Inexpensive polynomial-degree-robust equilibrated flux a posteriori estimates for isogeometric analysis

Gregor Gantner and **Martin Vohralík**

Inria Paris & Ecole des Ponts

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Outline

1. Introduction
   - The Poisson model problem and its Galerkin approximation
   - State of the art & goals
   - Equilibration in finite elements
   - Equilibration in IGA: a first idea
2. Inexpensive equilibration in IGA
   - Main idea
   - Hierarchical mesh in the parameter domain
   - Hierarchical B-splines in the parameter domain
   - Bi-Lipschitz mapping $\boldsymbol{F}$ and the physical domain Ω
3. Theoretical results
4. Numerical experiments
5. Conclusions

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The Poisson model problem and its Galerkin approximation

The Poisson problem

Find $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, *such that*

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Weak formulation

Find $u \in H^1_0(\Omega)$ *such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H^1_0(\Omega).$$

Galerkin approximation

Find $u_h \in V_h \subset H^1_0(\Omega)$ *such that*

$$V_h = \mathbb{Q}^p(\mathcal{T}_h) \cap C^{p-m}(\Omega)$$

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goals

A posteriori error estimates

a posteriori estimates

$$\|\nabla(u - u_h)\|_\Omega \leq \eta(u_h)$$

goals

A posteriori error estimates

Guaranteed a posteriori estimates

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goals

A posteriori error estimates

Guaranteed a posteriori estimates **efficient**

$$\|\nabla(u - u_h)\|_\Omega \leq \eta(u_h) \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_\Omega, \quad .$$

goals

A posteriori error estimates

Guaranteed a posteriori estimates **efficient** and **robust** wrt h and p:

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State of the art & goals

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Guaranteed a posteriori estimates **locally efficient** and **robust** wrt h and p:

$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\omega_K}, \quad C_{\text{eff}} \text{ only depends on } d, \kappa_{\mathcal{T}_h}.$$

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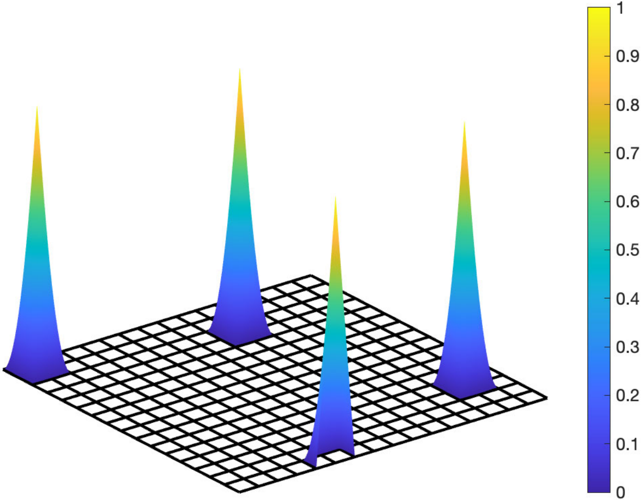

Hat basis functions $\psi_{\mathbf{a}} \in \mathbb{Q}^1(\mathcal{T}_h) \cap C^0(\Omega) \subset V_h$

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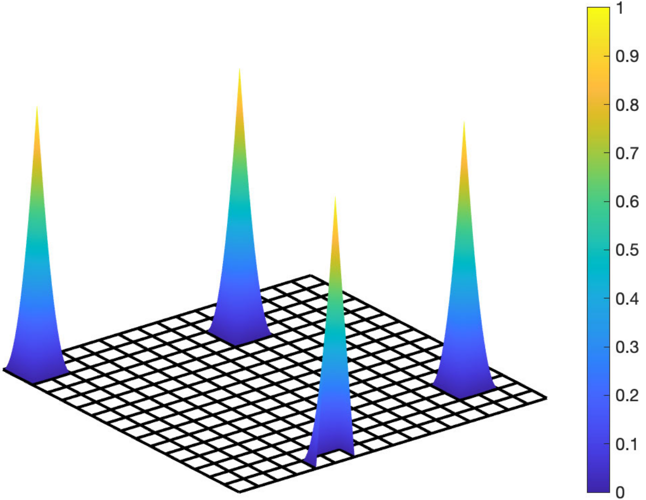

