Potential and flux reconstructions for optimal a priori and a posteriori error estimates

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Outline

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2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
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5. A posteriori estimates
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Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw polynomial
- *a posteriori* analysis of mixed and nonconforming FEs: show it can give polynomial degree-robustness
- show it can be used in *a priori* analysis of conforming FEs:

  global-best–local-best equivalence in H^1 Veeser (2016)

  approximation continuous pw pols $\approx_p$ discontinuous pw pols

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium → pw vector-valued polynomial with continuous normal trace and equilibrium
- *a posteriori* analysis of conforming FEs:

  estimate ≈ error

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Definition (Construction of s_h EV (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}}\xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}} s_h^{\boldsymbol{a}}.$$

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h(\psi_{\boldsymbol{a}}\xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}}\xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
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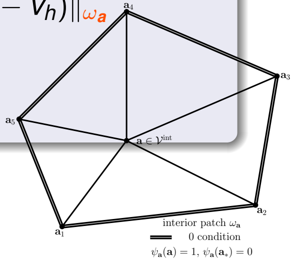

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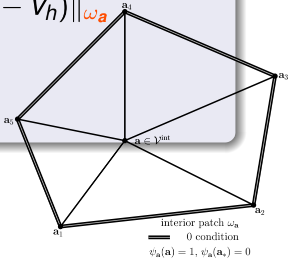

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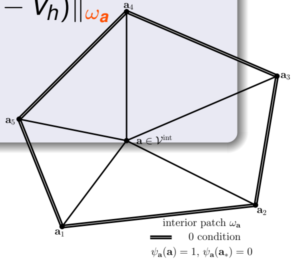

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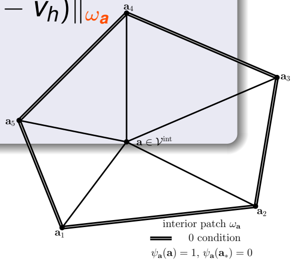

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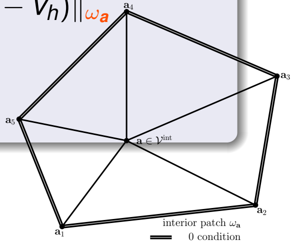

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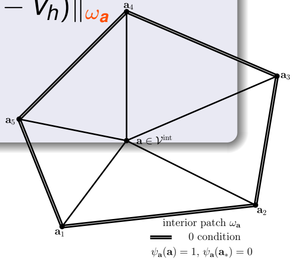

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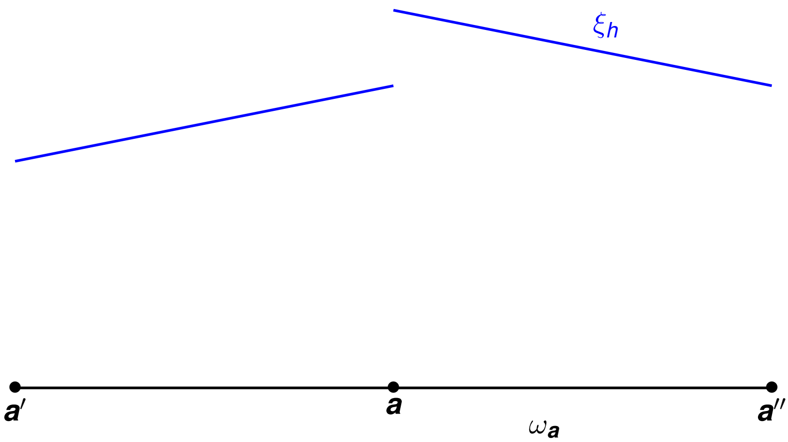

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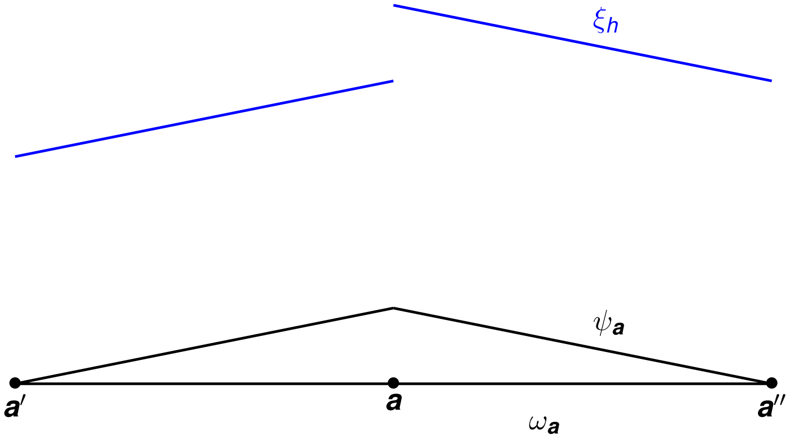

