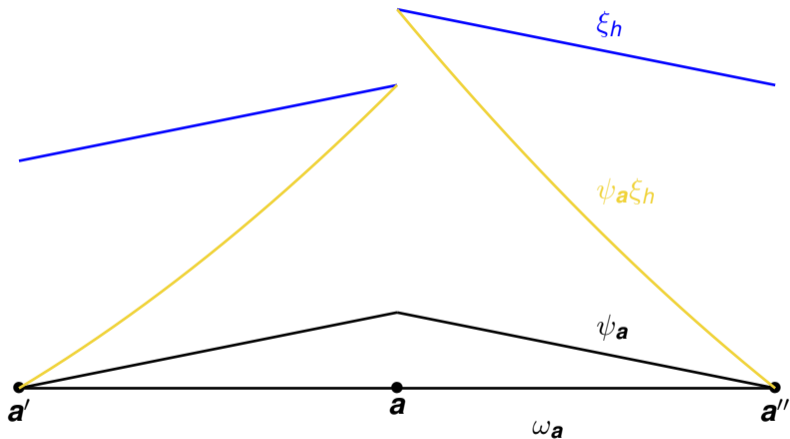
Potential reconstruction in 1D, $p = 1, p' = 2$



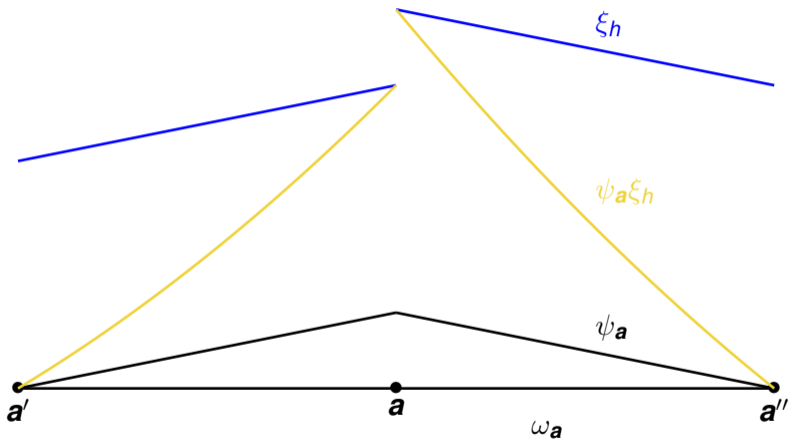
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Potential reconstruction in 1D, $p = 1, p' = 2$



Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T}), p \geq 1$

Definition (Construction of s_h Ern & V. (2015), \approx Carstensen and Merdon (2013))

For each vertex $a \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^a := \arg \min_{v_h \in V_h^a} \|\nabla_h(\psi_a \xi_h - v_h)\|_{\omega_a}$$

and combine

$$s_h = \sum_{a \in \mathcal{V}} s_h^a$$

Equivalent form: **conforming FEs**

Find $s_h^a \in V_h^a$ such that

$$(\nabla s_h^a, \nabla v_h)_{\omega_a} = (\nabla_h(\psi_a \xi_h), \nabla v_h)_{\omega_a} \quad \forall v_h \in V_h^a.$$

Key points

- localization to patches \mathcal{T}_a
- cut-off by hat basis functions ψ_a
- projection of the discontinuous $\psi_a \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_a$: $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$

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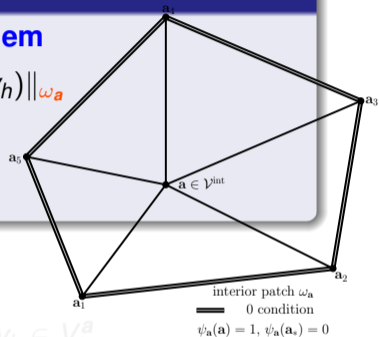
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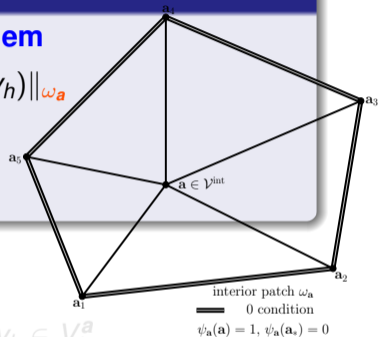
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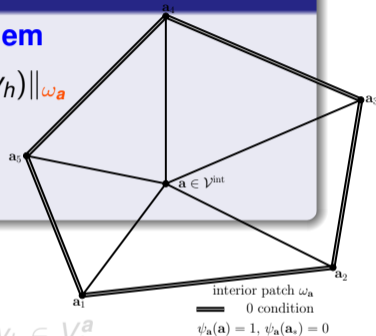
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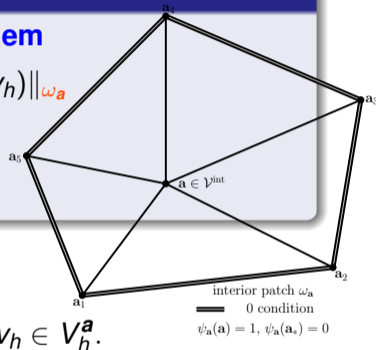
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Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h l_{p'}(\psi_{\mathbf{a}}\xi_h), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\mathbf{a}}$
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- projection of the discontinuous $\psi_{\mathbf{a}}\xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$ or $p' = p$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

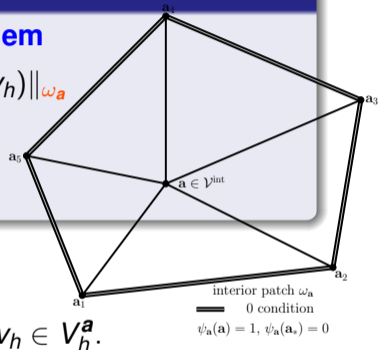
Definition (Construction of s_h Ern & V. (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(l_{p'}(\psi_{\mathbf{a}}\xi_h) - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