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1$$

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Equilibrated flux reconstruction

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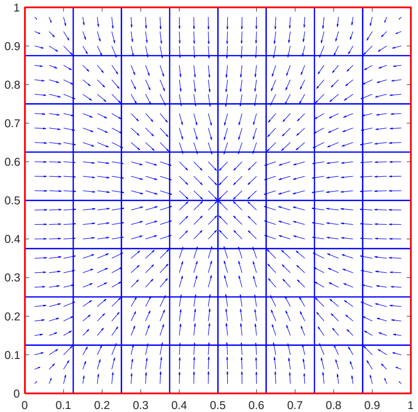

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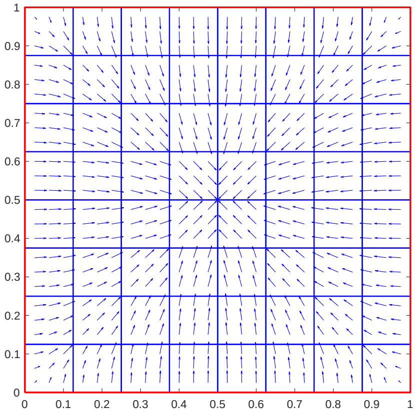

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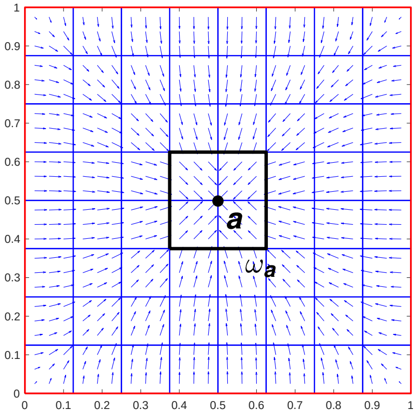

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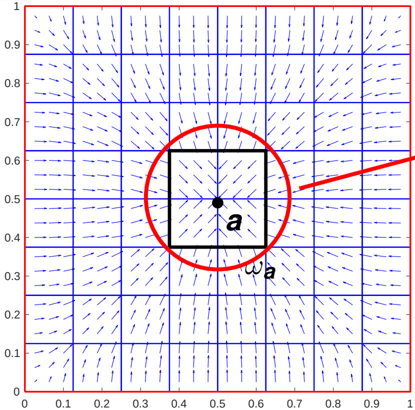

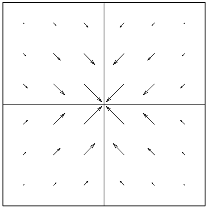

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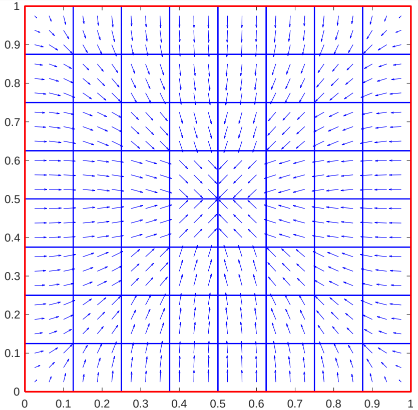

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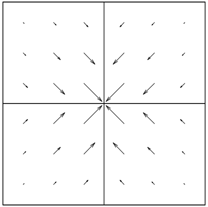

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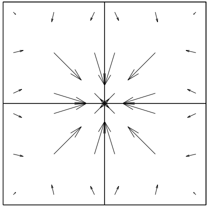

$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

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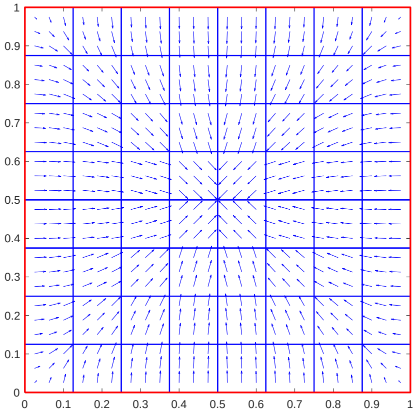