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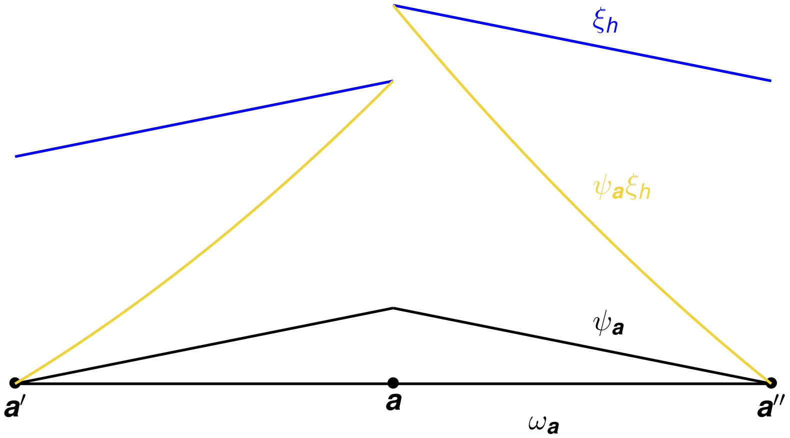

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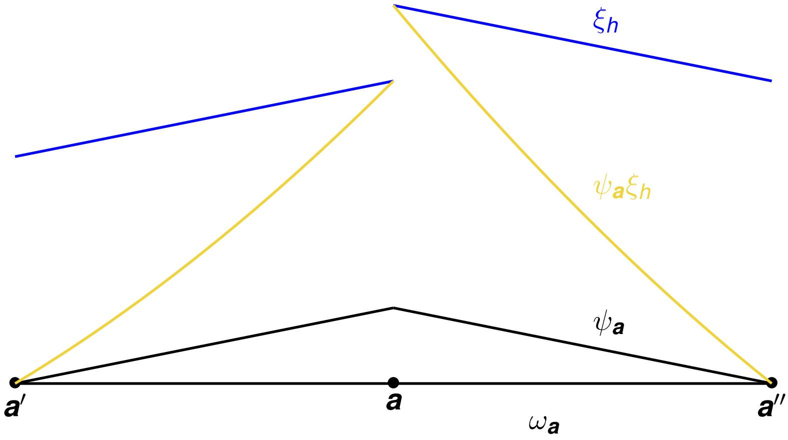
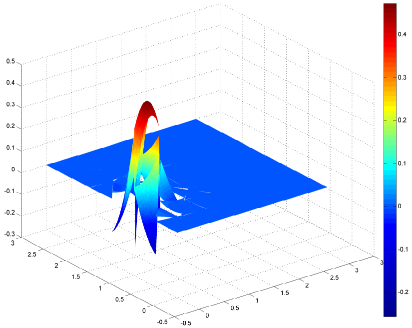


Potential reconstruction in 2D, $p = 2$, $p' = 2$



Potential ξ_h

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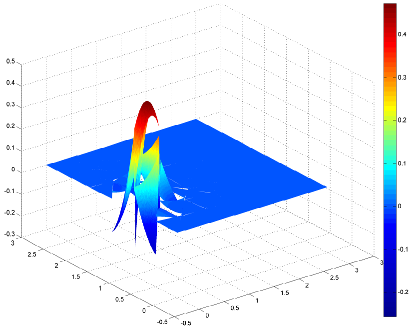

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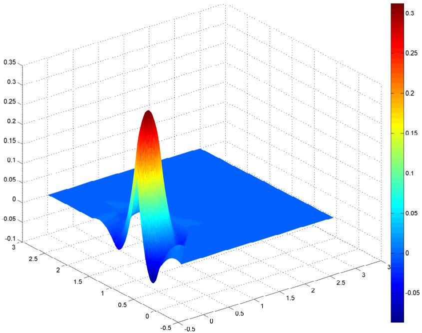

