A unified framework for a posteriori error estimation for the Stokes problem

Antti Hannukainen, Rolf Stenberg, and Martin Vohralík

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie, Paris

Glasgow, June 29, 2011

Outline

Outline

Previous results

A posteriori error estimation

- Prager and Synge (1947), energy error equality
- Babuška and Rheinboldt (1978), mathematical analysis
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates
- Hlaváček, Haslinger, Nečas, and Lovíšek (1988), equilibrated fluxes estimates
- Verfürth (1989), local efficiency
- Ainsworth and Oden (1993), equilibration
- Repin (1997), functional a posteriori error estimates
- Luce and Wohlmuth (2004), dual meshes estimates
- Dörfler and Ainsworth (2005), guaranteed upper bound in the Stokes setting
- Ainsworth (2005), unified framework
- Carstensen (2005–2009), unified framework

Outline

The Stokes problem

Stokes problem

Find **u** and p such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega \end{aligned}$$

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in Q \end{aligned}$$

- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega)$
- $a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})$, $b(\mathbf{v}, q) := -(q, \nabla\cdot \mathbf{v})$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta$$

The Stokes problem

Stokes problem

Find $\mathbf{u}$ and p such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega \end{aligned}$$

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 \qquad\quad \forall q \in Q \end{aligned}$$

- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega)$
- $a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})$, $b(\mathbf{v}, q) := -(q, \nabla\cdot\mathbf{v})$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta$$

The Stokes problem

Stokes problem

Find $\mathbf{u}$ and p such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega \end{aligned}$$

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 \qquad \forall q \in Q \end{aligned}$$

- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega)$
- $a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})$, $b(\mathbf{v}, q) := -(q, \nabla \cdot \mathbf{v})$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta$$

The Stokes problem

Stokes problem

Find $\mathbf{u}$ and p such that

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on} \quad \partial\Omega \end{aligned}$$

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 && \forall q \in Q \end{aligned}$$

- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega)$
- $a(\mathbf{u}, \mathbf{v}) := (\nabla\mathbf{u}, \nabla\mathbf{v})$, $b(\mathbf{v}, q) := -(q, \nabla\cdot\mathbf{v})$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\nabla\mathbf{v}\| \, \|q\|} = \beta$$

Babuška–Brezzi splitting

Energy norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V} \times Q$$

Babuška–Brezzi splitting

- $\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) := a(\mathbf{v}, \mathbf{z}) + b(\mathbf{z}, q) + b(\mathbf{v}, r)$
- equivalent formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q$$

- inf–sup condition on $\mathbf{V} \times Q$

$$\inf_{(\mathbf{v},q)\in\mathbf{V}\times Q} \sup_{(\mathbf{z},r)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r))}{|||(\mathbf{z}, r)|||\, |||(\mathbf{v}, q)|||} = \frac{\sqrt{5}-1}{2} =: C_{\mathrm{S}}$$

(β disappears thanks to the definition of the energy norm)
(value of C_{S} communicated to us by J.-F. Maître)

Babuška–Brezzi splitting

Energy norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V} \times Q$$

Babuška–Brezzi splitting

- $\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) := a(\mathbf{v}, \mathbf{z}) + b(\mathbf{z}, q) + b(\mathbf{v}, r)$
- equivalent formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q$$

- inf–sup condition on $\mathbf{V} \times Q$

$$\inf_{(\mathbf{v},q) \in \mathbf{V} \times Q} \sup_{(\mathbf{z},r) \in \mathbf{V} \times Q} \frac{\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r))}{|||(\mathbf{z}, r)||| \, |||(\mathbf{v}, q)|||} = \frac{\sqrt{5} - 1}{2} =: C_{\mathrm{S}}$$

(β disappears thanks to the definition of the energy norm)
(value of C_{S} communicated to us by J.-F. Maître)

Babuška–Brezzi splitting

Energy norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V} \times Q$$

Babuška–Brezzi splitting

- $\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) := a(\mathbf{v}, \mathbf{z}) + b(\mathbf{z}, q) + b(\mathbf{v}, r)$
- equivalent formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q$$

- inf–sup condition on $\mathbf{V} \times Q$

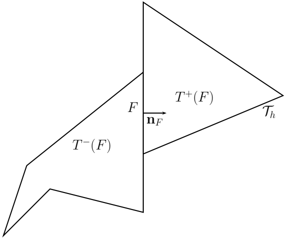
$$\inf_{(\mathbf{v},q)\in\mathbf{V}\times Q} \sup_{(\mathbf{z},r)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r))}{|||(\mathbf{z}, r)|||\ |||(\mathbf{v}, q)|||} = \frac{\sqrt{5}-1}{2} =: C_{\mathrm{S}}$$

(β disappears thanks to the definition of the energy norm)
(value of C_{S} communicated to us by J.-F. Maître)

Discrete setting

Mesh $\mathcal{T}_h$

- a polygonal (polyhedral) partition of Ω
- nonconvex, non star-shaped, elements, $\mathcal{T}_h$ nonmatching



Broken Sobolev space

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}$$

Energy semi-norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V}(\mathcal{T}_h) \times Q$$

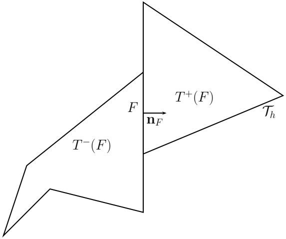
Approximate solution

- $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$

Discrete setting

Mesh $\mathcal{T}_h$

- a polygonal (polyhedral) partition of Ω
- nonconvex, non star-shaped, elements, $\mathcal{T}_h$ nonmatching



Broken Sobolev space

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}$$

Energy semi-norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V}(\mathcal{T}_h) \times Q$$

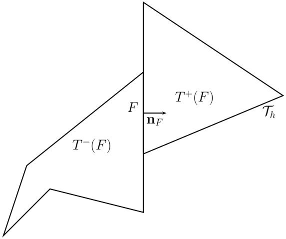
Approximate solution

- $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$

Discrete setting

Mesh $\mathcal{T}_h$

- a polygonal (polyhedral) partition of Ω
- nonconvex, non star-shaped, elements, $\mathcal{T}_h$ nonmatching



Broken Sobolev space

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}$$

Energy semi-norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V}(\mathcal{T}_h) \times Q$$

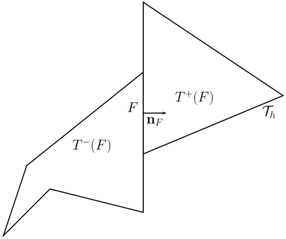
Approximate solution

- $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$

Discrete setting

Mesh $\mathcal{T}_h$

- a polygonal (polyhedral) partition of Ω
- nonconvex, non star-shaped, elements, $\mathcal{T}_h$ nonmatching



Broken Sobolev space

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}$$

Energy semi-norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V}(\mathcal{T}_h) \times Q$$

Approximate solution

- $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$

Outline

Outline

Assumption 1: velocity and stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Assumption 1: velocity and stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Assumption 1: velocity and stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Assumption 1: velocity and stress reconstructions