Equivalent form: **conforming FEs**

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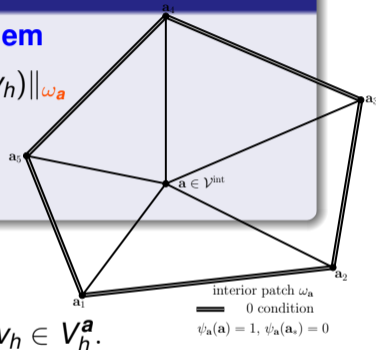
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Equivalent form: conforming FEs

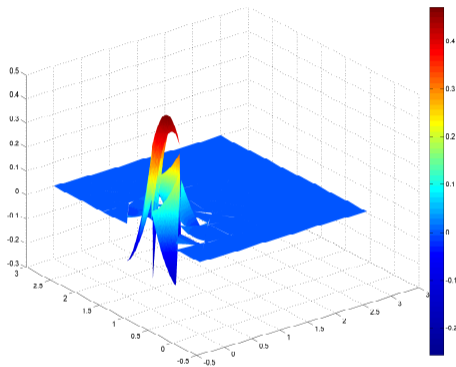
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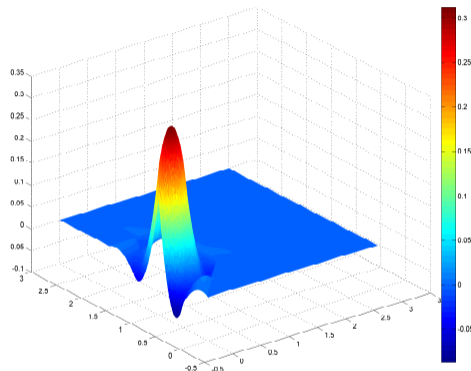
Key points

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- $p' = p + 1$ or $p' = p$

Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \rightarrow s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Stability of the potential reconstruction

Theorem (Local stability Ern & V. (2015, 2020), using Tools)

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v_h)\|_{\omega_a} \lesssim \min_{v \in H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v)\|_{\omega_a}.$$

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is *closer* to ξ_h than *any* $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F[\xi_h]\|_F^2 \right\}^{1/2}.$$

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Stability of the potential reconstruction

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s_h so good that no $u \in H_0^1(\Omega)$ can do better

Stability of the potential reconstruction

Corollary (Global stability; $p' = p$)

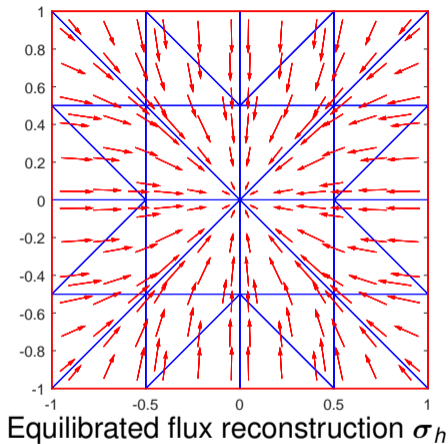
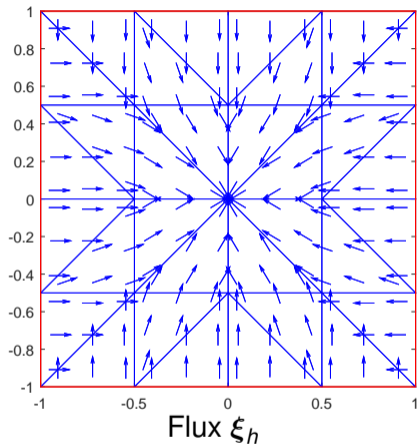
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Outline

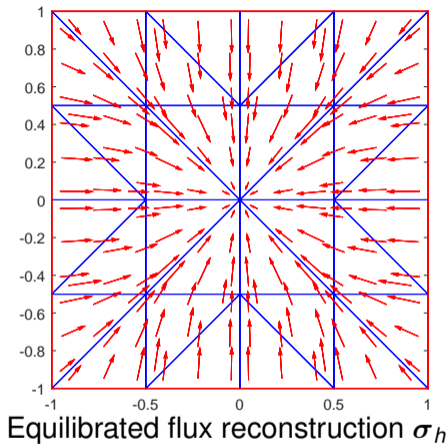
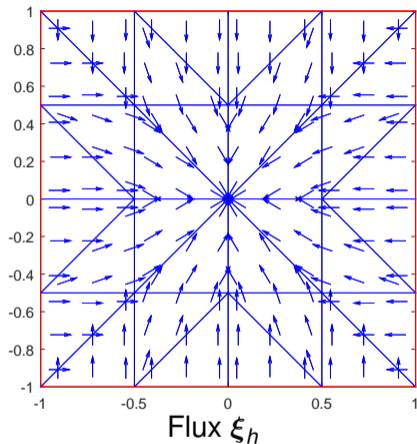
- 1 Introduction
- 2 Potential reconstruction
- 3 Flux reconstruction**
- 4 A priori estimates
 - Global-best – local-best equivalence in H^1
 - Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
 - Stable commuting local projector in $\mathbf{H}(\text{div})$
- 5 A posteriori estimates
 - Guaranteed upper bound
 - Polynomial-degree-robust local efficiency
 - Numerical illustration
- 6 Tools (*hp*-optimality, *p*-robustness)
- 7 Conclusions and outlook

Equilibrated flux reconstruction



$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}} \rightarrow \sigma_h \in \underbrace{RTN_{p'}(\mathcal{T})}_{p' = p \text{ or } p' = p+1} \cap H(\text{div}, \Omega), \nabla \cdot \sigma_h = \Pi_{p'} f$$

Equilibrated flux reconstruction



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Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds $(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$

Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^a \\ \nabla \cdot \mathbf{v}_h = 0}} \| \psi_a \xi_h - \mathbf{v}_h \|_{\omega_a}$$

