Estimations d'erreur *a priori* et *a posteriori* localisées sous régularité minimale

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en collaboration avec

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informatics mathematics

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Outline

Localization

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, open polygon/polyhedron
- $\mathcal{T}$ simplicial mesh of $\overline{\Omega}$, $\mathcal{V}$ set of vertices, $\omega_{\mathbf{a}}$ vertex patch
- $H^1(\mathcal{T})$ broken Sobolev space, ∇_h elementwise gradient

Localization

- localization of integral norms: for all $v \in L^2(\Omega)$

$$\|v\|^2 = \sum_{K \in \mathcal{T}} \|v\|_K^2$$

- localization of **dual norms**: for all $\mathcal{R} \in H^{-1}(\Omega)$

$$\|\mathcal{R}\|_{H^{-1}(\Omega)}^2 \approx \sum_{\mathbf{a} \in \mathcal{V}} \|\mathcal{R}\|_{H^{-1}(\omega_{\mathbf{a}})}^2$$

- localization of **distance** to $H_0^1(\Omega)$: for all $v \in H^1(\mathcal{T})$

$$\min_{\zeta \in H_0^1(\Omega)} \|\nabla_h(v - \zeta)\|^2 \approx \sum_{\mathbf{a} \in \mathcal{V}} \min_{\zeta \in H_{\partial\Omega}^1(\omega_{\mathbf{a}})} \|\nabla_h(v - \zeta)\|_{\omega_{\mathbf{a}}}^2$$

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A priori error estimates (appr. $u_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$ of u)

- **elementwise localized estimates**:

$$\|\nabla(u - u_h)\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{(\text{best approximation of } u \text{ on } K \text{ in } \|\nabla(\cdot)\|)^2}_{\substack{\text{no interface constraints} \\ \text{regularity only in } K \text{ counts} \\ \text{alternative to (quasi-)interpolation operators}}}$$

- Ciarlet (1978), Brenner & Scott (1994), Ern & Guermond (2004) …
- $\lesssim_p$: only depends on d, shape-regularity of $\mathcal{T}$, and p

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A posteriori error estimates (appr. $u_h \in \mathbb{P}_p(\mathcal{T})$ of u)

- no constant, **guaranteed upper bound**, fully computable:

$$\|\nabla_h(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}} \eta_K(u_h)^2$$

- local efficiency, **data-** / **polynomial-degree-robustness**:

$$\eta_K(u_h) \lesssim \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla_h(u - u_h)\|^2_{\omega_{\mathbf{a}}} \qquad \forall K \in \mathcal{T}$$

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Minimal regularity: $H_0^1(\Omega)$, $\boldsymbol{H}(\mathrm{div}, \Omega)$, $W_0^{1,\alpha}(\Omega)$.

Outline

1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (p-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: hp-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
   - Equilibrated flux reconstruction
7. Conclusions and outlook

Outline

Global-best approx. ≈ local-best approx., H^1

Theorem (Equivalence in H^1, Veeser (2016))

Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint}}}.$$

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Proof via potential reconstruction

- define discontinuous $\xi_h \in \mathbb{P}_p(\mathcal{T})$ by

$$\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K, \quad (\xi_h, 1)_K = (u, 1)_K \quad \forall K \in \mathcal{T}$$

- ξ_h: potential reconstruction $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)$
- global H^1 stability ($p' = p$), jump term efficiency + mean ξ_h

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{\sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2\right\}^{1/2} \lesssim \|\nabla_h(u - \xi_h)\|$$

- bound on minimum, triangle inequality

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \le \|\nabla(u - s_h)\| \le \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

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Global-best approx. ≈ local-best approx., $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint}}}$$

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Proof via flux reconstruction

- define discontinuous $\xi_h \in \boldsymbol{RTN}_p(\mathcal{T})$ as elementwise $[\boldsymbol{L}^2]^d$-orthogonal projection of $\boldsymbol{\sigma}$

$$\xi_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}$$

- since $\nabla \psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\forall \boldsymbol{a} \in \mathcal{V}_K$,

$$(\boldsymbol{\sigma} - \xi_h, \nabla \psi_{\boldsymbol{a}})_K = 0 \qquad \forall K \in \mathcal{T}$$

- as $\boldsymbol{\sigma}|_{\omega_{\boldsymbol{a}}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ ($\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$)
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with $f := \nabla\cdot\boldsymbol{\sigma}$

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- as $\boldsymbol{\sigma}|_{\omega_{\boldsymbol{a}}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ ($\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$) $\Rightarrow$ $\psi_{\boldsymbol{a}}$-orthogonality

$$(\boldsymbol{\sigma}, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = -(\nabla \cdot \boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \;\Rightarrow\; \boxed{(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\xi_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}}$$

with $f := \nabla \cdot \boldsymbol{\sigma}$

- ξ_h, f: flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

Proof via flux reconstruction

- define discontinuous $\xi_h \in \boldsymbol{RTN}_p(\mathcal{T})$ as elementwise $[\boldsymbol{L}^2]^d$-orthogonal projection of $\boldsymbol{\sigma}$

$$\xi_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\forall \boldsymbol{a} \in \mathcal{V}_K$,

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- ξ_h, f: ▸ flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

Proof continuation

- global ▸ $\boldsymbol{H}(\mathrm{div})$ stability ($p' = p$)

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2 \right\}^{1/2}$$

- bound on minimum, triangle inequality

$$\min_{\substack{\boldsymbol{v}_h\in \boldsymbol{RTN}_p(\mathcal{T})\cap\boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|$$

$$\lesssim_p \left\{ \sum_{K\in\mathcal{T}} \left[\|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\|_K^2 + h_K^2\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2\right] \right\}^{1/2}$$

- $[L^2(K)]^d$-orthogonal projection consequence

$$\|\boldsymbol{\sigma}-\boldsymbol{\xi}_h\|_K^2 + h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\xi}_h)\|_K^2 \lesssim_p \min_{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2 + h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{v}_h)\|_K^2\right]$$

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$$\sum_{K\in\mathcal{T}} h_K^2\|\nabla\cdot\boldsymbol{\sigma}-\Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2 \le \sum_{K\in\mathcal{T}} \min_{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2 + h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{v}_h)\|_K^2\right]$$

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Outline

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2. Global-best – local-best equivalences
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   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (*p*-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: *hp*-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
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7. Conclusions and outlook

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Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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Theorem (Localized *a priori* estimate)

From $H_0^1(\Omega)$ global – local, *there holds*

$$\underbrace{\|\nabla(u-u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u-v_h)\|^2} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u-v_h)\|_K^2}.$$

local-best approximation of u on each K
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Outline

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{v}) - (u, \nabla\cdot\boldsymbol{v}) &= 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega),\\
(\nabla\cdot\boldsymbol{\sigma}, q) &= (f, q) \quad \forall q \in L^2(\Omega)
\end{aligned}
$$

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Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

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Outline

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, Ern, Gudi, Smears, & V. (2018)**)**

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *from* ▸ construction *is locally defined,*

$$\Pi_p(\nabla\cdot\boldsymbol{\sigma}) = \nabla\cdot(P_p\boldsymbol{\sigma}) \quad \textit{commuting},$$