$-\Delta \mathbf{u} + \nabla p = \mathbf{f}$

$\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$ **constitutive law**

$\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$ **equilibrium**

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Assumption 1: velocity and stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Assumption 1: velocity and stress reconstructions

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} \quad \text{constitutive law}$$
$$\nabla\cdot\underline{\boldsymbol{\sigma}} - \nabla p + \mathbf{f} = \mathbf{0} \quad \text{equilibrium}$$

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

Outline

Guaranteed upper bound

Theorem (A posteriori error estimate (cf. Prager and Synge (1947), Repin (2002), Dörfler and Ainsworth (2005)))

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Let the velocity reconstruction $\mathbf{s}_h$ and the stress reconstruction $\underline{\boldsymbol{\sigma}}_h$ satisfy **Assumption 1***. Then,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \le \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NC},T}^2 \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2}.$$

Guaranteed upper bound

Theorem (A posteriori error estimate (cf. Prager and Synge (1947), Repin (2002), Dörfler and Ainsworth (2005)))

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Let the velocity reconstruction $\mathbf{s}_h$ and the stress reconstruction $\underline{\sigma}_h$ satisfy **Assumption 1**. *Then,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \leq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NC},T}^2 \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2}.$$

Guaranteed upper bound

Theorem (A posteriori error estimate (cf. Prager and Synge (1947), Repin (2002), Dörfler and Ainsworth (2005)))

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Let the velocity reconstruction $\mathbf{s}_h$ and the stress reconstruction $\underline{\boldsymbol{\sigma}}_h$ satisfy **Assumption 1**. *Then,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||$$

$$\leq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NC},T}^2 \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2}.$$

Guaranteed upper bound

Theorem (A posteriori error estimate (cf. Prager and Synge (1947), Repin (2002), Dörfler and Ainsworth (2005)))

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Let the velocity reconstruction $\mathbf{s}_h$ and the stress reconstruction $\underline{\boldsymbol{\sigma}}_h$ satisfy **Assumption 1**. *Then,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \leq \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NC},T}^2 \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2}.$$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla\cdot\mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{0}}$
- **equilibrium**: $\nabla\cdot\underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla\cdot\mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla \cdot \mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla \cdot \mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter
- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla \cdot \mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla \cdot \mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla\cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla\cdot \mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla\cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla\cdot \mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla{\cdot}\mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla{\cdot}\underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla{\cdot}\mathbf{u} = 0$

Estimators, $\underline{\sigma} = \nabla \mathbf{u}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - \underline{\sigma}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\sigma}_h - \nabla p_h + \mathbf{f}\|_T$$

- $\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ as before

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - \underline{\sigma} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\sigma} - \nabla p + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla \cdot \mathbf{u} = 0$

Estimators, $\underline{\sigma} = \nabla \mathbf{u}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - \underline{\sigma}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\sigma}_h - \nabla p_h + \mathbf{f}\|_T$$

- $\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ as before

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - \underline{\sigma} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\sigma} - \nabla p + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla \cdot \mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}\|_T$$

- $\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ as before

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\boldsymbol{\sigma}} - \nabla p + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla \cdot \mathbf{u} = 0$

Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - \underline{\boldsymbol{\sigma}}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla{\cdot}\underline{\boldsymbol{\sigma}}_h - \nabla p_h + \mathbf{f}\|_T$$

- $\eta_{\mathrm{NC},T}$ and $\eta_{\mathrm{D},T}$ as before

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla{\cdot}\underline{\boldsymbol{\sigma}} - \nabla p + \mathbf{f} = \mathbf{0}$
- **constraints**: $\mathbf{u} \in \mathbf{V}$ and $\nabla{\cdot}\mathbf{u} = 0$

Main steps of the proof, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\mathbf{I}}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

- triangle inequality:
$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\| + |||(\mathbf{u}-\mathbf{s}_h, p-p_h)|||$$
- stability estimate:
$$|||(\mathbf{u}-\mathbf{s}_h, p-p_h)||| \leq \frac{1}{C_S} \sup_{(\varphi,\psi)\in \mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi))}{|||(\varphi,\psi)|||}$$
- definition of $\mathcal{B}$ and of the weak solution:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\mathbf{f},\varphi) - (\nabla \mathbf{s}_h, \nabla\varphi) + (\nabla\cdot\varphi, p_h) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla\varphi)$, Green theorem:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f},\varphi) - (\nabla \mathbf{s}_h - p_h\underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla\varphi) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- **local cons. of $\underline{\boldsymbol{\sigma}}_h$**, Poincaré and Cauchy–Schwarz ineq.:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) \leq \left\{\sum_{T\in\mathcal{T}_h} \left\{(\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2\right\}\right\}^{1/2} |||(\varphi,\psi)|||$$

Main steps of the proof, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\mathbf{I}}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

- triangle inequality:
$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le \|\nabla(\mathbf{u}_h-\mathbf{s}_h)\| + |||(\mathbf{u}-\mathbf{s}_h, p-p_h)|||$$
- stability estimate:
$$|||(\mathbf{u}-\mathbf{s}_h, p-p_h)||| \le \frac{1}{C_{\mathrm{S}}} \sup_{(\varphi,\psi)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi))}{|||(\varphi,\psi)|||}$$
- definition of $\mathcal{B}$ and of the weak solution:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\mathbf{f},\varphi) - (\nabla\mathbf{s}_h, \nabla\varphi) + (\nabla\cdot\varphi, p_h) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla\varphi)$, Green theorem:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \varphi) - (\nabla\mathbf{s}_h - p_h\underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla\varphi) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- **local cons. of $\underline{\boldsymbol{\sigma}}_h$**, Poincaré and Cauchy–Schwarz ineq.:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) \le \left\{\sum_{T\in\mathcal{T}_h}\left\{(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2\right\}\right\}^{1/2} |||(\varphi,\psi)|||$$

Main steps of the proof, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\mathbf{I}}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

- triangle inequality:
  $$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \leq \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\| + |||(\mathbf{u} - \mathbf{s}_h, p - p_h)|||$$
- stability estimate:
  $$|||(\mathbf{u} - \mathbf{s}_h, p - p_h)||| \leq \frac{1}{C_{\mathrm{S}}} \sup_{(\varphi,\psi) \in \mathbf{V} \times Q} \frac{\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\varphi, \psi))}{|||(\varphi, \psi)|||}$$
- definition of $\mathcal{B}$ and of the weak solution:
  $$\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\varphi, \psi)) = (\mathbf{f}, \varphi) - (\nabla \mathbf{s}_h, \nabla \varphi) + (\nabla \cdot \varphi, p_h) + (\nabla \cdot \mathbf{s}_h, \psi)$$
- adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla \varphi)$, Green theorem:
  $$\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\varphi, \psi)) = (\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \varphi) - (\nabla \mathbf{s}_h - p_h \underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla \varphi) + (\nabla \cdot \mathbf{s}_h, \psi)$$
- **local cons. of $\underline{\boldsymbol{\sigma}}_h$**, Poincaré and Cauchy–Schwarz ineq.:
  $$\mathcal{B}((\mathbf{u} - \mathbf{s}_h, p - p_h), (\varphi, \psi)) \leq \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2} |||(\varphi, \psi)|||$$