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$$\sigma_h = \sum_{a \in \mathcal{V}} \sigma_h^a$$

Key points

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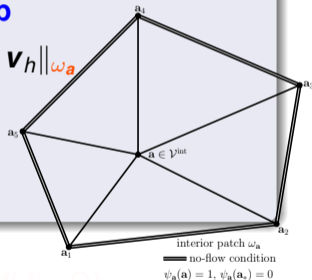
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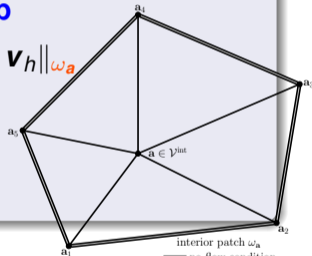
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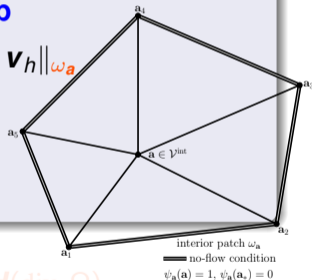
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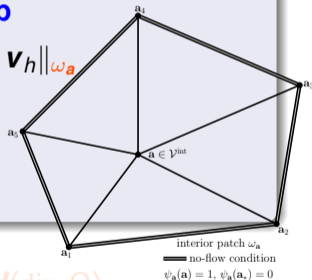
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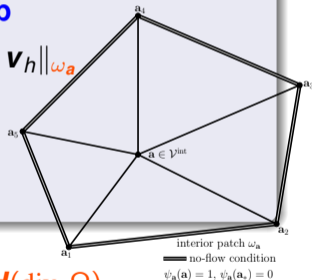
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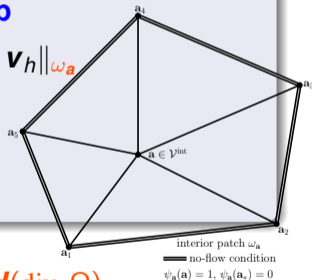
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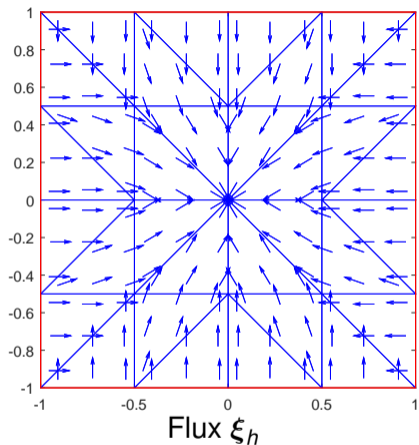
Equilibrated flux reconstruction

Equivalent form: mixed FEs

Find $(\boldsymbol{\sigma}_h^{\mathbf{a}}, \boldsymbol{\gamma}_h^{\mathbf{a}}) \in \mathbf{V}_h^{\mathbf{a}} \times \mathbb{P}_{p'}(\mathcal{T}_a)$ such that

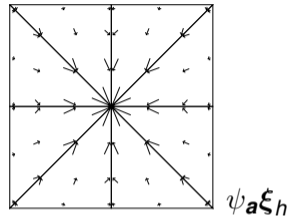
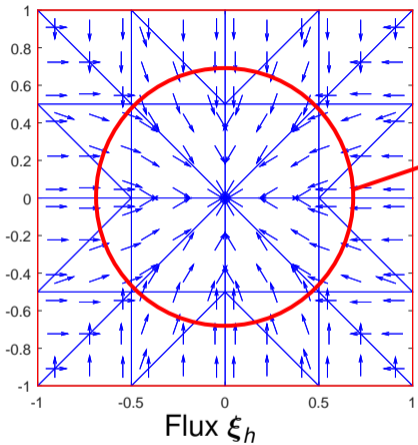
$$\begin{aligned} (\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_a} - (\boldsymbol{\gamma}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h)_{\omega_a} &= (\mathbf{I}_{p'}(\psi_{\mathbf{a}} \boldsymbol{\xi}_h), \mathbf{v}_h)_{\omega_a} & \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_a} &= (f \psi_{\mathbf{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\mathbf{a}}, q_h)_{\omega_a} & \forall q_h \in \mathbb{P}_{p'}(\mathcal{T}_a) \end{aligned}$$

Equilibrated flux reconstruction



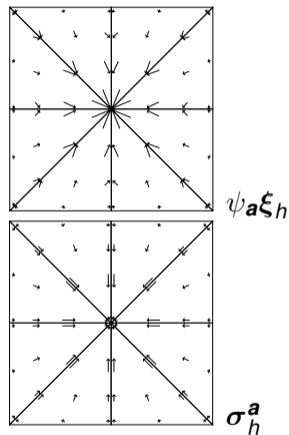
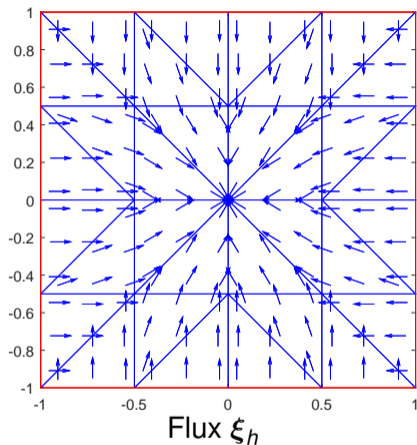
$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