$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \quad \textit{projector},$$

$$\|P_p\boldsymbol{\sigma}\| \lesssim \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2\right\}^{1/2} \quad \textit{stable}.$$

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Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *from* construction *is locally defined,*

$\Pi_p(\nabla{\cdot}\boldsymbol{\sigma}) = \nabla{\cdot}(P_p\boldsymbol{\sigma})$ *commuting,*

$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma}$ *if* $\boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *projector,*

$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K^2\right\}^{1/2} \quad \textit{stable.}$$

1. $\nabla{\cdot}\boldsymbol{\sigma}_h = \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})$ by construction
2. $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from construction, global $\boldsymbol{H}(\mathrm{div})$ stability $(p' = p)$
$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\xi}_h = \boldsymbol{\sigma}$$
3. triangle inequality $\|\boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\boldsymbol{\sigma}\|$ and stability
$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim_p \left\{\sum_{K\in\mathcal{T}} \left[\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K^2\right]\right\}^{1/2}$$

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, Ern, Gudi, Smears, & V. (2018)**)**

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ *from* ▸ construction *is locally defined,*

$$\Pi_p(\nabla\cdot\boldsymbol{\sigma}) = \nabla\cdot(P_p\boldsymbol{\sigma}) \quad \textit{commuting},$$

$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \quad \textit{projector},$$

$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2\right\}^{1/2} \quad \textit{stable}.$$

1. $\nabla\cdot\boldsymbol{\sigma}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})$ by construction
2. $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from ▸ construction, global ▸ $\boldsymbol{H}(\mathrm{div})$ stability $(p' = p)$

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\xi}_h = \boldsymbol{\sigma}$$

3. triangle inequality $\|\boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\boldsymbol{\sigma}\|$ and stability

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \lesssim_p \left\{\sum_{K\in\mathcal{T}} \left[\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2\right]\right\}^{1/2}$$

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$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\xi}_h = \boldsymbol{\sigma}$$

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Outline

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2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (*p*-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: *hp*-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
   - Equilibrated flux reconstruction
7. Conclusions and outlook

Localization of dual norms on $W_0^{1,\alpha}$ $(1/\beta := 1 - 1/\alpha)$

Theorem (Dual norms localization, Babuška & Miller (1987), Blechta, Málek, & V. (2018))

Let $\mathcal{R} \in [W_0^{1,\alpha}(\Omega)]'$, $1 \le \alpha \le \infty$, *be arbitrary subject to*

$$\underbrace{\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle = 0}_{\textit{lowest-modes orthogonality}} \qquad \forall \boldsymbol{a} \in \mathcal{V}^{\rm int}.$$

Then, when $1 < \alpha \le \infty$,

$$\underbrace{\|\mathcal{R}\|_{[W_0^{1,\alpha}(\Omega)]'}}_{\sup_{v \in W_0^{1,\alpha}(\Omega);\, \|\nabla v\|_\alpha = 1} \langle \mathcal{R}, v\rangle} \approx \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}} \underbrace{\|\mathcal{R}\|^\beta_{[W_0^{1,\alpha}(\omega_{\boldsymbol{a}})]'}}_{\sup_{v \in W_0^{1,\alpha}(\omega_{\boldsymbol{a}});\, \|\nabla v\|_{\alpha,\omega_{\boldsymbol{a}}} = 1} \langle \mathcal{R}, v\rangle} \right\}^{1/\beta},$$

and, when $\alpha = 1$,

$$\|\mathcal{R}\|_{[W_0^{1,\alpha}(\Omega)]'} \approx \max_{\boldsymbol{a} \in \mathcal{V}} \|\mathcal{R}\|_{[W_0^{1,\alpha}(\omega_{\boldsymbol{a}})]'}.$$

Condition $\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle = 0$ *is only needed in the left inequalities.*

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Proof ($\lesssim$): partition of unity & Poincaré–Friedrichs in.

- fix $v \in W_0^{1,\alpha}(\Omega)$ with $\|\nabla v\|_\alpha = 1$ $(1 < \alpha < \infty)$
- partition of unity $\sum_{\mathbf{a}\in\mathcal{V}} \psi_{\mathbf{a}} = 1$, linearity of $\mathcal{R}$, orthogonality:

$$\langle \mathcal{R}, v\rangle = \sum_{\mathbf{a}\in\mathcal{V}} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle = \sum_{\mathbf{a}\in\mathcal{V}^{\mathrm{int}}} \langle \mathcal{R}, \underbrace{\psi_{\mathbf{a}}(v - \underbrace{\Pi_{0,\omega_{\mathbf{a}}} v}_{\text{mean value}})}_{\in W_0^{1,\alpha}(\omega_{\mathbf{a}})}\rangle + \sum_{\mathbf{a}\in\mathcal{V}^{\mathrm{ext}}} \langle \mathcal{R}, \underbrace{\psi_{\mathbf{a}} v}_{\in W_0^{1,\alpha}(\omega_{\mathbf{a}})}\rangle$$

- $w \in W^{1,\alpha}(\omega_{\mathbf{a}})$ with mean value 0 on $\omega_{\mathbf{a}}$ or 0 on part of $\partial\omega_{\mathbf{a}}$:

$$\begin{aligned} \|\nabla(\psi_{\mathbf{a}} w)\|_{\alpha,\omega_{\mathbf{a}}} &= \|\nabla\psi_{\mathbf{a}} w + \psi_{\mathbf{a}}\nabla w\|_{\alpha,\omega_{\mathbf{a}}} \\ &\le \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|w\|_{\alpha,\omega_{\mathbf{a}}} + \|\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}\|\nabla w\|_{\alpha,\omega_{\mathbf{a}}} \\ &\le \underbrace{(1 + C_{\mathrm{PF},\alpha,\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}}\|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}})}_{\le C_{\mathrm{cont,PF}}}\|\nabla w\|_{\alpha,\omega_{\mathbf{a}}}, \end{aligned}$$

- Hölder inequality (finite overlapping to conclude):

$$\langle \mathcal{R}, v\rangle \le C_{\mathrm{cont,PF}} \left\{\sum_{\mathbf{a}\in\mathcal{V}} \|\mathcal{R}\|^{\beta}_{[W_0^{1,\alpha}(\omega_{\mathbf{a}})]'}\right\}^{1/\beta} \left\{\sum_{\mathbf{a}\in\mathcal{V}} \|\nabla v\|^{\alpha}_{\alpha,\omega_{\mathbf{a}}}\right\}^{1/\alpha}$$

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Proof ($\gtrsim$): local α-Laplacian liftings

- α-Laplacian lifting of $\mathcal{R}$ on patch $\omega_{\mathbf{a}}$: $\mathscr{r}^{\mathbf{a}} \in W_0^{1,\alpha}(\omega_{\mathbf{a}})$ s.t.

$$(|\nabla \mathscr{r}^{\mathbf{a}}|^{\alpha-2}\nabla \mathscr{r}^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v\rangle \qquad \forall v \in W_0^{1,\alpha}(\omega_{\mathbf{a}})$$