Main steps of the proof, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\mathbf{I}}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

- triangle inequality:
$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\| + |||(\mathbf{u}-\mathbf{s}_h, p-p_h)|||$$
- stability estimate:
$$|||(\mathbf{u}-\mathbf{s}_h, p-p_h)||| \leq \frac{1}{C_{\mathrm{S}}} \sup_{(\varphi,\psi)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi))}{|||(\varphi,\psi)|||}$$
- definition of $\mathcal{B}$ and of the weak solution:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\mathbf{f},\varphi) - (\nabla\mathbf{s}_h, \nabla\varphi) + (\nabla\cdot\varphi, p_h) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla\varphi)$, Green theorem:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \varphi) - (\nabla\mathbf{s}_h - p_h\underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h, \nabla\varphi) + (\nabla\cdot\mathbf{s}_h, \psi)$$
- local cons. of $\underline{\boldsymbol{\sigma}}_h$, Poincaré and Cauchy–Schwarz ineq.:
$$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) \leq \left\{\sum_{T\in\mathcal{T}_h} \left\{(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2\right\}\right\}^{1/2} |||(\varphi,\psi)|||$$

Main steps of the proof, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

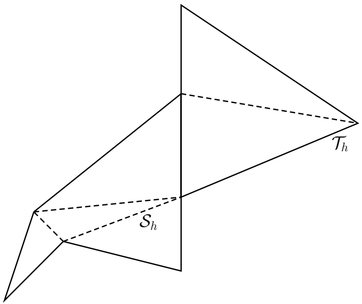
- triangle inequality:
  $$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le \|\nabla(\mathbf{u}_h-\mathbf{s}_h)\| + |||(\mathbf{u}-\mathbf{s}_h, p-p_h)|||$$
- stability estimate:
  $$|||(\mathbf{u}-\mathbf{s}_h, p-p_h)||| \le \frac{1}{C_{\mathrm{S}}} \sup_{(\varphi,\psi)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi))}{|||(\varphi,\psi)|||}$$
- definition of $\mathcal{B}$ and of the weak solution:
  $$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\mathbf{f},\varphi) - (\nabla\mathbf{s}_h,\nabla\varphi) + (\nabla\cdot\varphi, p_h) + (\nabla\cdot\mathbf{s}_h,\psi)$$
- adding and subtracting $(\underline{\boldsymbol{\sigma}}_h, \nabla\varphi)$, Green theorem:
  $$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) = (\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f},\varphi) - (\nabla\mathbf{s}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h,\nabla\varphi) + (\nabla\cdot\mathbf{s}_h,\psi)$$
- **local cons. of $\underline{\boldsymbol{\sigma}}_h$**, Poincaré and Cauchy–Schwarz ineq.:
  $$\mathcal{B}((\mathbf{u}-\mathbf{s}_h, p-p_h),(\varphi,\psi)) \le \left\{\sum_{T\in\mathcal{T}_h}\left\{(\eta_{\mathrm{R},T}+\eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2\right\}\right\}^{1/2} |||(\varphi,\psi)|||$$

Outline

Assumption 2

Technical aspects

- there exists a shape-regular matching simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$
- $\mathbf{u}_h$, p_h, $\mathbf{f}$, and $\underline{\boldsymbol{\sigma}}_h$ are piecewise k-th order polynomials



Nonmatching polygonal mesh $\mathcal{T}_h$ and its simplicial submesh $\mathcal{S}_h$

Assumption 3

Approximation property

There holds

$$\|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

or

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

where

$$\eta_{\mathrm{res},T}^2 := \sum_{T \in \mathfrak{T}_T} \left\{ h_T^2 \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T^2 + \|\nabla \cdot \mathbf{u}_h\|_T^2 \right\} + \sum_{F \in \mathfrak{F}_T^{\mathrm{int}}} h_F \|[\![(\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}) \mathbf{n}_F]\!]\|_F^2 + \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2$$

is the classical local residual error indicator.

Assumption 3

Approximation property

There holds

$$\|\nabla \mathbf{u}_h - p_h \underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

or

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

where

$$\eta_{\mathrm{res},T}^2 := \sum_{T \in \mathfrak{T}_T} \left\{ h_T^2 \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T^2 + \|\nabla \cdot \mathbf{u}_h\|_T^2 \right\} + \sum_{F \in \mathfrak{F}_T^{\mathrm{int}}} h_F \|[\![(\nabla \mathbf{u}_h - p_h \underline{\mathbf{I}})\mathbf{n}_F]\!]\|_F^2 + \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2$$

is the classical local residual error indicator.

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let **Assumptions 2** *and* **3** *hold. Let* $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1}\|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let **Assumptions 2** *and* **3** *hold. Let* $\mathbf{s}_h = \mathcal{I}_{\rm av}(\mathbf{u}_h)$*. Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\rm NC,\mathit{T}} + \eta_{\rm R,\mathit{T}} + \eta_{\rm DF,\mathit{T}} + \eta_{\rm D,\mathit{T}}$$
$$\lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1}\|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let **Assumptions 2** *and* **3** *hold. Let* $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$*. Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1} \|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

*Let **Assumptions 2** and **3** hold. Let $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$. Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1}\|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let Assumptions 2 and 3 hold. Let $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$*. Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1} \|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let Assumptions 2 and 3 hold. Let $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T}$$
$$\lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1}\|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1}\|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let **Assumptions 2** *and* **3** *hold. Let* $\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$. *Let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F$ can be bounded by $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T}$ in MFEs (Achdou, Bernardi, Coquel 2003) and DGs (Ainsworth 2007)
- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = h_F^{-1} \|[\![\mathbf{u} - \mathbf{u}_h]\!]\|_F$ can be added to the error measure to obtain both-sided estimates in the same norm

Outline

Outline

Discontinuous Galerkin method

Discontinuous approximation spaces

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

Bilinear and linear forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T\in\mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!]\rangle_F$$
$$- \sum_{F\in\partial\mathcal{T}_h} \left\{ \langle \{\!\{\nabla \mathbf{u}_h\}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!]\rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F \right\},$$

$$b_h(\mathbf{v}_h, q_h) := - \sum_{T\in\mathcal{T}_h} (q_h, \nabla\cdot\mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!]\cdot\mathbf{n}_F\rangle_F$$

Discontinuous Galerkin method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

Discontinuous Galerkin method

Discontinuous approximation spaces

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

Bilinear and linear forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T\in\mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F$$
$$- \sum_{F\in\partial\mathcal{T}_h} \left\{ \langle \{\!\{\nabla \mathbf{u}_h\}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!] \rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F \right\},$$

$$b_h(\mathbf{v}_h, q_h) := - \sum_{T\in\mathcal{T}_h} (q_h, \nabla\cdot\mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!]\cdot\mathbf{n}_F \rangle_F$$

Discontinuous Galerkin method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0 \qquad \forall q_h \in Q_h \end{aligned}$$

Discontinuous Galerkin method

Discontinuous approximation spaces

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

Bilinear and linear forms

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) := & \sum_{T\in\mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F \\ & - \sum_{F\in\partial\mathcal{T}_h} \{ \langle \{\!\{\nabla \mathbf{u}_h\}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!] \rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F \}, \end{aligned}$$

$$b_h(\mathbf{v}_h, q_h) := - \sum_{T\in\mathcal{T}_h} (q_h, \nabla\cdot \mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!] \cdot \mathbf{n}_F \rangle_F$$