Potential reconstruction s_h

Stability of the potential reconstruction

Theorem (Local stability EV (2015, 2016), using ▸ Tools**)**

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

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For any $u \in H^1_0(\Omega)$,

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

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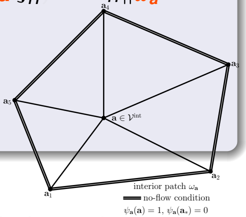

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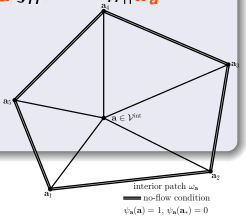

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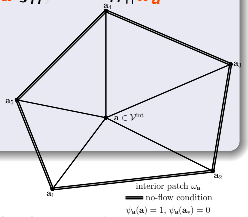

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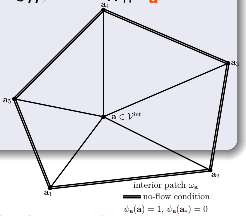

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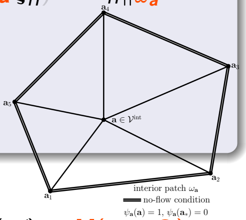

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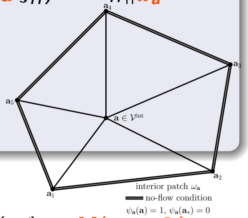

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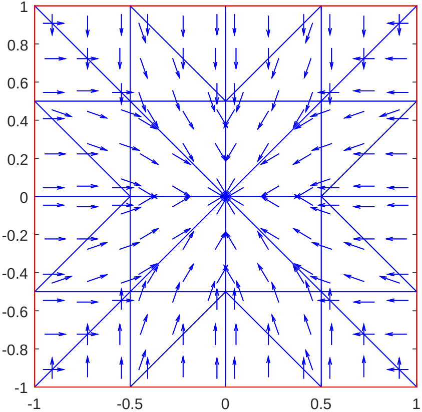
Equilibrated flux reconstruction

Equivalent form: mixed FEs

Find $(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \gamma_h^{\boldsymbol{a}}) \in \boldsymbol{V}_h^{\boldsymbol{a}} \times \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}})$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} - (\gamma_h^{\boldsymbol{a}}, \nabla\cdot\boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h), \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}},\\
(\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} &= (f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} && \forall q_h \in \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}})
\end{aligned}
$$

Equilibrated flux reconstruction in 2D, $p = 0$, $p' = 1$



Flux ξ_h

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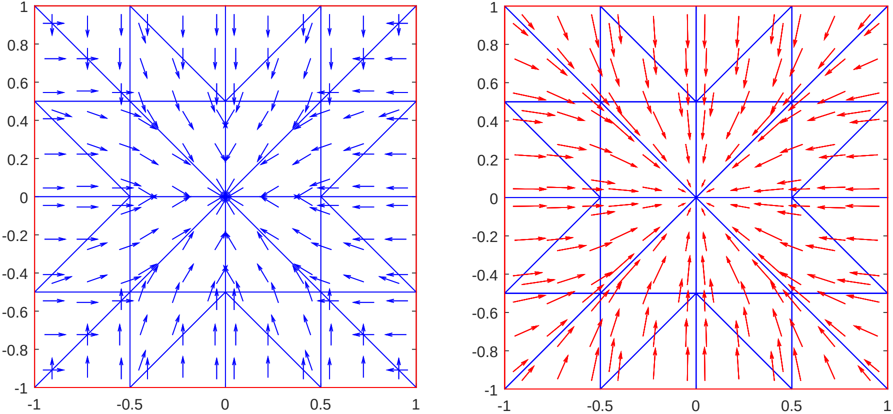

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Stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), EV (2016; 3D), using Tools**)**

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Corollary (Global stability; $p' = p + 1$)

For any $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div},\Omega)$ *such that* $\nabla\cdot\boldsymbol{\sigma} = f$,

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2}\|f - \Pi_p f\|_K^2\right\}^{1/2}.$$

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1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
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Global-best – local-best equivalence in H^1

Theorem (Equivalence in H^1, $p \geq 1$ Veeser (2016))

Let $u \in H^1_0(\Omega)$ be arbitrary. Then,

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|^2_K.$$

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Proof via potential reconstruction.