- energy equality:

$$\|\nabla \mathscr{r}^{\mathbf{a}}\|_{\alpha,\omega_{\mathbf{a}}}^{\alpha} = (|\nabla \mathscr{r}^{\mathbf{a}}|^{\alpha-2}\nabla \mathscr{r}^{\mathbf{a}}, \nabla \mathscr{r}^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, \mathscr{r}^{\mathbf{a}}\rangle = \|\mathcal{R}\|_{[W_0^{1,\alpha}(\omega_{\mathbf{a}})]'}^{\beta}$$

- setting $\mathscr{r} := \sum_{\mathbf{a}\in\mathcal{V}} \mathscr{r}^{\mathbf{a}} \in W_0^{1,\alpha}(\Omega)$:

$$\sum_{\mathbf{a}\in\mathcal{V}} \|\mathcal{R}\|_{[W_0^{1,\alpha}(\omega_{\mathbf{a}})]'}^{\beta} = \sum_{\mathbf{a}\in\mathcal{V}} \langle \mathcal{R}, \mathscr{r}^{\mathbf{a}}\rangle = \langle \mathcal{R}, \mathscr{r}\rangle \leq \|\mathcal{R}\|_{[W_0^{1,\alpha}(\Omega)]'} \|\nabla \mathscr{r}\|_{\alpha}$$

- overlapping of the patches:

$$\|\nabla \mathscr{r}\|_{\alpha}^{\alpha} \leq (d+1)^{\alpha-1} \sum_{\mathbf{a}\in\mathcal{V}} \|\nabla \mathscr{r}^{\mathbf{a}}\|_{\alpha,\omega_{\mathbf{a}}}^{\alpha}$$

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- α-Laplacian lifting of $\mathcal{R}$ on patch $\omega_{\mathbf{a}}$: $r^{\mathbf{a}} \in W_0^{1,\alpha}(\omega_{\mathbf{a}})$ s.t.

$$(|\nabla r^{\mathbf{a}}|^{\alpha-2}\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v\rangle \qquad \forall v \in W_0^{1,\alpha}(\omega_{\mathbf{a}})$$

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- setting $r := \sum_{\mathbf{a}\in\mathcal{V}} r^{\mathbf{a}} \in W_0^{1,\alpha}(\Omega)$:

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Localization of distances to $H_0^1(\Omega)$

Theorem (Localization of distance to $H_0^1(\Omega)$, Ciarlet & V. (2018))

Let $v \in H^1(\mathcal{T})$ *with* $\langle [\![v]\!], 1 \rangle_F = 0$ *for all* $F \in \mathcal{F}$ *be arbitrary. Then*

$$\underbrace{\min_{\zeta \in H_0^1(\Omega)} \|\nabla_h (v - \zeta)\|^2}_{\text{global distance to } H_0^1(\Omega)} \approx \sum_{\mathbf{a} \in \mathcal{V}} \underbrace{\min_{\zeta \in H^1_{\#}(\omega_{\mathbf{a}})} \|\nabla_h (v - \zeta)\|^2_{\omega_{\mathbf{a}}}}_{\text{local distance to } H^1_{\#}(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm int} \text{ and } H^1_{\partial\Omega}(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm ext}}.$$

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Localization of distances to $H_0^1(\Omega)$

Theorem (Localization of distance to $H_0^1(\Omega)$, Ciarlet & V. (2018))

Let $v \in H^1(\mathcal{T})$ with $\langle [\![v]\!], 1\rangle_F = 0$ for all $F \in \mathcal{F}$ be arbitrary. Then

$$\underbrace{\min_{\zeta \in H_0^1(\Omega)} \|\nabla_h (v - \zeta)\|^2}_{\text{global distance to } H_0^1(\Omega)} \approx \sum_{\mathbf{a} \in \mathcal{V}} \underbrace{\min_{\zeta \in H_\#^1(\omega_{\mathbf{a}})} \|\nabla_h (v - \zeta)\|_{\omega_{\mathbf{a}}}^2}_{\text{local distance to } H_\#^1(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm int} \text{ and } H_{\partial\Omega}^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm ext}}.$$

Proof ($\lesssim$) when $\langle [\![v]\!], 1\rangle_F = 0$ for all $F \in \mathcal{F}$

- define $s \in H_0^1(\Omega)$ by

$$s^{\boldsymbol{a}} := \arg \min_{\zeta \in H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}} v - \zeta)\|_{\omega_{\boldsymbol{a}}}, \qquad s := \sum_{\boldsymbol{a} \in \mathcal{V}} s^{\boldsymbol{a}}$$

- minimum, partition of unity:

$$\min_{\zeta \in H_0^1(\Omega)} \|\nabla_h(v - \zeta)\|^2 \leq \|\nabla_h(v - s)\|^2$$
$$\leq (d+1) \sum_{\boldsymbol{a} \in \mathcal{V}} \|\nabla_h(\psi_{\boldsymbol{a}} v - s^{\boldsymbol{a}})\|^2_{\omega_{\boldsymbol{a}}}$$

- $\psi_{\boldsymbol{a}} \zeta \in H_0^1(\omega_{\boldsymbol{a}})$ for any $\zeta \in H^1_{\#}(\omega_{\boldsymbol{a}})$, definition of $s^{\boldsymbol{a}}$:

$$\|\nabla_h(\psi_{\boldsymbol{a}} v - s^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \leq \inf_{\zeta \in H^1_{\#}(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}}(v - \zeta))\|_{\omega_{\boldsymbol{a}}}$$

- broken Poincaré–Friedrichs inequality:

$$\inf_{\zeta \in H^1_{\#}(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}}(v - \zeta))\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{cont,bPF}} \min_{\zeta \in H^1_{\#}(\omega_{\boldsymbol{a}})} \|\nabla_h(v - \zeta)\|_{\omega_{\boldsymbol{a}}}$$

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Outline

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1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (*p*-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: *hp*-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
   - Equilibrated flux reconstruction
7. Conclusions and outlook

Nonlinear Laplacian

Quasi-linear elliptic problem

$$
\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

- $\alpha > 1$, $\beta := \frac{\alpha}{\alpha-1}$, $f \in L^{\beta}(\Omega)$
- example: α-Laplacian with $\boldsymbol{\sigma}(u,\nabla u) = |\nabla u|^{\alpha-2}\nabla u$

Weak formulation

Find $u \in W_0^{1,\alpha}(\Omega)$ such that

$$(\boldsymbol{\sigma}(u,\nabla u),\nabla v) = (f,v) \qquad \forall v \in W_0^{1,\alpha}(\Omega)$$

Residual $\mathcal{R}(u_h) \in [W_0^{1,\alpha}(\Omega)]'$ **of** $u_h \in W_0^{1,\alpha}(\Omega)$,

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localizes from dual norms for finite element discretizations

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Localized α-robust *a posteriori* error estimates

Theorem (Localized α-robust estimate El Alaoui, Ern, & V. (2011)**)**

- *Let* $\boldsymbol{\sigma}(u, \nabla u) = \boldsymbol{\sigma}(\nabla u)$ *and* $f \in \mathbb{P}_0(\mathcal{T})$ *for simplicity;*
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- $\boldsymbol{\xi}_h := -\boldsymbol{\sigma}(\nabla u_h)$, f: $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_1(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ ▸ flux reconstruction.