Discontinuous Galerkin method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

Velocity and stress reconstructions in DG

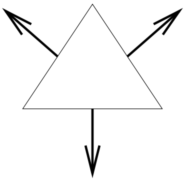
Reconstructed velocity $\mathbf{s}_h$

$$\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$$

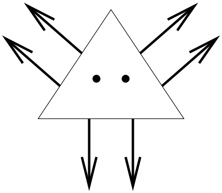
Reconstructed stress $\underline{\sigma}_h$

$$\underline{\sigma}_h \in \underline{\Sigma}^l(\mathcal{T}_h) := \{\underline{v}_h \in \underline{\boldsymbol{H}}(\mathrm{div},\Omega);\ \underline{v}_h|_T \in \underline{\Sigma}^l(T) \quad \forall T \in \mathcal{T}_h\},$$
$$\underline{\Sigma}^l(T) := [\mathbb{P}_l(T)]^{d\times d} + [\mathbb{P}_l(T)]^d \otimes \mathbf{x},$$

Raviart–Thomas–Nédélec space of tensor functions of order l, $l = k-1$ or k (simplicial meshes)



$l = 0$



$l = 1$

Velocity and stress reconstructions in DG

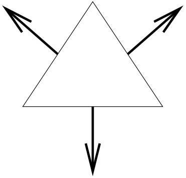
Reconstructed velocity $\mathbf{s}_h$

$$\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$$

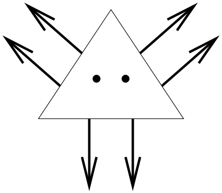
Reconstructed stress $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega);\ \underline{\boldsymbol{v}}_h|_T \in \underline{\boldsymbol{\Sigma}}^l(T) \quad \forall T \in \mathcal{T}_h\},$$
$$\underline{\boldsymbol{\Sigma}}^l(T) := [\mathbb{P}_l(T)]^{d\times d} + [\mathbb{P}_l(T)]^d \otimes \mathbf{x},$$

Raviart–Thomas–Nédélec space of tensor functions of order l, $l = k - 1$ or k (simplicial meshes)



$l = 0$



$l = 1$

Application of the framework to DG

Specification of degrees of freedom of $\underline{\boldsymbol{\sigma}}_h$

- for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$

$$\langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F$$

- for all $\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d \times d}$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \underline{\boldsymbol{\tau}}_h)_T - \theta \sum_{F \in \mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F$$

Lemma (Reconstructed stress in the DG method)

For all $T \in \mathcal{T}_h$, there holds

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Application of the framework to DG

Specification of degrees of freedom of $\underline{\boldsymbol{\sigma}}_h$

- for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$

$$\langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F$$

- for all $\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d \times d}$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \underline{\boldsymbol{\tau}}_h)_T - \theta \sum_{F \in \mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F$$

Lemma (Reconstructed stress in the DG method)

For all $T \in \mathcal{T}_h$, there holds

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Application of the framework to DG

Specification of degrees of freedom of $\underline{\boldsymbol{\sigma}}_h$

- for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$

$$\langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F$$

- for all $\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d \times d}$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \underline{\boldsymbol{\tau}}_h)_T - \theta \sum_{F \in \mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F$$

Lemma (Reconstructed stress in the DG method)

For all $T \in \mathcal{T}_h$, there holds

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Application of the framework to DG

Specification of degrees of freedom of $\underline{\boldsymbol{\sigma}}_h$

- for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$

$$\langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F$$

- for all $\underline{\boldsymbol{\tau}}_h \in [\mathbb{P}_{l-1}(T)]^{d \times d}$

$$(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}, \underline{\boldsymbol{\tau}}_h)_T - \theta \sum_{F \in \mathcal{F}_T} \langle \omega_F \underline{\boldsymbol{\tau}}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F$$

Lemma (Reconstructed stress in the DG method)

For all $T \in \mathcal{T}_h$, there holds

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Outline

Conforming and conforming stabilized methods

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- t_h and s_h: stabilization terms
- Brezzi–Pitkäranta family
- Hughes–Franca–Balestra family
- Brezzi–Douglas family

Conforming and conforming stabilized methods

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h$$

- t_h and s_h: stabilization terms
- Brezzi–Pitkäranta family
- Hughes–Franca–Balestra family
- Brezzi–Douglas family

Conforming and conforming stabilized methods

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- t_h and s_h: stabilization terms
- Brezzi–Pitkäranta family
- Hughes–Franca–Balestra family
- Brezzi–Douglas family

Conforming and conforming stabilized methods

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- t_h and s_h: stabilization terms
- Brezzi–Pitkäranta family
- Hughes–Franca–Balestra family
- Brezzi–Douglas family

Conforming and conforming stabilized methods

$\mathbf{s}_h := \mathbf{u}_h$

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

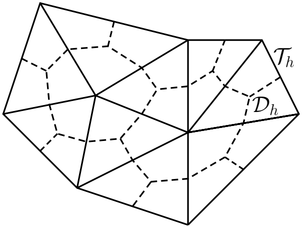
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + t_h(\mathbf{u}_h, p_h; \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$s_h(\mathbf{u}_h, p_h; q_h) + b(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in Q_h$$

- t_h and s_h: stabilization terms
- Brezzi–Pitkäranta family
- Hughes–Franca–Balestra family
- Brezzi–Douglas family

Local conservativity, lowest-order conforming methods

Dual mesh



Normal flux functions

$$\Upsilon_F(\mathbf{u}_h) := (\nabla \mathbf{u}_h \mathbf{n}_F)|_F \qquad F \subset \partial D,\ D \in \mathcal{D}_h$$

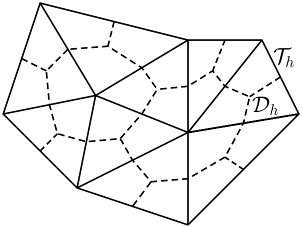
Lemma (Conservativity on $\mathcal{D}_h^{\rm int}$; $\sim$ Luce and Wohlmuth (2004))

For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$*, there holds*

$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h)\mathbf{n}_D {\cdot} \mathbf{n}_F, \mathbf{e}_i \rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D = 0, \qquad i = 1, \ldots, d \quad \forall D \in \mathcal{D}_h^{\rm int}.$$

Local conservativity, lowest-order conforming methods

Dual mesh



Normal flux functions

$$\Upsilon_F(\mathbf{u}_h) := (\nabla \mathbf{u}_h \mathbf{n}_F)|_F \qquad F \subset \partial D,\ D \in \mathcal{D}_h$$

Lemma (Conservativity on $\mathcal{D}_h^{\rm int}$; $\sim$ Luce and Wohlmuth (2004))