- define discontinuous $\xi_h \in \mathbb{P}_p(\mathcal{T})$ by

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- ξ_h: ▸ potential reconstruction $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)$
- global ▸ H^1 stability ($p' = p$),

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}$$

- bound on minimum, triangle inequality

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

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Theorem (Equivalence in H^1, $p \geq 1$ Veeser (2016))

Let $u \in H_0^1(\Omega)$ be arbitrary. Then,

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$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2} \lesssim \|\nabla_h(u - \xi_h)\|$$

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$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\| \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

Global-best – local-best equivalence in H^1

Theorem (Equivalence in H^1, $p \geq 1$ Veeser (2016))

Let $u \in H^1_0(\Omega)$ be arbitrary. Then,

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|^2_K.$$

Proof via potential reconstruction.

- define discontinuous $\xi_h \in \mathbb{P}_p(\mathcal{T})$ by

$$\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K, \quad (\xi_h, 1)_K = (u, 1)_K \quad \forall K \in \mathcal{T}$$

- ξ_h: potential reconstruction $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)$
- global H^1 stability ($p' = p$), jump term efficiency + mean ξ_h

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2} \lesssim \|\nabla_h(u - \xi_h)\|$$

- bound on minimum, triangle inequality

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

Global-best – local-best equivalence in H^1

Theorem (Equivalence in H^1, $p \geq 1$ Veeser (2016))

Let $u \in H_0^1(\Omega)$ be arbitrary. Then,

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2.$$

Proof via potential reconstruction.

- define discontinuous $\xi_h \in \mathbb{P}_p(\mathcal{T})$ by

$$\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K, \quad (\xi_h, 1)_K = (u, 1)_K \quad \forall K \in \mathcal{T}$$

- ξ_h: potential reconstruction $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$
- global H^1 stability ($p' = p$), jump term efficiency + mean ξ_h

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2} \lesssim \|\nabla_h(u - \xi_h)\|$$

- bound on minimum, triangle inequality

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\| \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

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Let $u \in H^1_0(\Omega)$ be arbitrary. Then,

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$$\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K, \quad (\xi_h, 1)_K = (u, 1)_K \quad \forall K \in \mathcal{T}$$

- ξ_h: ▸ potential reconstruction $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)$
- global ▸ H^1 stability ($p' = p$), jump term efficiency + mean ξ_h

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2} \lesssim \|\nabla_h(u - \xi_h)\|$$

- bound on minimum, triangle inequality

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \leq \|\nabla(u - s_h)\| \leq \|\nabla_h(u - \xi_h)\| + \|\nabla_h(\xi_h - s_h)\|$$

Outline

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $f := \nabla{\cdot}\boldsymbol{\sigma}$ be arbitrary. Then,

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \left[\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2 \right].$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $f := \nabla\cdot\boldsymbol{\sigma}$ be arbitrary. Then,

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \left[\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2 \right].$$

Proof.

- define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ by

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\boldsymbol{a} \in \mathcal{V}_K$,

$$(\boldsymbol{\sigma} - \boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_K + h_K^2 (\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h), \overbrace{\nabla\cdot(\nabla\psi_{\boldsymbol{a}})}^{0})_K = 0 \qquad \forall K \in \mathcal{T};$$

- as $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ $\Longrightarrow$ orthogonality

$$(\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = -(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \Longrightarrow$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $f := \nabla{\cdot}\boldsymbol{\sigma}$ be arbitrary. Then,

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \left[\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2 \right].$$

Proof.

- define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ by

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\boldsymbol{a} \in \mathcal{V}_K$,

$$(\boldsymbol{\sigma} - \boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_K + h_K^2 (\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{\xi}_h), \overbrace{\nabla{\cdot}(\nabla\psi_{\boldsymbol{a}})}^{0})_K = 0 \qquad \forall K \in \mathcal{T};$$

- as $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}}) \Rightarrow \psi_{\boldsymbol{a}}$-orthogonality

$$(\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = -(\nabla{\cdot}\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \Rightarrow \boxed{(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}}$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $f := \nabla{\cdot}\boldsymbol{\sigma}$ be arbitrary. Then,

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 \lesssim_p \sum_{K \in \mathcal{T}} \left[\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2 \right].$$

Proof.