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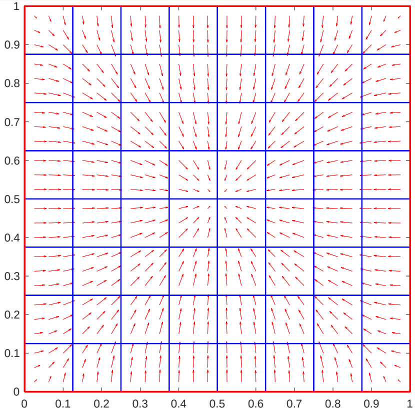

Equilibrated flux $\boldsymbol{\sigma}_h$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathbb{Q}^{p-1}(\mathcal{T}_h)} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{\sigma}_h = f$$

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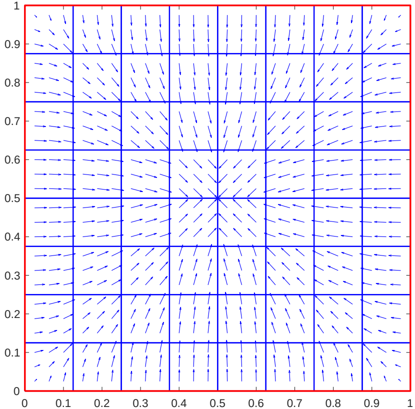

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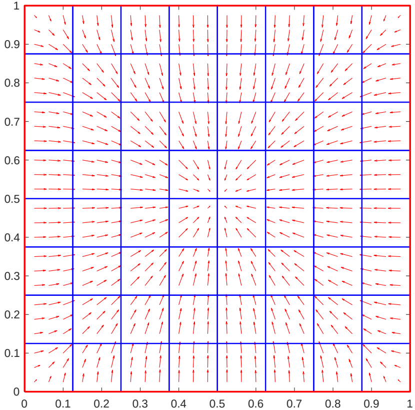
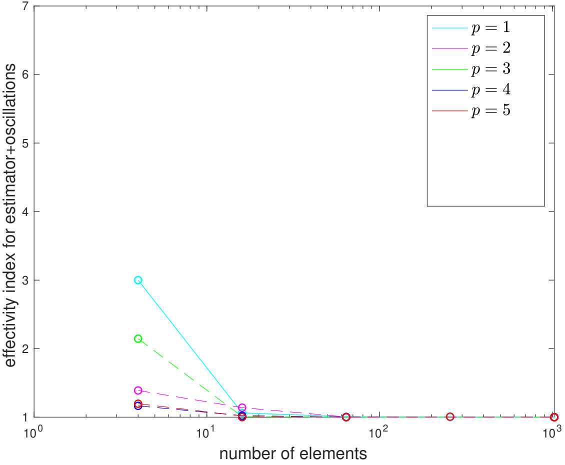
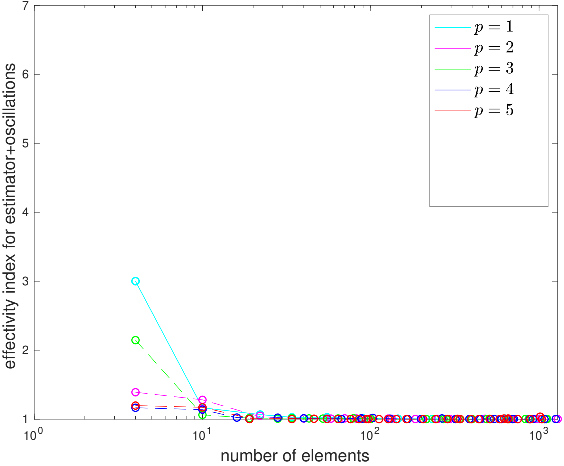


Equilibrated flux $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_h = f$

How **large** is the error? (effectivity indices)