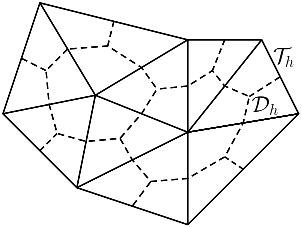
For $\mathbf{f}$ piecewise constant on $\mathcal{T}_h$, there holds

$$\sum_{F\in\mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h)\mathbf{n}_D{\cdot}\mathbf{n}_F, \mathbf{e}_i\rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D = 0,$$

$$i = 1,\ldots,d \quad \forall D \in \mathcal{D}_h^{\rm int}.$$

Local conservativity, lowest-order conforming methods

Dual mesh



Normal flux functions

$$\Upsilon_F(\mathbf{u}_h) := (\nabla \mathbf{u}_h \mathbf{n}_F)|_F \qquad F \subset \partial D,\ D \in \mathcal{D}_h$$

Lemma (Conservativity on $\mathcal{D}_h^{\text{int}}$; $\sim$ Luce and Wohlmuth (2004))

For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$*, there holds*

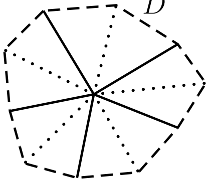
$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h)\mathbf{n}_D \cdot \mathbf{n}_F, \mathbf{e}_i \rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D = 0,$$

$$i = 1, \ldots, d \quad \forall D \in \mathcal{D}_h^{\text{int}}.$$

Stress reconstruction, lowest-order conf. methods

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^{0}(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^{0}(\mathcal{S}_D);\ \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h) \quad \forall F \subset \partial D\}$$



Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

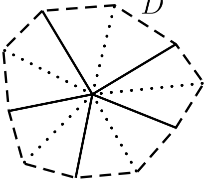
$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^{0}(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = \nabla p_h - \mathbf{f}} \|\nabla \mathbf{u}_h - \underline{\boldsymbol{v}}_h\|_D$$

- **local** Raviart–Thomas–Nédélec **MFE problem** on $\mathcal{S}_D$ (Neumann BC given by $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ on $\partial D \setminus \partial\Omega$, homogeneous Dirichlet BC given on $\partial D \cap \partial\Omega$)
- complementary energy minimization with constraints

Stress reconstruction, lowest-order conf. methods

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\mathbf{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0(\mathcal{S}_D);\ \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h) \quad \forall F \subset \partial D\}$$



Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = \nabla p_h - \mathbf{f}} \|\nabla \mathbf{u}_h - \underline{\boldsymbol{v}}_h\|_D$$

- **local** Raviart–Thomas–Nédélec **MFE problem** on $\mathcal{S}_D$ (Neumann BC given by $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$ on $\partial D \setminus \partial\Omega$, homogeneous Dirichlet BC given on $\partial D \cap \partial\Omega$)
- complementary energy minimization with constraints

Application of the framework to CG

Lemma (Reconstructed stress in the CG method)

For $\mathbf{f}$ *piecewise constant, there holds,*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = (\nabla p_h - \mathbf{f})|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Main elements of the proof

- construction of $\underline{\boldsymbol{\sigma}}_h$ from $\mathbf{u}_h$ and p_h
- local postprocessing of potentials in MFEs
- duality
- properties of Raviart–Thomas–Nédélec spaces, scaling arguments, equivalence of norms
- inverse, discrete Poincaré, and Friedrichs inequalities

Application of the framework to CG

Lemma (Reconstructed stress in the CG method)

For $\mathbf{f}$ *piecewise constant, there holds,*

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h)|_T = (\nabla p_h - \mathbf{f})|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Main elements of the proof

- construction of $\underline{\boldsymbol{\sigma}}_h$ from $\mathbf{u}_h$ and p_h
- local postprocessing of potentials in MFEs
- duality
- properties of Raviart–Thomas–Nédélec spaces, scaling arguments, equivalence of norms
- inverse, discrete Poincaré, and Friedrichs inequalities

Application of the framework to CG

Lemma (Reconstructed stress in the CG method)

For $\mathbf{f}$ *piecewise constant, there holds,*

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h)|_T = (\nabla p_h - \mathbf{f})|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Main elements of the proof

- construction of $\underline{\boldsymbol{\sigma}}_h$ from $\mathbf{u}_h$ and p_h
- local postprocessing of potentials in MFEs
- duality
- properties of Raviart–Thomas–Nédélec spaces, scaling arguments, equivalence of norms
- inverse, discrete Poincaré, and Friedrichs inequalities

Outline

Crouzeix–Raviart nonconforming method

Discontinuous approximation space

$$\mathbf{V}_h := \{\mathbf{v}_h \in [\mathbb{P}_1(\mathcal{T}_h)]^d;\ \langle [\![\mathbf{v}_h]\!], \mathbf{e}_i \rangle_F = 0, \quad i = 1, \ldots, d, \quad \forall F \in \partial\mathcal{T}_h\},$$
$$Q_h := \mathbb{P}_0(\mathcal{T}_h) \cap Q$$

Crouzeix–Raviart nonconforming method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 & \forall q_h \in Q_h \end{aligned}$$

Crouzeix–Raviart nonconforming method

Discontinuous approximation space

$$\mathbf{V}_h := \{\mathbf{v}_h \in [\mathbb{P}_1(\mathcal{T}_h)]^d;\ \langle [\![\mathbf{v}_h]\!], \mathbf{e}_i \rangle_F = 0, \quad i = 1, \ldots, d, \quad \forall F \in \partial\mathcal{T}_h\},$$
$$Q_h := \mathbb{P}_0(\mathcal{T}_h) \cap Q$$

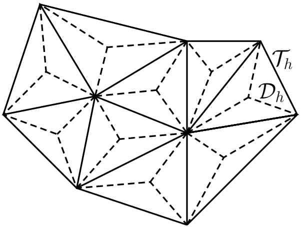
Crouzeix–Raviart nonconforming method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad &\forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 \qquad &\forall q_h \in Q_h \end{aligned}$$

Local conservativity, Crouzeix–Raviart method

Dual mesh



Dual mesh $\mathcal{D}_h$ and simplicial submesh $\mathcal{S}_h$

Normal flux functions

$$\Upsilon_F(\mathbf{u}_h, p_h) := (\nabla \mathbf{u}_h - p_h \underline{\mathbf{I}})\mathbf{n}_F \quad F \subset \partial D,\ D \in \mathcal{D}_h$$

Lemma (Local conservativity on $\mathcal{D}_h^{\mathrm{int}}$)

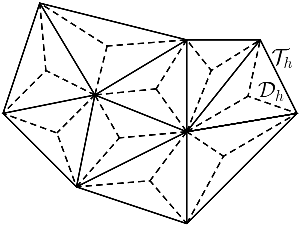
For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$, *there holds*

$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h, p_h)\mathbf{n}_D \cdot \mathbf{n}_F, \mathbf{e}_i \rangle_F + (\mathbf{f}, \mathbf{e}_i)_D = 0,$$

$$i = 1, \ldots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Local conservativity, Crouzeix–Raviart method

Dual mesh



Dual mesh $\mathcal{D}_h$ and simplicial submesh $\mathcal{S}_h$

Normal flux functions

$$\Upsilon_F(\mathbf{u}_h, p_h) := (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}})\mathbf{n}_F \quad F \subset \partial D,\ D \in \mathcal{D}_h$$

Lemma (Local conservativity on $\mathcal{D}_h^{\mathrm{int}}$)