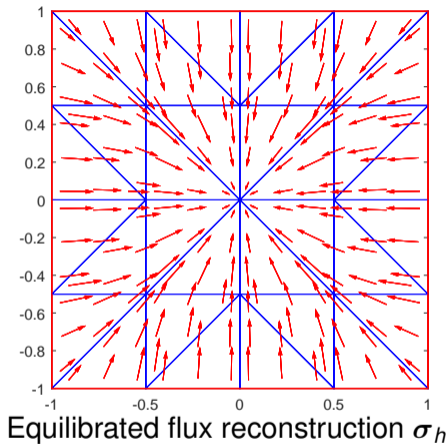
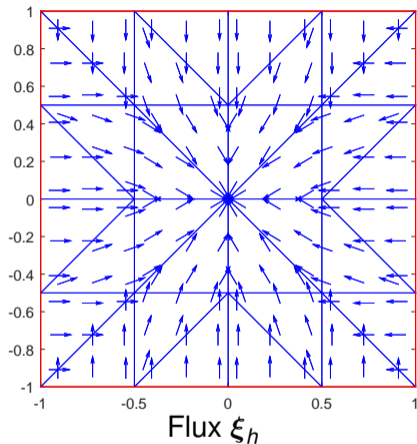
$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



$$\underbrace{\xi_h \in \mathbf{RTN}_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}^{\text{int}}} \rightarrow \sigma_h \in \underbrace{\mathbf{RTN}_{p'}(\mathcal{T})}_{p' = p \text{ or } p' = p+1} \cap \mathbf{H}(\text{div}, \Omega), \nabla \cdot \sigma_h = \Pi_{p'} f$$

Stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern & V. (2020; 3D), using Tools)

There holds

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f\psi_a + \xi_h \cdot \nabla \psi_a)}} \|\mathbf{I}_{p'}(\psi_a \xi_h) - \mathbf{v}_h\|_{\omega_a} \lesssim \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v} = \Pi_{p'}(f\psi_a + \xi_h \cdot \nabla \psi_a)}} \|\mathbf{I}_{p'}(\psi_a \xi_h) - \mathbf{v}\|_{\omega_a}.$$

Corollary (Global stability; $p' = p + 1$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

Corollary (Global stability; $p' = p$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim_p \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|f - \nabla \cdot \xi_h\|_K^2 \right\}^{1/2}.$$

Stability of the flux reconstruction

Corollary (Global stability; $p' = p + 1$)

σ_h is *closer* to ξ_h than *any* $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

σ_h so good that no $\sigma \in \mathbf{H}(\text{div}, \Omega)$ with $\nabla \cdot \sigma = f$ can do better

Stability of the flux reconstruction

Corollary (Global stability; $p' = p$)

σ_h is *closer* to ξ_h than *any* $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim_p \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|f - \nabla \cdot \xi_h\|_K^2 \right\}^{1/2}.$$

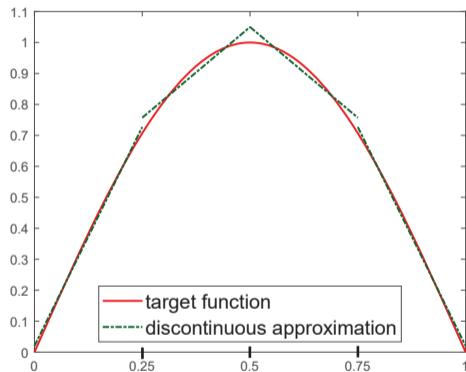
Outline

- 1 Introduction
- 2 Potential reconstruction
- 3 Flux reconstruction
- 4 A priori estimates**
 - Global-best – local-best equivalence in H^1
 - Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
 - Stable commuting local projector in $\mathbf{H}(\text{div})$
- 5 A posteriori estimates
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 - Polynomial-degree-robust local efficiency
 - Numerical illustration
- 6 Tools (*hp*-optimality, *p*-robustness)
- 7 Conclusions and outlook

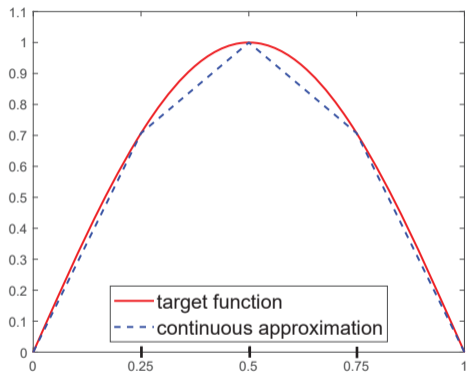
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- 1 Introduction
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 - **Global-best – local-best equivalence in H^1**
 - Constrained global-best – local-best equivalence in $H(\text{div})$
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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

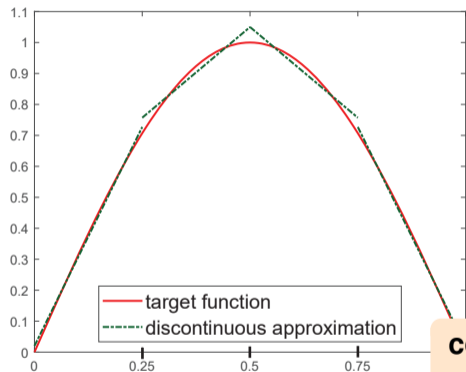


Approximation by **discontinuous** piecewise polynomials

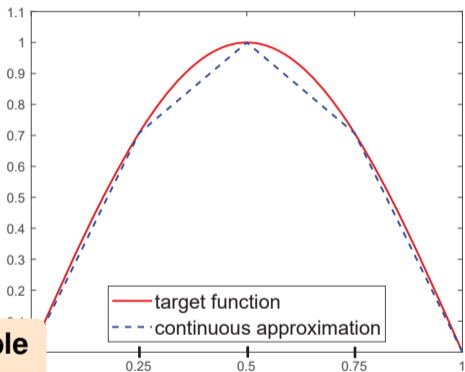


Approximation by **continuous** piecewise polynomials

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



**comparable
energy error**



Approximation by **discontinuous** piecewise polynomials

Approximation by **continuous** piecewise polynomials

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016))

bigger \approx smaller

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016))

$$\min_{\text{smaller space}} \approx \min_{\text{bigger space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016))

$$\min_{CG \text{ space}} \approx \min_{DG \text{ space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1), Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016)

Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

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- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying [Theorem 1.1](#) with $p' = p$, and using its [Corollary 1.2](#)

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016))