- define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ by

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \qquad \forall K \in \mathcal{T}$$

- since $\nabla \psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\boldsymbol{a} \in \mathcal{V}_K$,

$$(\boldsymbol{\sigma} - \boldsymbol{\xi}_h, \nabla \psi_{\boldsymbol{a}})_K + h_K^2 (\nabla{\cdot}(\boldsymbol{\sigma} - \boldsymbol{\xi}_h), \overbrace{\nabla{\cdot}(\nabla \psi_{\boldsymbol{a}})}^{0})_K = 0 \qquad \forall K \in \mathcal{T};$$

- as $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ $\Rightarrow$ $\psi_{\boldsymbol{a}}$-orthogonality

$$(\boldsymbol{\sigma}, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = -(\nabla{\cdot}\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \Rightarrow \boxed{(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\xi}_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}}$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ with $f := \nabla\cdot\boldsymbol{\sigma}$ be arbitrary. Then,

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 \lesssim_p \sum_{K\in\mathcal{T}} \left[\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2 \right].$$

Proof.

- define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ by

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\boldsymbol{a} \in \mathcal{V}_K$,

$$(\boldsymbol{\sigma} - \boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_K + h_K^2 (\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h), \overbrace{\nabla\cdot(\nabla\psi_{\boldsymbol{a}})}^{0})_K = 0 \qquad \forall K \in \mathcal{T};$$

- as $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}}) \Rightarrow \psi_{\boldsymbol{a}}$-orthogonality

$$(\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = -(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \Rightarrow \boxed{(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}}$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Proof continuation.

- $\boldsymbol{\xi}_h, f$: ▸ flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
- global ▸ $\boldsymbol{H}(\mathrm{div})$ stability $(p' = p)$

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} h_K^2 \|f - \nabla\cdot\boldsymbol{\xi}_h\|_K^2 \right\}^{1/2}$$

- bound on minimum, triangle inequality

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|$$

$$\lesssim_p \left\{ \sum_{K\in\mathcal{T}} \left[\|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\|_K^2 + h_K^2 \|f - \nabla\cdot\boldsymbol{\xi}_h\|_K^2 \right] \right\}^{1/2}$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Proof continuation.

- ξ_h, f: ▸ flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
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$$\lesssim_p \left\{ \sum_{K \in \mathcal{T}} \left[\|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\|_K^2 + h_K^2 \|f - \nabla\cdot \boldsymbol{\xi}_h\|_K^2 \right] \right\}^{1/2}$$

- introducing the constraint

$$\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \nabla\cdot \boldsymbol{v}_h\|_K^2 \right] \le \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot \boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

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- $\boldsymbol{\xi}_h, f$: ▸ flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega)$
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- bound on minimum, triangle inequality

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(\mathcal{T})\cap\boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|$$

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$$\min_{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2 + h_K^2\|f-\nabla\cdot\boldsymbol{v}_h\|_K^2\right] \le \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_p f}} \|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2 + h_K^2\|f-\Pi_p f\|_K^2$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Proof continuation.

- $\boldsymbol{\xi}_h, f$: flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
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$$\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \nabla\cdot\boldsymbol{v}_h\|_K^2 \right] \le \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2$$

Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$

Proof continuation.

- $\boldsymbol{\xi}_h, f$: ▸ flux reconstruction $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
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- bound on minimum, triangle inequality

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|$$

$$\lesssim_p \left\{ \sum_{K \in \mathcal{T}} \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \nabla{\cdot}\boldsymbol{v}_h\|_K^2 \right] \right\}^{1/2}$$

- introducing the constraint

$$\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \nabla{\cdot}\boldsymbol{v}_h\|_K^2 \right] \le \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|f - \Pi_p f\|_K^2$$

Outline

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ be arbitrary. Then, $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ is locally constructed, such that

$$\Pi_p(\nabla\cdot\boldsymbol{\sigma}) = \nabla\cdot(P_p\boldsymbol{\sigma}) \quad \text{commuting},$$

$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \quad \text{projector},$$

$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \|h\nabla\cdot\boldsymbol{\sigma}\| \quad \text{stable}.$$

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ be arbitrary. Then, $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ is locally constructed, such that

$$\Pi_p(\nabla\cdot\boldsymbol{\sigma}) = \nabla\cdot(P_p\boldsymbol{\sigma}) \quad \textit{commuting},$$

$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \quad \textit{projector},$$

$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \|h\nabla\cdot\boldsymbol{\sigma}\| \quad \textit{stable}.$$

Proof.