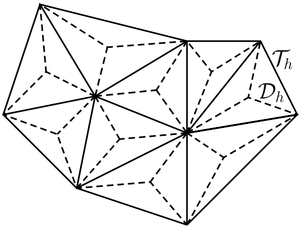
For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$*, there holds*

$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h, p_h)\mathbf{n}_D \cdot \mathbf{n}_F, \mathbf{e}_i \rangle_F + (\mathbf{f}, \mathbf{e}_i)_D = 0,$$

$$i = 1, \ldots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Local conservativity, Crouzeix–Raviart method

Dual mesh



Dual mesh $\mathcal{D}_h$ and simplicial submesh $\mathcal{S}_h$

Normal flux functions

$$\Upsilon_F(\mathbf{u}_h, p_h) := (\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}})\mathbf{n}_F \quad F \subset \partial D,\ D \in \mathcal{D}_h$$

Lemma (Local conservativity on $\mathcal{D}_h^{\mathrm{int}}$)

For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$*, there holds*

$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h, p_h)\mathbf{n}_D{\cdot}\mathbf{n}_F, \mathbf{e}_i \rangle_F + (\mathbf{f}, \mathbf{e}_i)_D = 0,$$

$$i = 1, \ldots, d, \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Stress reconstruction, Crouzeix–Raviart method

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\Sigma}^0_{\mathrm{N}}(\mathcal{S}_D) := \{\underline{v}_h \in \underline{\Sigma}^0(\mathcal{S}_D);\ \underline{v}_h \mathbf{n}_F = \Upsilon_F(\mathbf{u}_h, p_h) \quad \forall F \subset \partial D\}$$

Stress reconstruction $\underline{\sigma}_h$

$$\underline{\sigma}_h|_D := \arg \inf_{\underline{v}_h \in \underline{\Sigma}^0_{\mathrm{N}}(\mathcal{S}_D),\, \nabla \cdot \underline{v}_h = -\mathbf{f}} \|\nabla \mathbf{u}_h - p_h \underline{I} - \underline{v}_h\|_D$$

Lemma (Reconstructed stress in the CR method)

For $\mathbf{f}$ piecewise constant, there holds

$$(\nabla \cdot \underline{\sigma}_h)|_T = -\mathbf{f}|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Stress reconstruction, Crouzeix–Raviart method

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\mathbf{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0(\mathcal{S}_D);\ \underline{\boldsymbol{v}}_h \mathbf{n}_F = \Upsilon_F(\mathbf{u}_h, p_h) \quad \forall F \subset \partial D\}$$

Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = -\mathbf{f}} \|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{v}}_h\|_D$$

Lemma (Reconstructed stress in the CR method)

For $\mathbf{f}$ piecewise constant, there holds

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = -\mathbf{f}|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Stress reconstruction, Crouzeix–Raviart method

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_D);\ \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) \quad \forall F \subset \partial D\}$$

Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}_{\mathrm{N}}^0(\mathcal{S}_D),\, \nabla\cdot\underline{\boldsymbol{v}}_h = -\mathbf{f}} \|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{v}}_h\|_D$$

Lemma (Reconstructed stress in the CR method)

For $\mathbf{f}$ *piecewise constant, there holds*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = -\mathbf{f}|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Stress reconstruction, Crouzeix–Raviart method

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\boldsymbol{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{S}_D);\, \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h, p_h) \quad \forall F \subset \partial D\}$$

Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D),\, \nabla \cdot \underline{\boldsymbol{v}}_h = -\mathbf{f}} \|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{v}}_h\|_D$$

Lemma (Reconstructed stress in the CR method)

For $\mathbf{f}$ *piecewise constant, there holds*

$$(\nabla \cdot \underline{\boldsymbol{\sigma}}_h)|_T = -\mathbf{f}|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Lemma (Approximation property)

Assumption 3 *holds true.*

Outline

A general locally conservative method

A general locally conservative method

$$\sum_{F \in \mathcal{F}_T} \Upsilon^i_F(\mathbf{n}_T \cdot \mathbf{n}_F) + (\mathbf{f}, \mathbf{e}_i)_T = 0, \qquad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

- side fluxes Υ_F, veloc. $\mathbf{u}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^d$, pressures $p_h \in \mathbb{P}_0(\mathcal{T}_h)$

Stress reconstruction

$\underline{\sigma}_h \in \underline{\Sigma}^0(\mathcal{T}_h)$ such that $\underline{\sigma}_h \mathbf{n}_F|_F := \frac{\Upsilon_F}{|F|}$

Elementwise postprocessing of the velocity

$\tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ such that

$$\nabla \tilde{\mathbf{u}}_h|_T - p_h \underline{\boldsymbol{I}}|_T = \underline{\sigma}_h|_T,$$

$$(\tilde{\mathbf{u}}_h, \mathbf{e}_i)_T / |T| = \mathbf{u}^i_h|_T, \qquad i = 1, \ldots, d$$

Application of the framework

- definition of $\underline{\sigma}_h$ and local conservation $\Rightarrow$ **Assumption 1**
- $\|\nabla \tilde{\mathbf{u}}_h - p_h \underline{\boldsymbol{I}} - \underline{\sigma}_h\|_T = 0$ by the def. of $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

A general locally conservative method

A general locally conservative method

$$\sum_{F \in \mathcal{F}_T} \Upsilon^i_F(\mathbf{n}_T \cdot \mathbf{n}_F) + (\mathbf{f}, \mathbf{e}_i)_T = 0, \qquad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

- side fluxes Υ_F, veloc. $\mathbf{u}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^d$, pressures $p_h \in \mathbb{P}_0(\mathcal{T}_h)$

Stress reconstruction

$\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{T}_h)$ such that $\underline{\boldsymbol{\sigma}}_h \mathbf{n}_F|_F := \frac{\Upsilon_F}{|F|}$

Elementwise postprocessing of the velocity

$\tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ such that

$$\nabla \tilde{\mathbf{u}}_h|_T - p_h \underline{\boldsymbol{I}}|_T = \underline{\boldsymbol{\sigma}}_h|_T,$$

$$(\tilde{\mathbf{u}}_h, \mathbf{e}_i)_T / |T| = \mathbf{u}^i_h|_T, \qquad i = 1, \ldots, d$$

Application of the framework

- definition of $\underline{\boldsymbol{\sigma}}_h$ and local conservation $\Rightarrow$ **Assumption 1**
- $\|\nabla \tilde{\mathbf{u}}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T = 0$ by the def. of $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

A general locally conservative method

A general locally conservative method

$$\sum_{F\in\mathcal{F}_T} \Upsilon_F^i(\mathbf{n}_T\cdot\mathbf{n}_F) + (\mathbf{f},\mathbf{e}_i)_T = 0, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h$$

- side fluxes Υ_F, veloc. $\mathbf{u}_h\in[\mathbb{P}_0(\mathcal{T}_h)]^d$, pressures $p_h\in\mathbb{P}_0(\mathcal{T}_h)$

Stress reconstruction

$\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{T}_h)$ such that $\underline{\boldsymbol{\sigma}}_h\mathbf{n}_F|_F := \frac{\Upsilon_F}{|F|}$