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Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized a priori error estimate)

$$\underbrace{\|\nabla(u - u_h)\|}^2 \leq \min_{v_h \in V_h} \|\nabla(u - v_h)\| \|\nabla(u - u_h)\|$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

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Corollary (Localized a priori error estimate)

From [Theorem 10.10](#), there holds

$$\underbrace{\|\nabla(u - u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2} \lesssim \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best approximation of } u \text{ on each } K \\ \text{no interface constraints} \\ \text{regularity only in } K \text{ counts}}} \lesssim |P|.$$

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Corollary (Localized a priori error estimate)

From $H_0^1(\Omega)$ global - local, there holds

$$\underbrace{\|\nabla(u - u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best approximation of } u \text{ on each } K \\ \text{no interface constraints} \\ \text{regularity only in } K \text{ counts}}} \lesssim_u h^p.$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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Corollary (Localized a priori error estimate)

From $\triangleright H_0^1(\Omega)$ global – local, there holds

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Conforming finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

- no interpolate
- avoids the stability of the interpolate
- holds in $H_0^1(\Omega)$ (minimal regularity)
- leads to optimal hp estimates

Corollary (Localized a priori error estimate)

From $\triangleright H_0^1(\Omega)$ global – local, there holds

$$\underbrace{\|\nabla(u - u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best approximation of } u \text{ on each } K \\ \text{no interface constraints} \\ \text{regularity only in } K \text{ counts}}} \lesssim_u h^p.$$

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hp interpolation/stable local commuting projectors

hp interpolation estimates

- Demkowicz and Buffa (2005): $\log(p)$ factors
- Bespalov and Heuer (2011): low regularity but still not $H(\text{div})$
- Ern and Guermond (2017): $H(\text{div})$ regularity but not commuting and only optimal in h
- Melenk and Rojik (2019): optimal hp approximation estimates (no $\log(p)$ factors) but higher regularity requested

Stable local commuting projectors defined on $H(\text{div})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Falk and Winther (2014): local and $H(\text{div})$ -stable but not L^2 -stable
- Ern and Guermond (2016): not local
- Licht (2019): essential boundary conditions on part of $\partial\Omega$

hp interpolation/stable local commuting projectors

hp interpolation estimates

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- Licht (2019): essential boundary conditions on part of $\partial\Omega$

Global-best approximation \approx local-best approximation in $H(\text{div})$

Theorem (Constrained equivalence in $H(\text{div})$), Ern, Gudi, Smears, & V. (2021)

bigger \approx smaller

Global-best approximation \approx local-best approximation in $H(\text{div})$

Theorem (Constrained equivalence in $H(\text{div})$), Ern, Gudi, Smears, & V. (2021)

$$\min_{\text{smaller space with constraints}} \approx \min_{\text{bigger space without constraints}}$$

Global-best approximation \approx local-best approximation in $H(\text{div})$

Theorem (Constrained equivalence in $H(\text{div})$, Ern, Gudi, Smears, & V. (2021))

$$\min_{\text{MFE space with constraints}} \approx \min_{\text{broken MFE space without constraints}}$$

Global-best approximation \approx local-best approximation in $\mathbf{H}(\text{div})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2021))

Let $\sigma \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \text{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \sigma)}} \|\sigma - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \sigma - \Pi_p \nabla \cdot \sigma\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\left[\min_{\mathbf{v}_h \in \text{RTN}_p(K)} \|\sigma - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \sigma - \Pi_p \nabla \cdot \sigma\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- \approx_p : only depends on d , shape-regularity of \mathcal{T} , and p
- proof using $\text{RTN}_p(K)$ with $p' = p$ & $\text{RTN}_p(K)$

Global-best approximation \approx local-best approximation in $\mathbf{H}(\text{div})$

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global-best on Ω
 normal trace-continuity constraint
 divergence constraint
 MFE space (much smaller)

$$\approx_p \sum_{K \in \mathcal{T}} \left[\min_{\mathbf{v}_h \in \mathbf{RTN}_p(K)} \|\sigma - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \sigma - \Pi_p \nabla \cdot \sigma\|_K^2 \right].$$

local-best on each K
 no normal trace-continuity constraint
 no divergence constraint
 broken MFE space (much bigger)

- \approx_p : only depends on d , shape-regularity of \mathcal{T} , and p
- proof using Flux reconstruction with $p' = p$ & $\mathbf{H}(\text{div})$ stability

Global-best approximation \approx local-best approximation in $H(\text{div})$

Theorem (Constrained equivalence in $H(\text{div})$, Ern, Gudi, Smears, & V. (2021))

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Theorem (Constrained equivalence in $H(\text{div})$, Ern, Gudi, Smears, & V. (2021))

Let $\sigma \in H(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \text{RTN}_p(\mathcal{T}) \cap H(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \sigma)}} \|\sigma - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \sigma - \Pi_p \nabla \cdot \sigma\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\left[\min_{\mathbf{v}_h \in \text{RTN}_p(K)} \|\sigma - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \sigma - \Pi_p \nabla \cdot \sigma\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- \approx_p : only depends on d , shape-regularity of \mathcal{T} , and p
- proof using flux reconstruction with $p' = p$ & $H(\text{div})$ stability

Optimal hp approximation estimate

Theorem (Localized hp approximation, Ern, Gudi, Smears, & V. (2021))

For any $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ s.t., *locally* on all $K \in \mathcal{T}$,

$$\mathbf{v}|_K \in \mathbf{H}^s(K), \quad s > 0,$$

there holds

$$\min_{\substack{\mathbf{v}_h \in \text{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v}\|_K^2 & \text{if } s < 1, \\ \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 & \text{if } s \geq 1. \end{cases}$$

- \lesssim_s : only depends on d , shape-regularity of \mathcal{T} , and s
- $\mathbf{H}(\text{div})$ stability of flux reconstruction with $p' = p$ & $p' = p + 1$
- **fully optimal** hp approximation estimate (minimal **elementwise** regularity, **no** logarithmic factor in p)