1. $\nabla\cdot\boldsymbol{\sigma}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})$ by construction
2. $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from ▸ construction, global ▸ $\boldsymbol{H}(\mathrm{div})$ stability ($p' = p$)

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|f - \nabla\cdot\boldsymbol{\xi}_h\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\sigma}$$

3. using $\boldsymbol{v}_h = 0$ in ▸ equivalence proof

$$\|\boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot\boldsymbol{\sigma}\|_K^2\right\}^{1/2}$$

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ be arbitrary. Then, $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ is locally constructed, such that

$$\Pi_p(\nabla{\cdot}\boldsymbol{\sigma}) = \nabla{\cdot}(P_p\boldsymbol{\sigma}) \quad \textit{commuting},$$
$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \quad \textit{projector},$$
$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \|h\nabla{\cdot}\boldsymbol{\sigma}\| \quad \textit{stable}.$$

Proof.

1. $\nabla{\cdot}\boldsymbol{\sigma}_h = \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})$ by construction
2. $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from ▸ construction, global ▸ $\boldsymbol{H}(\mathrm{div})$ stability ($p' = p$)

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|f - \nabla{\cdot}\boldsymbol{\xi}_h\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\sigma}$$

3. using $\boldsymbol{v}_h = 0$ in ▸ equivalence proof

$$\|\boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|\nabla{\cdot}\boldsymbol{\sigma}\|_K^2\right\}^{1/2}$$

Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable commuting local projector, $p \geq 0$ EGSV (2018))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ be arbitrary. Then, $P_p \boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ is locally constructed, such that

$$\Pi_p(\nabla{\cdot}\boldsymbol{\sigma}) = \nabla{\cdot}(P_p\boldsymbol{\sigma}) \quad \textit{commuting},$$
$$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma} \text{ if } \boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \quad \textit{projector},$$
$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \|h \nabla{\cdot}\boldsymbol{\sigma}\| \quad \textit{stable}.$$

Proof.

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2. $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from ▸ construction, global ▸ $\boldsymbol{H}(\mathrm{div})$ stability ($p' = p$)
$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} h_K^2 \|f - \nabla{\cdot}\boldsymbol{\xi}_h\|_K^2\right\}^{1/2} = 0 \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\sigma}$$
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Outline

1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
5. A posteriori estimates
   - Guaranteed upper bound
   - Polynomial-degree-robust local efficiency
   - Applications and numerical results
6. Tools
7. Conclusions and outlook

Outline

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), . . . **)**

- *Let* $u \in H_0^1(\Omega)$ *be the weak solution;*
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, *be arbitrary subject to*

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}^{\rm int};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ potential reconstruction;
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Then

$$\|\nabla_h(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}} \Big(\underbrace{\|\nabla_h u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \Big)^2 + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

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Outline

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity Braess, Pillwein, and Schöberl (2009), EV (2015, 2016)**)**

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![u_h]\!]\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \boldsymbol{\sigma}_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of ‣ H^1 stability and ‣ $H(\mathrm{div})$ stability
- p-robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

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- immediate consequence of H^1 stability and $\boldsymbol{H}(\mathrm{div})$ stability
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Outline

Applications

Discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ mixed finite elements

Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform h refinement

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Uniform refinement: asymptotic exactness

h	p	$\|\nabla_{\mathrm{d}}(u-u_h)\|$	$\|\nabla_{\mathrm{d}} u_h+\sigma_h\|$	η_{osc}	$\|\nabla_{\mathrm{d}}(u_h-s_h)\|$	η	I^{eff}
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
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$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
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Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d} u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
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h_0	**2**	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	**1.06**
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	**1.04**
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	**1.03**
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	**1.03**
h_0	**3**	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	**1.03**
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	**1.01**
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	**1.01**
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	**1.01**
h_0	**4**	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	**1.02**
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	**1.01**
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	**1.01**
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	**1.01**
h_0	**5**	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	**1.02**
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	**1.01**
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	**1.00**
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	**1.00**
h_0	**6**	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	**1.02**
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	**1.01**
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	**1.01**