Elementwise postprocessing of the velocity

$\tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ such that

$$\nabla\tilde{\mathbf{u}}_h|_T - p_h\underline{\boldsymbol{I}}|_T = \underline{\boldsymbol{\sigma}}_h|_T,$$
$$(\tilde{\mathbf{u}}_h,\mathbf{e}_i)_T/|T| = \mathbf{u}_h^i|_T, \qquad i=1,\ldots,d$$

Application of the framework

- definition of $\underline{\boldsymbol{\sigma}}_h$ and local conservation $\Rightarrow$ **Assumption 1**
- $\|\nabla\tilde{\mathbf{u}}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T = 0$ by the def. of $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

A general locally conservative method

A general locally conservative method

$$\sum_{F\in\mathcal{F}_T} \Upsilon^i_F(\mathbf{n}_T\cdot\mathbf{n}_F) + (\mathbf{f},\mathbf{e}_i)_T = 0, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h$$

- side fluxes Υ_F, veloc. $\mathbf{u}_h\in[\mathbb{P}_0(\mathcal{T}_h)]^d$, pressures $p_h\in\mathbb{P}_0(\mathcal{T}_h)$

Stress reconstruction

$\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^0(\mathcal{T}_h)$ such that $\underline{\boldsymbol{\sigma}}_h\mathbf{n}_F|_F := \frac{\Upsilon_F}{|F|}$

Elementwise postprocessing of the velocity

$\tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ such that

$$\nabla\tilde{\mathbf{u}}_h|_T - p_h\underline{\boldsymbol{I}}|_T = \underline{\boldsymbol{\sigma}}_h|_T,$$
$$(\tilde{\mathbf{u}}_h,\mathbf{e}_i)_T/|T| = \mathbf{u}^i_h|_T, \qquad i=1,\ldots,d$$

Application of the framework

- definition of $\underline{\boldsymbol{\sigma}}_h$ and local conservation $\Rightarrow$ **Assumption 1**
- $\|\nabla\tilde{\mathbf{u}}_h - p_h\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T = 0$ by the def. of $\tilde{\mathbf{u}}_h$ $\Rightarrow$ **Assumption 3**

Outline

Mixed finite element methods

Mixed finite element method

Find $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h, p_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p such that

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h, \nabla\cdot\underline{\boldsymbol{\tau}}_h) &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h,\\
-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
(\mathbf{u}_h, \nabla q_h) &= 0 && \forall q_h \in Q_h
\end{aligned}
$$

Approximation spaces

$\underline{\boldsymbol{\Sigma}}_h := \underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$, $\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d$, and $Q_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega) \cap Q$, $k \geq 0$

Application of the framework

- $\underline{\boldsymbol{\sigma}}_h$ directly constructed by the MFE $\Rightarrow$ **Assumption 1**
- local postprocessing of $\mathbf{u}_h$ into $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

Mixed finite element methods

Mixed finite element method

Find $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h, p_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p such that

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h, \nabla\cdot\underline{\boldsymbol{\tau}}_h) &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h,\\
-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
(\mathbf{u}_h, \nabla q_h) &= 0 && \forall q_h \in Q_h
\end{aligned}
$$

Approximation spaces

$\underline{\boldsymbol{\Sigma}}_h := \underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$, $\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d$, and $Q_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega) \cap Q$, $k \geq 0$

Application of the framework

- $\underline{\boldsymbol{\sigma}}_h$ directly constructed by the MFE $\Rightarrow$ **Assumption 1**
- local postprocessing of $\mathbf{u}_h$ into $\tilde{\mathbf{u}}_h$ $\Rightarrow$ **Assumption 3**

Mixed finite element methods

Mixed finite element method

Find $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h, p_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p such that

$$\begin{aligned} (\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h, \nabla\cdot\underline{\boldsymbol{\tau}}_h) &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h, \\ -(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathbf{u}_h, \nabla q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

Approximation spaces

$\underline{\boldsymbol{\Sigma}}_h := \underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$, $\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d$, and $Q_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega) \cap Q$, $k \geq 0$

Application of the framework

- $\underline{\boldsymbol{\sigma}}_h$ directly constructed by the MFE $\Rightarrow$ **Assumption 1**
- local postprocessing of $\mathbf{u}_h$ into $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

Outline

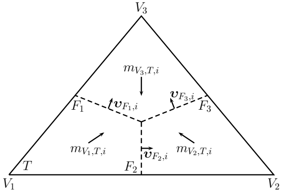
Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

“Nonconservative” schemes

- locally conservative fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- higher-order schemes: equilibration (cf. Ainsworth and Oden (1993)); here a **fixed** small **size** $(d+1) \times (d+1)$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_{F_1,i} \\ v_{F_2,i} \\ v_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

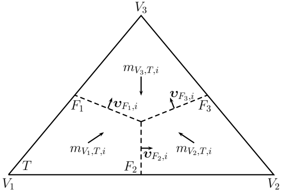
Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

“Nonconservative” schemes

- locally conservative fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- higher-order schemes: equilibration (cf. Ainsworth and Oden (1993)); here a **fixed** small **size** $(d+1) \times (d+1)$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

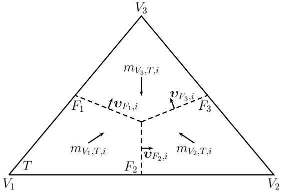
Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

"Nonconservative" schemes

- locally conservative fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- higher-order schemes: equilibration (cf. Ainsworth and Oden (1993)); here a **fixed** small **size** $(d+1)\times(d+1)$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

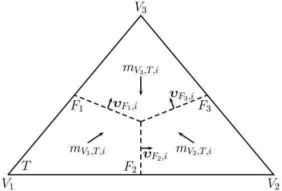
Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

"Nonconservative" schemes

- locally conservative fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- higher-order schemes: equilibration (cf. Ainsworth and Oden (1993)); here a **fixed** small **size** $(d+1) \times (d+1)$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

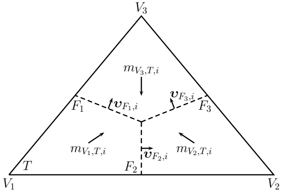
Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

“Nonconservative” schemes

- locally conservative fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- higher-order schemes: equilibration (cf. Ainsworth and Oden (1993)); here a **fixed** small **size** $(d+1)\times(d+1)$



$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

Outline

Setting

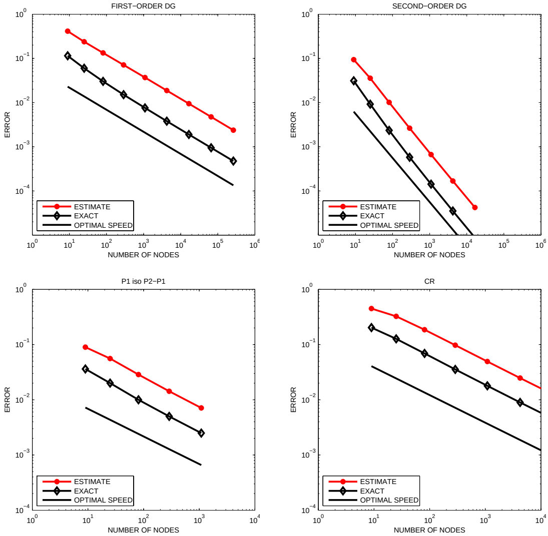
Model problem

- $\Omega = (0,1) \times (0,1)$
- $\mathbf{f}$ chosen according to the solution

$$\mathbf{u} = \nabla \times (x-1)^2 x^{1+\alpha} (y-1)^2 y^2 \mathbf{e}_3, \quad p = x + y - 1$$