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Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\boldsymbol{\sigma}, u) \in \mathbf{H}(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) &= 0 & \forall \mathbf{v} \in \mathbf{H}(\text{div}, \Omega), \\ (\nabla \cdot \boldsymbol{\sigma}, q) &= (f, q) & \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h := \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\boldsymbol{\sigma}_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \boldsymbol{\sigma}_h, q_h) &= (f, q_h) & \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal hp a priori error estimate, Ern, Gudi, Smears, & V. (2021))

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \mathbf{v}_h\|$$

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From \mathbf{V}_h , there holds

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \mathbf{v}_h\| \lesssim \frac{h^{\min(s, p+1)}}{(p+1)^s}$$

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From $\mathbf{H}(\text{div}, \Omega)$ hp approximation, there holds

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$$(\sigma, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) = 0$$

$$(\nabla \cdot \sigma, q) = (f, q)$$

- no interpolate
- avoids the stability of the interpolate
- holds in $H(\text{div}, \Omega)$ (minimal regularity)
- leads to optimal hp estimates

Mixed finite elements

Find $(\sigma_h, u_h) \in \mathbf{V}_h := \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

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Stable local commuting projector in $H(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2021))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \boldsymbol{\sigma}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) =$ *flux reconstruction* of $\boldsymbol{\xi}_h|_K := \arg \min_{\mathbf{v}_h \in \mathbf{RTN}_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$ with $p' = p$ is locally defined,

$$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v}) \quad \text{commuting,}$$

$$P_p \mathbf{v} = \mathbf{v} \text{ if } \mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \quad \text{projector,}$$

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Comments

- P_p defined on the entire $\mathbf{H}(\text{div}, \Omega)$ (no additional regularity)
- \lesssim_p : only depends on d , shape-regularity of \mathcal{T} , and p
- $h_K \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K / (p+1)$: data oscillation term, disappears when $\nabla \cdot \mathbf{v}$ is a piecewise p -degree polynomial

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Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed a posteriori error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), V. (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_a)_{\omega_a} = (f, \psi_a)_{\omega_a} \quad \forall a \in \mathcal{V}^{\text{int}};$$
- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ *potential reconstruction*;
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ *stress constraint*.

Then

$$\|\nabla_h(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

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Theorem (A guaranteed a posteriori error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), V. (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}^{\text{int}};$$
- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ potential reconstruction;
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ flux reconstruction.

Then

$$\begin{aligned} \|\nabla_h(u - u_h)\|^2 &\leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 \\ &\quad + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}. \end{aligned}$$

Outline

- 1 Introduction
- 2 Potential reconstruction
- 3 Flux reconstruction
- 4 A priori estimates
 - Global-best – local-best equivalence in H^1
 - Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
 - Stable commuting local projector in $\mathbf{H}(\text{div})$
- 5 A posteriori estimates**
 - Guaranteed upper bound
 - Polynomial-degree-robust local efficiency**
 - Numerical illustration
- 6 Tools (*hp*-optimality, *p*-robustness)
- 7 Conclusions and outlook

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity Braess, Pillwein, and Schöberl (2009), Ern & V. (2015, 2020))

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F \llbracket u_h \rrbracket\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \sigma_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of H^1 stability and $H(\text{div})$ stability with $p' = p + 1$
- p -robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

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Remarks

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How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$p^* = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{-1}\%$	1.1	$2.4 \times 10^{-1}\%$	1.17
$\approx h_0/2$	2	0.3×10^{-1}	$1.1 \times 10^{-1}\%$	0.5	$0.4 \times 10^{-1}\%$	1.17
$\approx h_0/4$	3	0.1×10^{-1}	$0.4 \times 10^{-1}\%$	0.25	$0.2 \times 10^{-1}\%$	1.17
$\approx h_0/8$	4	0.03×10^{-1}	$0.1 \times 10^{-1}\%$	0.125	$0.1 \times 10^{-1}\%$	1.17
$\approx h_0/2$	2	0.3×10^{-1}	$1.1 \times 10^{-1}\%$	0.5	$0.4 \times 10^{-1}\%$	1.17
$\approx h_0/4$	3	0.1×10^{-1}	$0.4 \times 10^{-1}\%$	0.25	$0.2 \times 10^{-1}\%$	1.17
$\approx h_0/8$	4	0.03×10^{-1}	$0.1 \times 10^{-1}\%$	0.125	$0.1 \times 10^{-1}\%$	1.17

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
 V. Daligault, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$\rho^H = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.1 \times 10^1\%$	5.8×10^{-1}	$5.1 \times 10^0\%$	1.15
$\approx h_0/4$		3.1×10^{-1}	$5.7 \times 10^0\%$	2.9×10^{-1}	$2.6 \times 10^0\%$	1.12
$\approx h_0/8$		1.5×10^{-1}	$2.8 \times 10^0\%$	1.4×10^{-1}	$1.3 \times 10^0\%$	1.10
$\approx h_0/2$	2	4.2×10^{-2}	$1.5 \times 10^0\%$	4.1×10^{-2}	$3.7 \times 10^{-2}\%$	1.08
$\approx h_0/4$	3	2.6×10^{-2}	$9.4 \times 10^{-1}\%$	2.6×10^{-2}	$2.4 \times 10^{-2}\%$	1.07
$\approx h_0/8$	4	2.6×10^{-2}	$9.4 \times 10^{-1}\%$	2.6×10^{-2}	$2.4 \times 10^{-2}\%$	1.07

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 V. Doležal, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$\rho^{eff} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.8×10^{-1}	$3 \times 10^0\%$	
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$			
$\approx h_0/4$	3	2.6×10^{-2}	$5.9 \times 10^{-1}\%$			
$\approx h_0/8$	4	2.6×10^{-2}	$5.9 \times 10^{-1}\%$			

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How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{eff} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.25
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.5%	1.35
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.45
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$1.0 \times 10^0\%$	1.55
$\approx h_0/4$	3	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$6.5 \times 10^{-1}\%$	1.65
$\approx h_0/8$	4	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$6.5 \times 10^{-1}\%$	1.75