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|u-u_h\|_{\rm DG}$	$\|\nabla_{\rm d}u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$\eta_{\rm DG}$	$I^{\rm eff}$	$I^{\rm eff}_{\rm DG}$
h_0	1	1.07E-00	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
$\approx h_0/2$		5.56E-01	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
$\approx h_0/4$		2.92E-01	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
$\approx h_0/8$		1.39E-01	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
h_0	2	1.54E-01	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
$\approx h_0/2$		4.07E-02	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
$\approx h_0/4$		1.10E-02	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
$\approx h_0/8$		2.50E-03	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
h_0	3	1.37E-02	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
h_0	4	9.87E-04	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
$\approx h_0/2$		6.92E-05	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
$\approx h_0/8$		2.58E-07	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
h_0	5	5.64E-05	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
$\approx h_0/4$		7.74E-08	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
h_0	6	2.85E-06	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

Numerics: singular case

Model problem

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2,\\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$u(r,\phi) = r^{2/3}\sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

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Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
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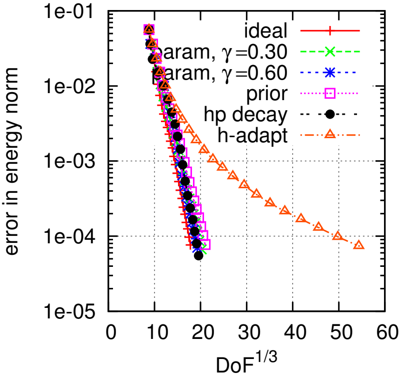
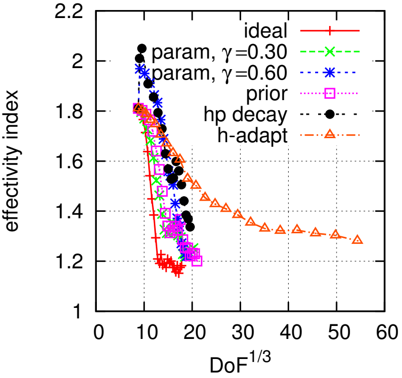
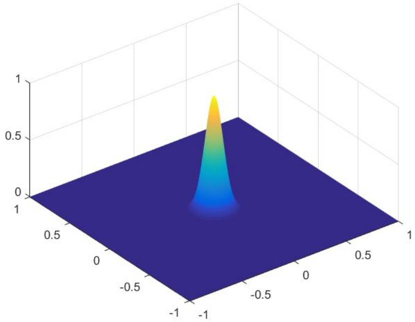
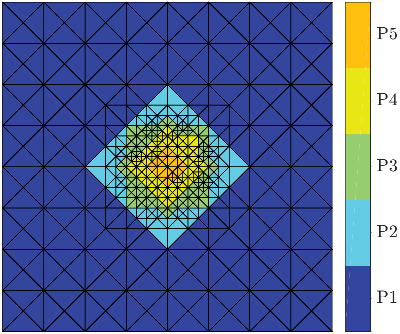
hp-adaptive refinement: **exponential convergence**





Numerics: example of *hp*-approximation





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Potentials

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, Schöberl (2009)**)**

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials

Theorem (Broken H^1 polynomial extension on a patch EV (2015, 2016))

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility

$$\begin{aligned} r_F|_{F\cap\partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}, \\ \sum_{F\in\mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}. \end{aligned}$$

Then

$$\min_{\substack{v_h\in\mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_h=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v_h]\!]=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v\in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v]\!]=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.$$

Fluxes

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel, McIntosh (2010); Ainsworth, Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, Schöberl (2012); EV (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

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Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. *Let* $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, *satisfying* $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ *if* $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. *Then*

$$\|\varphi_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a patch Braess, Pillwein, & Schöberl (2009; 2D), EV (2016; 3D))

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

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Conclusions and outlook

Conclusions

- simple proof of global-best – local-best equivalence in H^1
- constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
- incidentally leads to stable commuting local projectors
- optimal *a priori* error estimates
- *p*-robust *a posteriori* error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, and others carried out

Ongoing work

- *p*-robust global-best – local-best equivalence

Conclusions and outlook

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- *p*-robust global-best – local-best equivalence

References

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Thank you for your attention!