Then

$$\|\mathcal{R}\|_{[W_0^{1,\alpha}(\Omega)]'} \leq \left\{ \sum_{K \in \mathcal{T}} \|\boldsymbol{\sigma}(\nabla u_h) + \boldsymbol{\sigma}_h\|_{\beta,K}^{\beta} \right\}^{1/\beta},$$

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Outline

Transmission problems with sign-changing coefficients

Model problem

$$-\nabla\cdot(\underline{\boldsymbol{\Sigma}}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\underline{\boldsymbol{\Sigma}}$ **not positive definite** (and symmetric)
- example: $\Omega = \Omega_+ \cup \Omega_-$, $\sigma_+ > 0$ and $\sigma_- < 0$,

$$\underline{\boldsymbol{\Sigma}}|_{\Omega_+} = \sigma_+ \underline{\boldsymbol{I}}, \quad \underline{\boldsymbol{\Sigma}}|_{\Omega_-} = \sigma_- \underline{\boldsymbol{I}}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

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- well-posed (T-coercivity) Bonnet-Ben Dhia, Chesnel, Ciarlet Jr. (2012), numerical discretization following e.g. Chesnel and Ciarlet Jr. (2013)

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Intrinsic norm and its localization

Energy norm $\|v\|^2_{\rm en} := (\underline{\Sigma}\nabla v, \nabla v)$ for $v \in H^1_0(\Omega)$

- not well-defined: $(\underline{\Sigma}\nabla v, \nabla v) < 0$ may happen

Broken H^1 seminorm when $\underline{\Sigma} = \underline{I}$, $v \in H^1(\mathcal{T})$

$$\|\nabla_h v\|^2 = \underbrace{\max_{\varphi \in H^1_0(\Omega);\, \|\nabla\varphi\|=1} (\nabla_h v, \nabla\varphi)^2}_{\text{dual norm}} + \underbrace{\min_{\zeta \in H^1_0(\Omega)} \|\nabla_h (v - \zeta)\|^2}_{\text{distance to } H^1_0(\Omega)}$$

- localizes from and for finite element discretizations

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Intrinsic norm of error

$$|||u - u_h|||^2 = \max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\underline{\Sigma}\nabla_h (u - u_h), \nabla\varphi)^2 + \min_{\zeta \in H_0^1(\Omega)} \|\nabla_h (u_h - \zeta)\|^2$$

$$+ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_F^0 \underbrace{[\![u - u_h]\!]}_{[\![-u_h]\!]}\|_F^2 \qquad u_h \in H^1(\mathcal{T})$$

- localizes from ▸ dual norms and ▸ distance norms for finite element discretizations

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$$|||v|||^2 := \max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\underline{\Sigma}\nabla_h v, \nabla\varphi) + \min_{\zeta \in H_0^1(\Omega)} \|\nabla_h(v-\zeta)\|^2$$

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- localizes from ▸ dual norms and ▸ distance norms for finite element discretizations

Intrinsic norm and its localization

Energy norm $\|v\|^2_{\mathrm{en}} := (\underline{\Sigma}\nabla v, \nabla v)$ for $v \in H^1_0(\Omega)$

- not well-defined: $(\underline{\Sigma}\nabla v, \nabla v) < 0$ may happen

Broken H^1 seminorm when $\underline{\Sigma} = \underline{I}$, $v \in H^1(\mathcal{T})$

$$\|\nabla_h v\|^2 = \underbrace{\max_{\varphi \in H^1_0(\Omega);\, \|\nabla\varphi\|=1} (\nabla_h v, \nabla\varphi)^2}_{\text{dual norm}} + \underbrace{\min_{\zeta \in H^1_0(\Omega)} \|\nabla_h(v-\zeta)\|^2}_{\text{distance to } H^1_0(\Omega)}$$

Intrinsic norm

$$|||v|||^2 := \max_{\varphi \in H^1_0(\Omega);\, \|\nabla\varphi\|=1} (\underline{\Sigma}\nabla_h v, \nabla\varphi) + \min_{\zeta \in H^1_0(\Omega)} \|\nabla_h(v-\zeta)\|^2 + \sum_{F\in\mathcal{F}} h_F^{-1} \|\Pi^0_F [\![v]\!]\|^2_F \qquad v \in H^1(\mathcal{T})$$

- localizes from ▸ dual norms and ▸ distance norms for finite element discretizations

Intrinsic norm and its localization

Energy norm $\|v\|_{\mathrm{en}}^2 := (\underline{\Sigma}\nabla v, \nabla v)$ for $v \in H_0^1(\Omega)$

- not well-defined: $(\underline{\Sigma}\nabla v, \nabla v) < 0$ may happen

Broken H^1 seminorm when $\underline{\Sigma} = \underline{I}$, $v \in H^1(\mathcal{T})$

$$\|\nabla_h v\|^2 = \underbrace{\max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla_h v, \nabla\varphi)^2}_{\text{dual norm}} + \underbrace{\min_{\zeta \in H_0^1(\Omega)} \|\nabla_h(v - \zeta)\|^2}_{\text{distance to } H_0^1(\Omega)}$$

Intrinsic norm of error

$$|||u - u_h|||^2 = \max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\underline{\Sigma}\nabla_h(u - u_h), \nabla\varphi)^2 + \min_{\zeta \in H_0^1(\Omega)} \|\nabla_h(u_h - \zeta)\|^2$$

$$+ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_F^0 \underbrace{[\![u - u_h]\!]}_{[\![-u_h]\!]}\|_F^2 \qquad u_h \in H^1(\mathcal{T})$$

- localizes from dual norms and distance norms for finite element discretizations

Localized $\underline{\Sigma}$-robust *a posteriori* error estimates

Theorem (Localized $\underline{\Sigma}$-robust *a posteriori* estimate Ciarlet & V. (2018))

- *Let* $\underline{\Sigma} \in [\mathbb{P}_0(\mathcal{T})]^{d\times d}$ *and* $f \in \mathbb{P}_{p-1}(\mathcal{T})$, $p \geq 1$, *for simplicity;*
- *let* $u \in H_0^1(\Omega)$ *be the weak solution;*
- $u_h \in \mathbb{P}_p(\mathcal{T})$ *be arbitrary subject to*

$$(\underline{\Sigma}\nabla_h u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ potential reconstruction;
- $\xi_h := -\underline{\Sigma}\nabla_h u_h, f$: $\sigma_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ flux reconstruction.