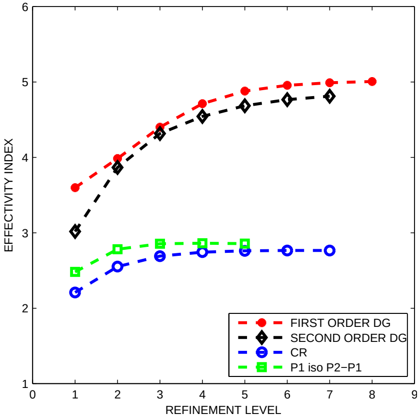
- regularity: $[H^{\frac{1}{2}+\alpha}(\Omega)]^d$ for $\alpha \notin \mathbb{N}$ and $[C^\infty(\Omega)]^d$ for $\alpha \in \mathbb{N}$

Errors and estimates



Estimated and exact errors, smooth case

Effectivity indices



Effectivity indices, smooth case

Estimated and exact error distributions

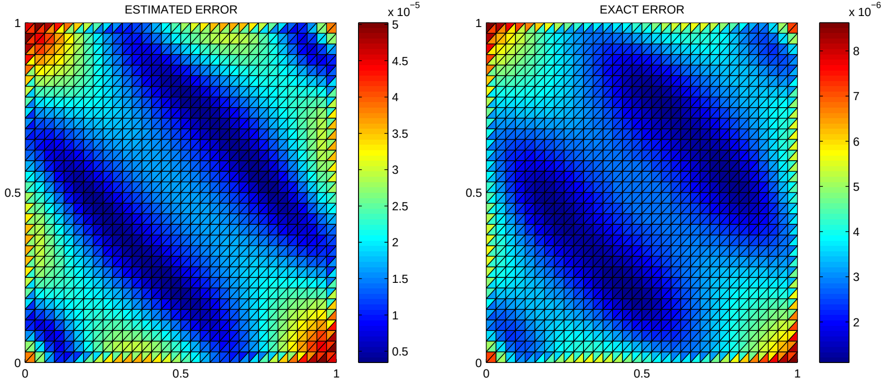


Figure: Estimated (left) and exact (right) error distributions, 2nd order DG method, smooth test case

Estimated and exact error distributions

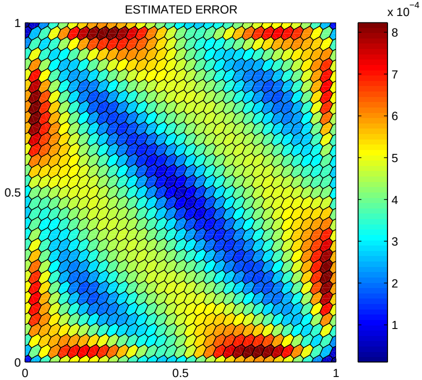


Figure: Estimated (left) and exact (right) error distributions, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method, smooth test case

Estimated and exact error distributions

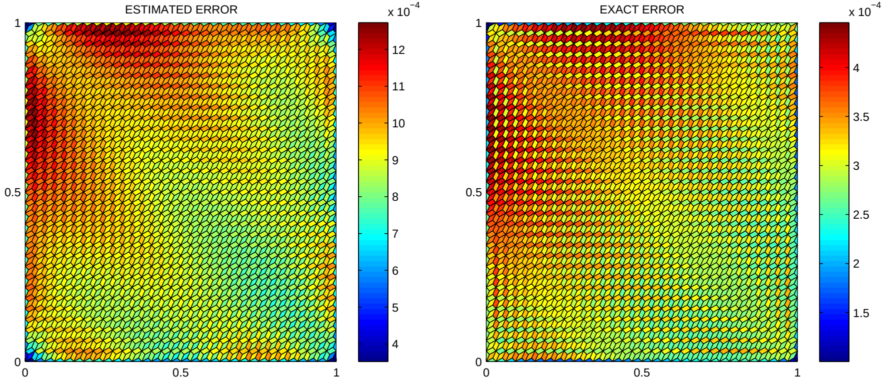
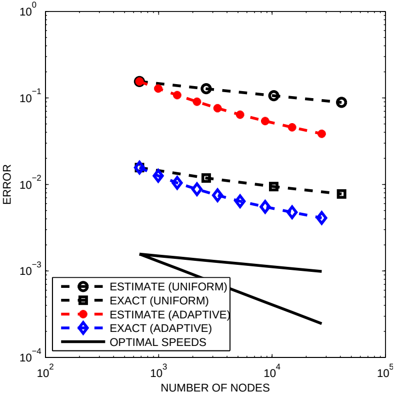


Figure: Estimated (left) and exact (right) error distributions, Crouzeix–Raviart, smooth test case

Singular cases and adaptivity



Estimated and exact errors in uniform/adaptive refinement, first-order DG method, singular test case

Adaptive mesh refinement

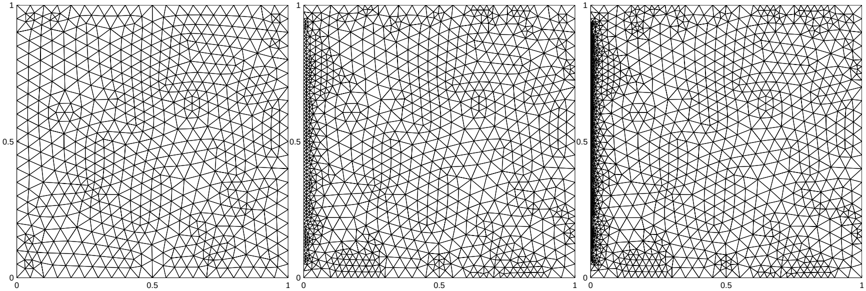


Figure: Adaptively refined meshes, 1st order DG method, singular test case

Outline

Conclusions and future work

Conclusions

- **unified framework** for major numerical methods
- **no discrete** inf–sup **condition** needed
- easily and **fully computable** estimates, **locally efficient**
- estimates **physically relevant**
- based on **local conservation**, built-in in any(?) scheme (directly or after equilibration)

Future work

- instationary Stokes problem
- Navier–Stokes problem

Conclusions and future work

Conclusions

- **unified framework** for major numerical methods
- **no discrete** inf–sup **condition** needed
- easily and **fully computable** estimates, **locally efficient**
- estimates **physically relevant**
- based on **local conservation**, built-in in any(?) scheme (directly or after equilibration)

Future work

- instationary Stokes problem
- Navier–Stokes problem

Bibliography

Papers

- HANNUKAINEN A., STENBERG R., AND VOHRALÍK M., A unified framework for a posteriori error estimation for the Stokes problem, HAL preprint 00470131, April 3, 2010, in review for *Numer. Math.*
- ERN A., VOHRALÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.* **48** (2010), 198–223.

Thank you for your attention!