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 V. Doležal, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$j^{eff} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.05
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.03
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.02
$\approx h_0/4$	3	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	1.01
$\approx h_0/8$	4	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$5.8 \times 10^{-1}\%$	1.01

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h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	1.01
$\approx h_0/8$	4	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$5.8 \times 10^{-1}\%$	1.01

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$\approx h_0/4$	3	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	2.6×10^{-2}	$5.9 \times 10^{-1}\%$	1.01
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$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

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V. Dolejší, A. Ern, M. Vohralik, SIAM Journal on Scientific Computing (2016)

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$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralik, SIAM Journal on Numerical Analysis (2015)

V. Dolejší, A. Ern, M. Vohralik, SIAM Journal on Scientific Computing (2016)

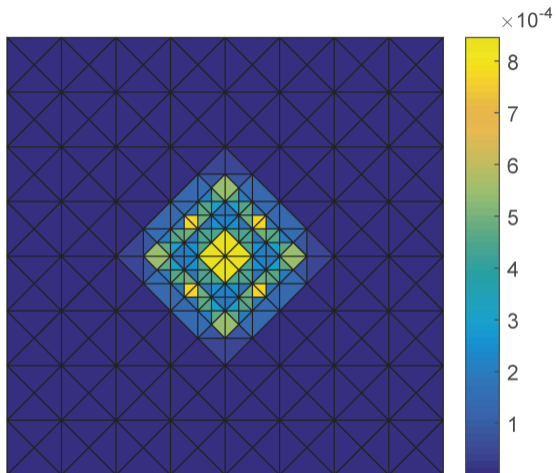
How large is the overall error? (model pb, known smooth solution)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{eff} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
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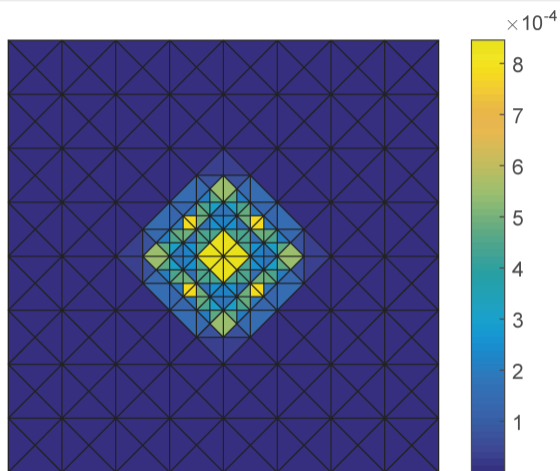
A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

Where (in space) is the error localized?



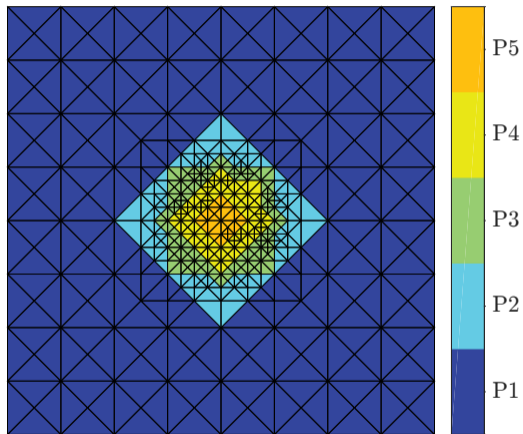
Estimated error distribution $\eta_K(u_h)$



Exact error distribution $\|\nabla(u - u_h)\|_K$

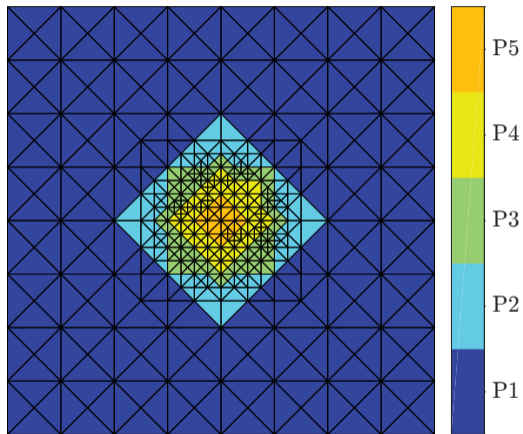
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**smooth** solution)

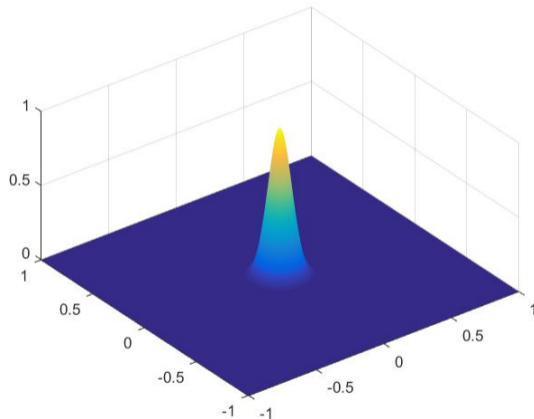


Mesh \mathcal{T}_ℓ and pol. degrees p_K

Can we decrease the error efficiently? *hp* adaptivity, (**smooth** solution)