Then, $\underline{\Sigma}$*- and* p*-robust localized equivalence holds:*

$$|||u - u_h|||^2 \leq \sum_{K\in\mathcal{T}} \left[\|\underline{\Sigma}\nabla_h u_h + \sigma_h\|_K^2 + \|\nabla_h(u_h - s_h)\|_K^2\right] + \sum_{F\in\mathcal{F}} h_F^{-1}\|\Pi_F^0 [\![u_h]\!]\|_F^2,$$

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Applications

Unified framework for all classical discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ mixed finite elements
- ✓ various finite volumes

Numerics: regular solution

Data

- $\Omega := (-1,1) \times (-1,1)$
- $\Omega_+ := (0,1) \times (-1,1)$, $\Omega_- := (-1,0) \times (-1,1)$
- $\sigma_+ = 1$, $\sigma_- < 0$

Exact solution

$$u(x,y) = \sigma_- x(x+1)(x-1)(y+1)(y-1) \text{ for } (x,y) \in \Omega_+,$$
$$u(x,y) = x(x+1)(x-1)(y+1)(y-1) \text{ for } (x,y) \in \Omega_-$$

Discretization

- conforming finite elements: $u_h \in H_0^1(\Omega)$
- unstructured triangular grids
- uniform h refinement

Numerics: regular solution

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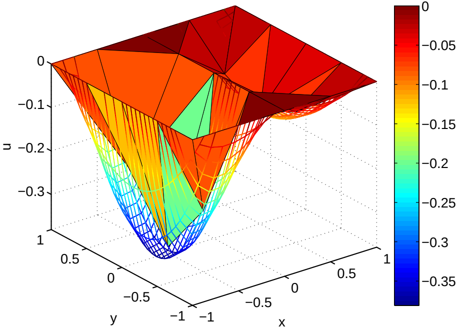
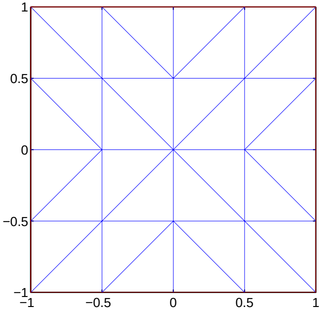
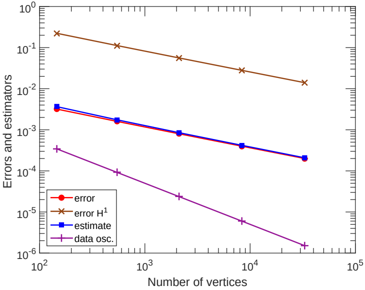
Exact solution, approximate solution, and mesh

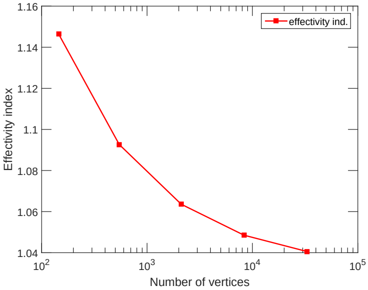




Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -0.01$



Error and estimate



Effectivity index

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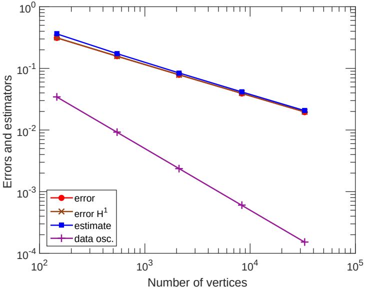
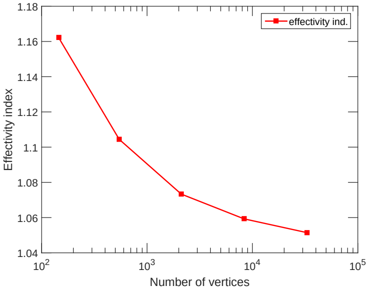
Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -0.99$



Error $|||u - u_h|||$ and estimate



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Numerics: singular solution

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Exact solution

$$u(x,y) = r^\lambda(c_1 \sin(\lambda\phi) + c_2 \sin(\lambda(\pi/2-\phi))) \text{ for } (x,y)\in\Omega_+,$$
$$u(x,y) = r^\lambda(d_1 \sin(\lambda(\phi-\pi/2)) + d_2 \sin(\lambda(2\pi-\phi))) \text{ for } (x,y)\in\Omega_-$$

- $u \in H^{1+\lambda}(\Omega)$
- $\sigma_- = -5$: $\lambda \approx 0.4601069123$
- $\sigma_- = -3.1$: $\lambda \approx 0.1391989493$

Discretization

- conforming finite elements: $u_h \in H_0^1(\Omega)$
- unstructured triangular grids
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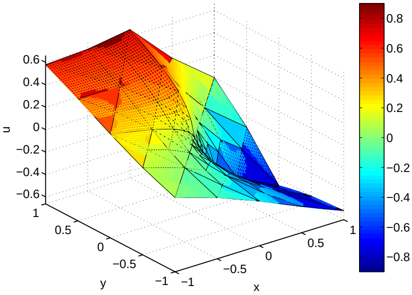
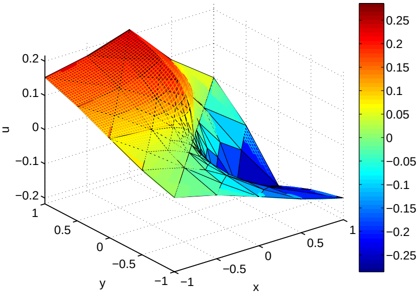
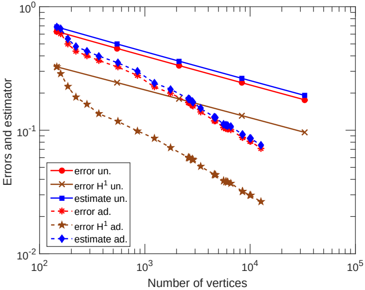
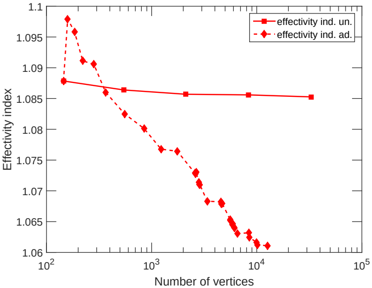




Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -5$



Error $|||u - u_h|||$ and estimate



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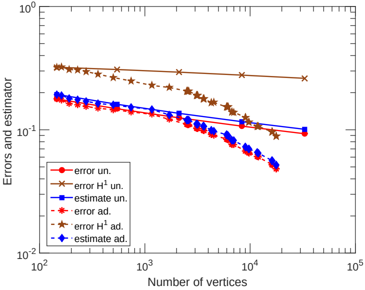
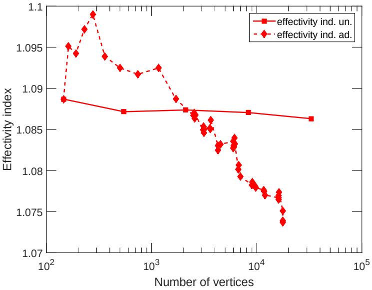
Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -3.1$



Error and estimate



Effectivity index

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Outline

1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (*p*-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - **Heat: space-time localization and *T*-robustness**
   - Laplace: *hp*-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
   - Equilibrated flux reconstruction
7. Conclusions and outlook

Heat equation

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

Spaces

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) \,\mathrm{d}t = \int_0^T (f, v) \,\mathrm{d}t \qquad \forall v \in X.$$

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Error and residual

Theorem (Parabolic inf–sup identity)

For every $v \in Y$, we have

$$\|v\|_Y^2 = \left[\sup_{\varphi \in X,\, \|\varphi\|_X = 1} \int_0^T \langle \partial_t v, \varphi \rangle + (\nabla v, \nabla \varphi)\, \mathrm{d}t\right]^2 + \|v(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v)\, \mathrm{d}t$$

- dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

Y norm error is the dual X norm of the residual + IC error

$$\|u - u_{h\tau}\|_Y^2 = \|\mathcal{R}(u_{h\tau})\|_{X'}^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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A posteriori estimate

Guaranteed upper bound

- ✓ $\|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- ✓ no undetermined constant: **error control**

Local space-time efficiency

- ✓ $\eta_K^n(u_{h\tau}) \lesssim \|u - u_{h\tau}\|_{\mathcal{E}_Y,\text{neighbors of } K\times(t^{n-1},t^n)}$
- ✓ optimal space-time mesh refinement
- ✓ **local** in **time** and in **space** error lower bound using dual norms localization

Robustness

- ✓ $\lesssim$ independent of data, domain Ω, **final time** T, meshes, solution u, **polynomial degrees** of $u_{h\tau}$ in space and in time

Small evaluation cost

- ✓ estimators can be evaluated cheaply (locally)

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Outline

Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method: $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
- uniform h and p refinement

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How large is the overall error? (model pb, known sol.)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	1.3	$2.8 \times 10^{1}\%$	1.1	$2.4 \times 10^{1}\%$	1.17
$\approx h_0/2$		[illegible]				
$\approx h_0/4$		[illegible]				
$\approx h_0/8$		[illegible]				
h_0	2	[illegible]				
$\approx h_0/2$	2	[illegible]				
h_0	3	[illegible]				
$\approx h_0/4$	3	[illegible]				
h_0	4	[illegible]				
$\approx h_0/8$	4	[illegible]				

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V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error?

h	p	$\eta(u_h)$
h_0	1	1.3

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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\lVert\nabla u_h\rVert}$	$\lVert\nabla(u-u_h)\rVert$	rel. error $\frac{\lVert\nabla(u-u_h)\rVert}{\lVert\nabla u_h\rVert}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\lVert\nabla(u-u_h)\rVert}$
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$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1}\%$	5.6×10^{-1}	$1.3 \times 10^{1}\%$	
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	

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$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	1.02
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	[illegible]

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$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	**1.06**
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	**1.04**
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	**1.06**
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	**1.04**
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	**1.03**
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	**1.01**
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	**1.02**
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known sol.)

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
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V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

Numerics: smooth case with localized features

Model problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega := (-1,1)^2, \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$u(x,y) = (x^2-1)(y^2-1)\exp\left(-100(x^2+y^2)\right)$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
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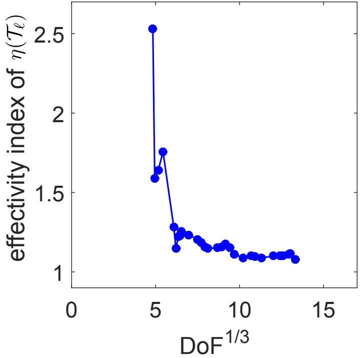
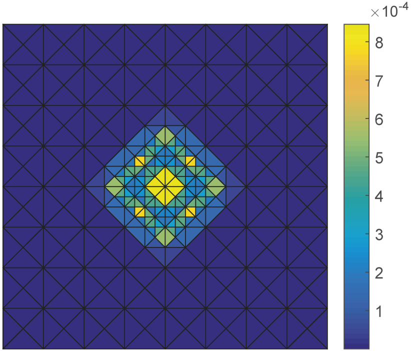
How precise are the estimates?



Effectivity indices on *hp* meshes

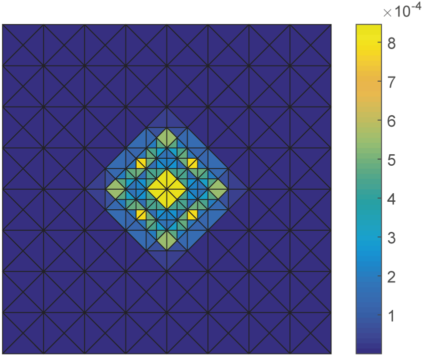
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Where (in space) is the error localized?



Estimated error distribution

$\eta_K(u_h)$

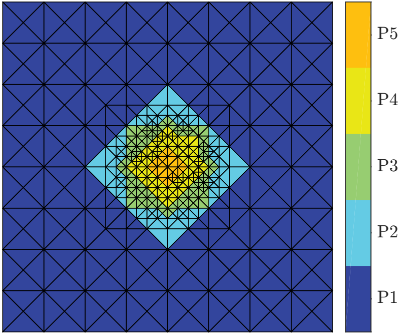


Exact error distribution

$\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently?



Mesh $\mathcal{T}$ and pol. degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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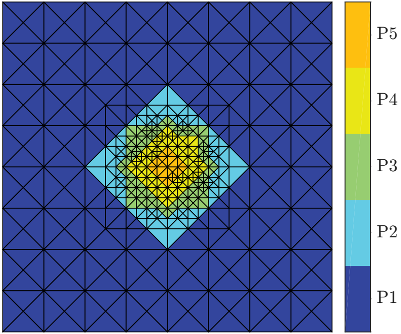

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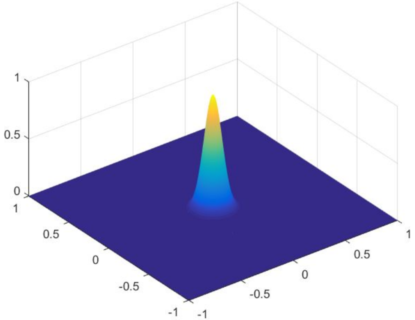

Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Numerics: singular case

Model problem

$$-\Delta u = 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

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Discretization

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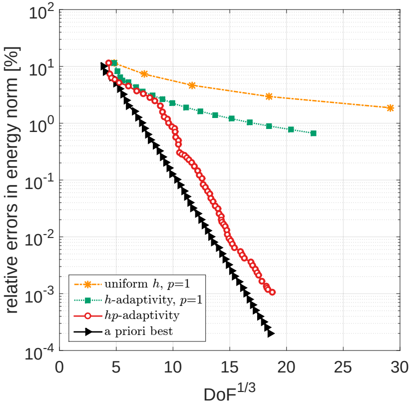

Relative error as a function of no. of unknowns

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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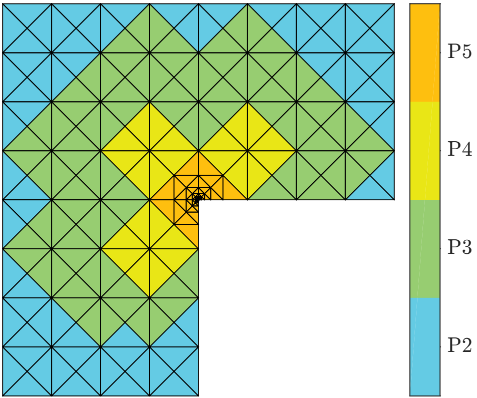
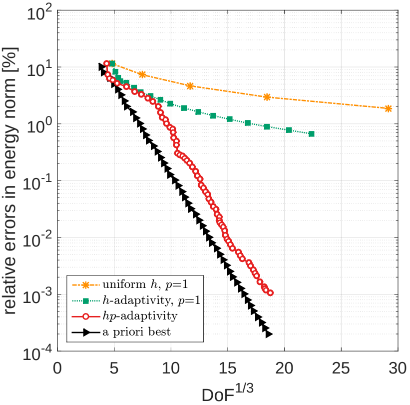