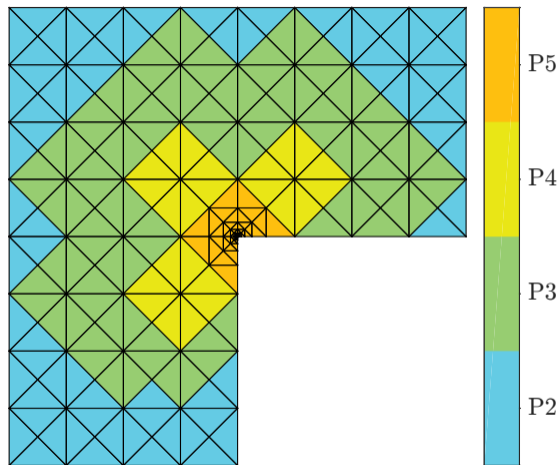
Mesh \mathcal{T}_ℓ and pol. degrees p_K



Exact solution

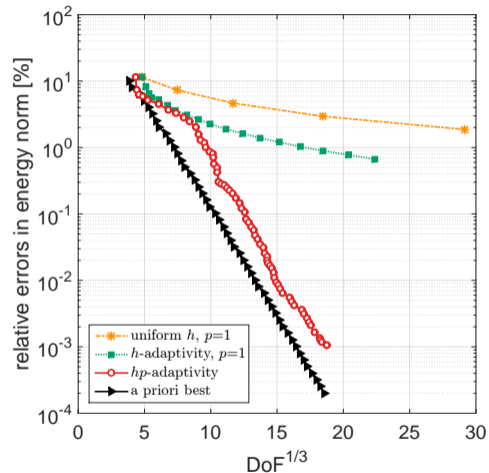
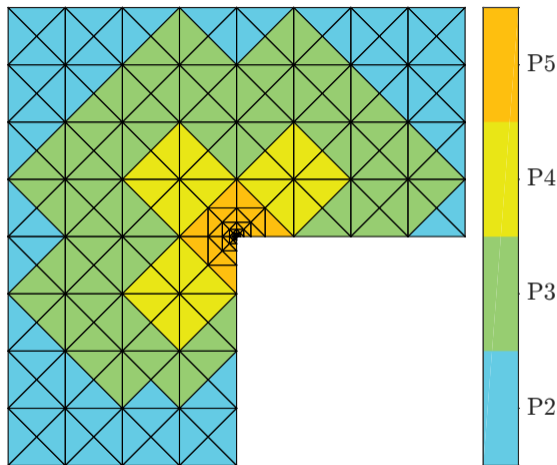
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**singular** solution)



Mesh \mathcal{T}_ℓ and polynomial degrees p_K

Can we decrease the error efficiently? *hp* adaptivity, (singular solution)



Mesh \mathcal{T}_ℓ and polynomial degrees p_K

Relative error as a function of DoF

Outline

- 1 Introduction
- 2 Potential reconstruction
- 3 Flux reconstruction
- 4 A priori estimates
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 - Stable commuting local projector in $\mathbf{H}(\text{div})$
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 - Guaranteed upper bound
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- 6 **Tools (*hp*-optimality, *p*-robustness)**
- 7 Conclusions and outlook

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^D \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^D)$ be continuous on \mathcal{F}_K^D . Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} .$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^D, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^D. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \stackrel{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials: patch

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2020))

For $p \geq 1$ and $\mathbf{a} \in \mathcal{V}^{\text{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_\mathbf{a}^{\text{int}})$. Suppose the *compatibility*

$$\begin{aligned} r_F|_{F \cap \partial \omega_\mathbf{a}} &= 0 & \forall F \in \mathcal{F}_\mathbf{a}^{\text{int}}, \\ \sum_{F \in \mathcal{F}_e} \iota_{F,e} r_F|_e &= 0 & \forall e \in \mathcal{E}_\mathbf{a}. \end{aligned}$$

Then

$$\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_\mathbf{a}) \\ v_h=0 \ \forall F \in \mathcal{F}_\mathbf{a}^{\text{ext}} \\ \llbracket v_h \rrbracket = r_F \ \forall F \in \mathcal{F}_\mathbf{a}^{\text{int}}}} \|\nabla_h v_h\|_{\omega_\mathbf{a}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_\mathbf{a}) \\ v=0 \ \forall F \in \mathcal{F}_\mathbf{a}^{\text{ext}} \\ \llbracket v \rrbracket = r_F \ \forall F \in \mathcal{F}_\mathbf{a}^{\text{int}}}} \|\nabla_h v\|_{\omega_\mathbf{a}}.$$

Fluxes: one element

Lemma ($\mathbf{H}(\text{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2020)

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^N \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^N) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, \mathbf{1})_F = (r_K, \mathbf{1})_K$ if $\mathcal{F}_K^N = \mathcal{F}_K$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \lesssim \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K .$$

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^N, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^N. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

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$$\sum_{K \in \mathcal{T}_\mathbf{a}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_\mathbf{a}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}_\mathbf{a}) \\ \mathbf{v}_h \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_\mathbf{a}^{\text{ext}} \\ \llbracket \mathbf{v}_h \cdot \mathbf{n}_F \rrbracket = r_F \quad \forall F \in \mathcal{F}_\mathbf{a}^{\text{int}} \\ \nabla_h \cdot \mathbf{v}_h|_K = r_K \quad \forall K \in \mathcal{T}_\mathbf{a}}} \|\mathbf{v}_h\|_{\omega_\mathbf{a}} \lesssim \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, \mathcal{T}_\mathbf{a}) \\ \mathbf{v} \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_\mathbf{a}^{\text{ext}} \\ \llbracket \mathbf{v} \cdot \mathbf{n}_F \rrbracket = r_F \quad \forall F \in \mathcal{F}_\mathbf{a}^{\text{int}} \\ \nabla_h \cdot \mathbf{v}|_K = r_K \quad \forall K \in \mathcal{T}_\mathbf{a}}} \|\mathbf{v}\|_{\omega_\mathbf{a}}.$$

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Conclusions and outlook

Conclusions

- simple proof of **global-best – local-best equivalence** in H^1
- **global-best – local-best equivalence** in $\mathbf{H}(\text{div})$, removing constraints
- incidentally leads to **stable local commuting projectors**
- **optimal hp** a priori error estimates
- **elementwise localized** a priori error estimates **under minimal regularity**
- **p -robust** a posteriori error estimates (**unified framework** for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, and others carried out

Ongoing work

- extensions to other settings

Conclusions and outlook





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Merci de votre attention !