Mesh $\mathcal{T}$ and polynomial degrees p_K



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P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Outline

1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (p-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: hp-adaptivity and exponential convergence
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   - Potential reconstruction
   - Equilibrated flux reconstruction
7. Conclusions and outlook

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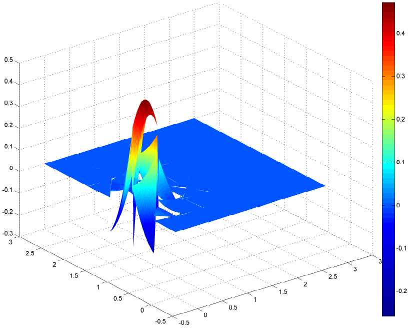
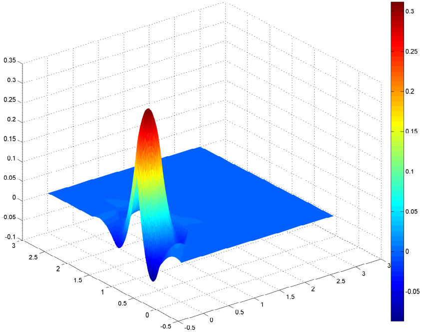
Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \to s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\ \ \psi_{\boldsymbol{a}}\xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}} s_h^{\boldsymbol{a}}.$$

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h \ \ (\psi_{\boldsymbol{a}}\xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
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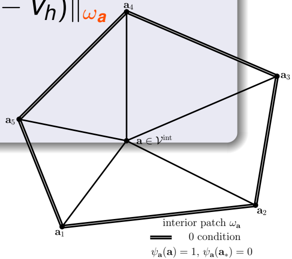

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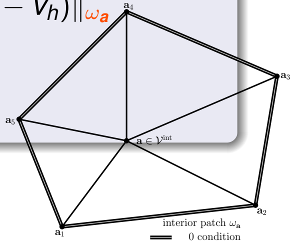

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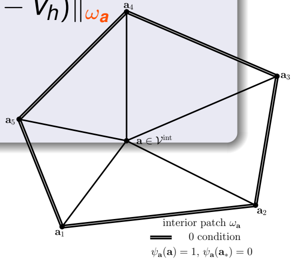

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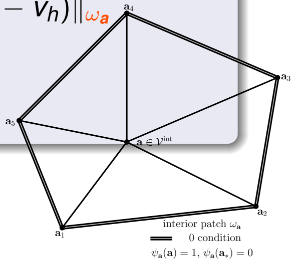

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}}\xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
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Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

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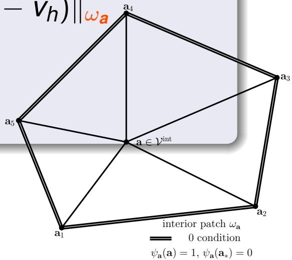

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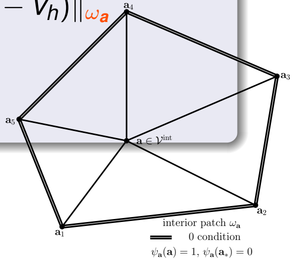

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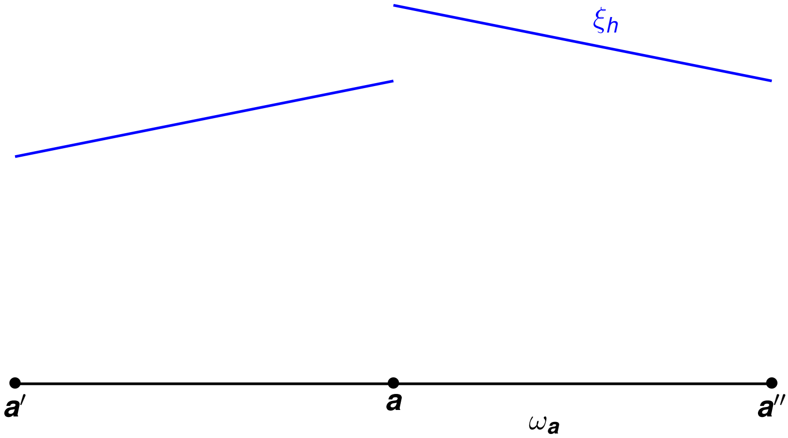
Potential reconstruction in 1D, $p = 1$, $p' = 2$



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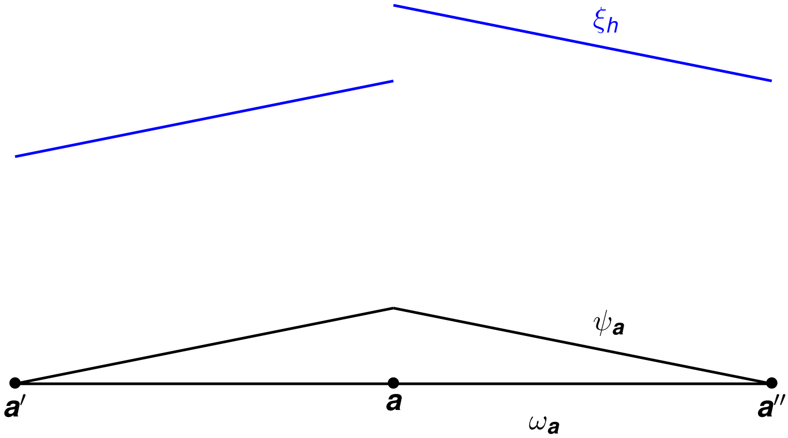

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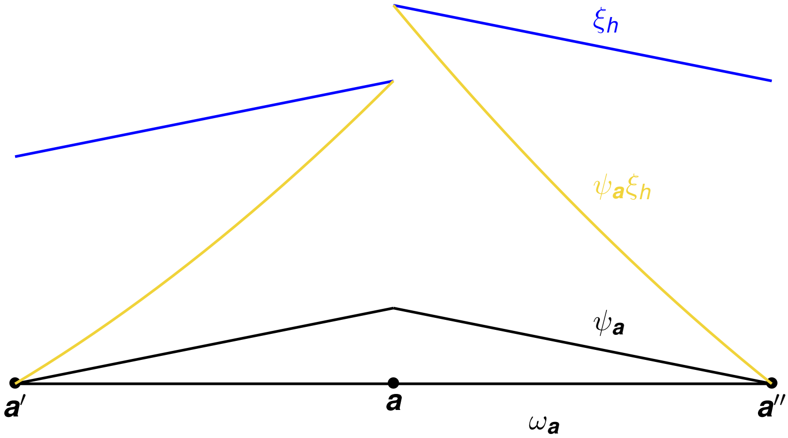

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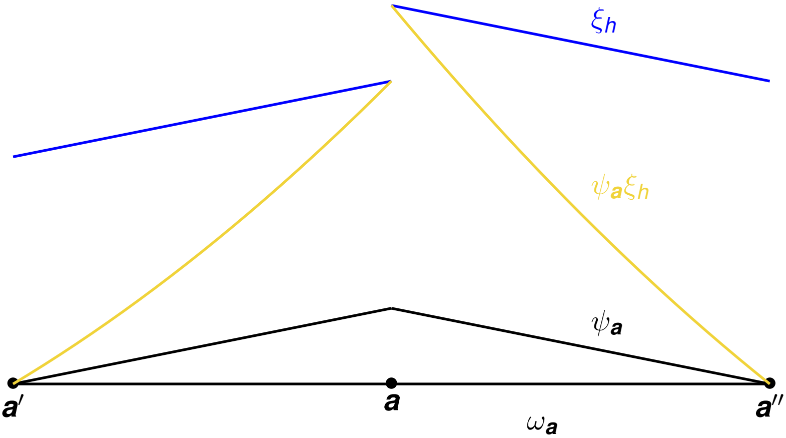

Stability of the potential reconstruction

Theorem (Local stability Ern & V. (2015, 2016), using Tools**)**

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

Stability of the potential reconstruction

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

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Corollary (Global stability; $p' = p$)

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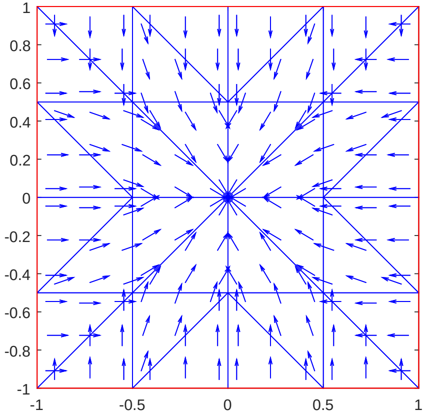
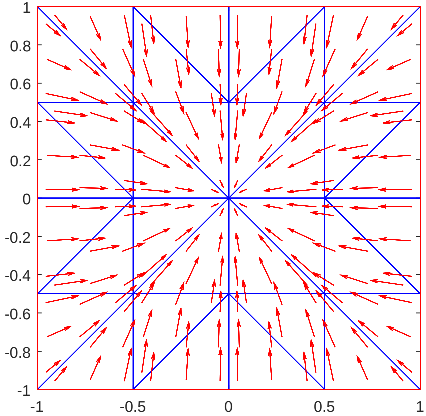
Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathbf{RTN}_p(\mathcal{T}),\ f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+(\boldsymbol{\xi}_h,\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\ \forall \mathbf{a}\in\mathcal{V}^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathbf{RTN}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap \mathbf{H}(\mathrm{div},\Omega),\ \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Outline

1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (p-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: *hp*-adaptivity and exponential convergence
6. **Tools**
   - Potential reconstruction
   - **Equilibrated flux reconstruction**
7. Conclusions and outlook

Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}}.$$

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder & Métivet (1999), Braess & Schöberl (2008))

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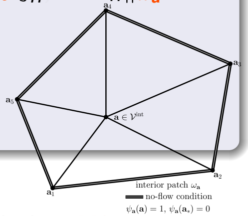

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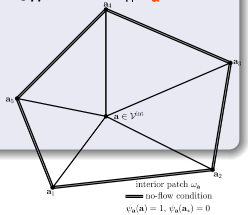

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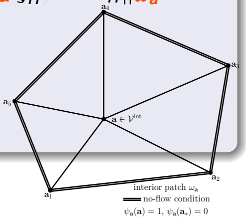

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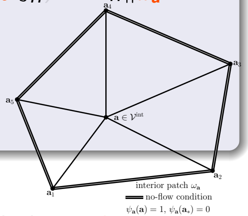

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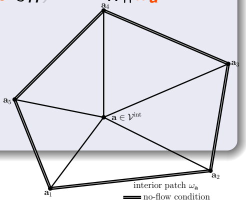

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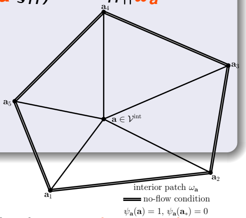

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Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern, V. (2016; 3D), using Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Stability of the flux reconstruction

Corollary (Global stability; $p' = p+1$)

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ such that $\nabla{\cdot}\boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

Stability of the flux reconstruction

Corollary (Global stability; $p' = p$)

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Outline

1. Introduction
2. Global-best – local-best equivalences
3. *A priori* estimates (elementwise localized)
   - Conforming finite elements for the Laplace equation
   - Mixed finite elements for the Laplace equation
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
4. Localization of dual and distance norms
5. *A posteriori* estimates (p-robust)
   - Nonlinear Laplace: localization and α-robustness
   - Non-coercive transmission: localization and Σ-robustness
   - Heat: space-time localization and T-robustness
   - Laplace: hp-adaptivity and exponential convergence
6. Tools
   - Potential reconstruction
   - Equilibrated flux reconstruction
7. **Conclusions and outlook**

Conclusions and outlook

Conclusions

- global-best – local-best equivalences: optimal *a priori* error estimates
- localization of dual and distance norms: optimal *a posteriori* error estimates
- broken polynomial extension operators: p-robustness
- unified framework for all classical numerical schemes

Ongoing work

- extensions to other settings

Conclusions and outlook

Conclusions

- global-best – local-best equivalences: optimal *a priori* error estimates
- localization of dual and distance norms: optimal *a posteriori* error estimates
- broken polynomial extension operators: p-robustness
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Merci de votre attention !

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

Potentials

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Fluxes

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

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Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. *Let* $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, *satisfying* $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ *if* $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

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Fluxes

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$$\|\varphi_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

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Set $\varphi_K := -\nabla \zeta_K$.

Potentials

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2016)**)**

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\rm int}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\rm int})$. Suppose the compatibility

$$
\begin{aligned}
r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int},\\
\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.
\end{aligned}
$$

Then

$$
\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext}\\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext}\\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.
$$

Fluxes

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2016; 3D)**)**

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Theorem (Localization of distance to $H^1_0(\Omega)$, Ciarlet & V. (2018))

Let $v \in H^1(\mathcal{T})$ be arbitrary. Then

$$\underbrace{\min_{\zeta \in H^1_0(\Omega)} \|\nabla_\theta (v - \zeta)\|^2 + \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_F^0 [\![v]\!]\|_F^2}_{\text{global distance to } H^1_0(\Omega)}$$

$$\approx \sum_{\mathbf{a} \in \mathcal{V}} \Bigg\{ \underbrace{\min_{\zeta \in H^1_\#(\omega_{\mathbf{a}})} \|\nabla_\theta (v - \zeta)\|^2_{\omega_{\mathbf{a}}} + \sum_{F \in \mathcal{F},\, \mathbf{a} \in F} h_F^{-1} \|\Pi_F^0 [\![v]\!]\|_F^2}_{\text{local distance to } H^1_\#(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm int} \text{ and } H^1_{\partial\Omega}(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\rm ext}} \Bigg\},$$

where, for $\theta \in \{-1, 0, 1\}$,

$$\underbrace{\nabla_\theta v}_{\text{discrete gradient}} := \nabla_h v - \theta \sum_{F \in \mathcal{E}} \underbrace{\iota_F([\![v]\!])}_{\text{lifting of the jumps}}.$$