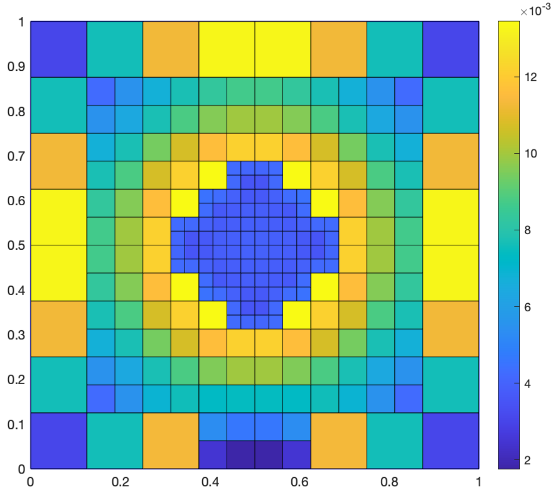


$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.})/\|\nabla(u - u_h)\|_\Omega$
(uniform mesh refinement)

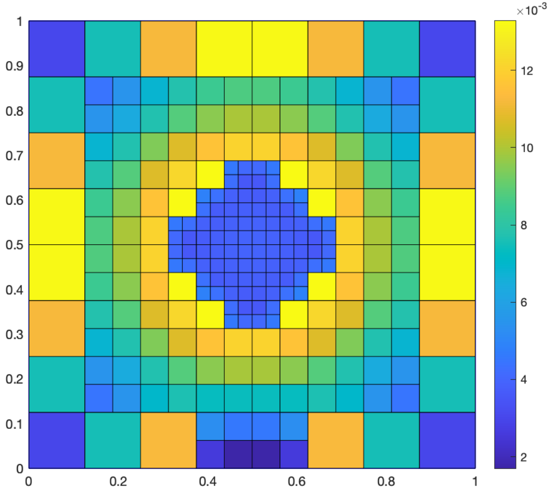


$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.})/\|\nabla(u - u_h)\|_\Omega$
(adaptive mesh refinement)

Where is the error **localized**?



Estimator distribution $\eta_K(u_h) = \|\nabla u_h + \boldsymbol{\sigma}_h\|_K$



Error distribution $\|\nabla(u - u_h)\|_K$

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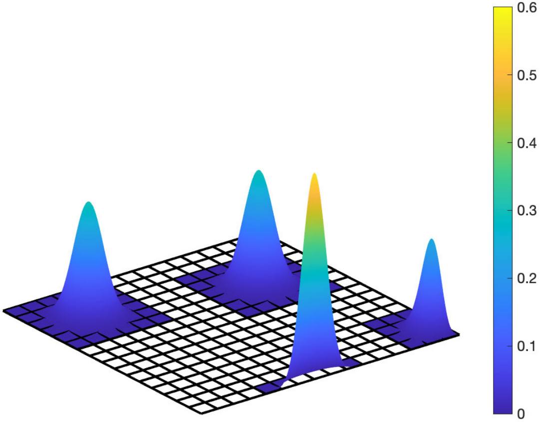

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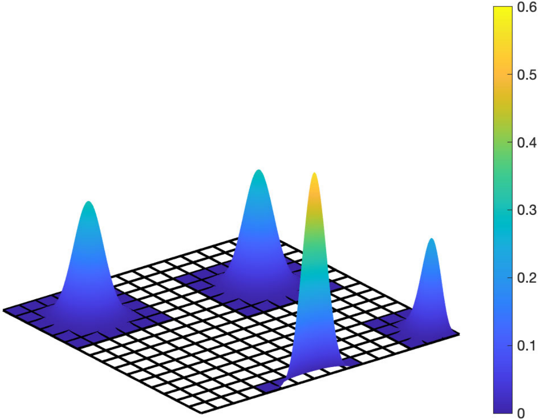

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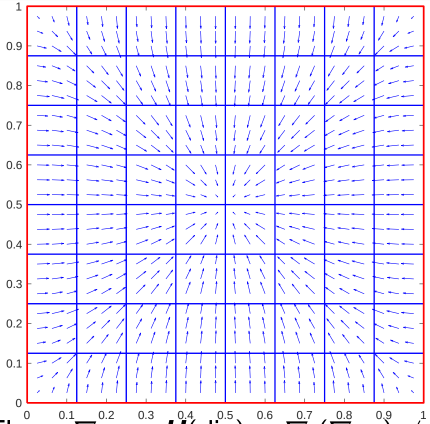

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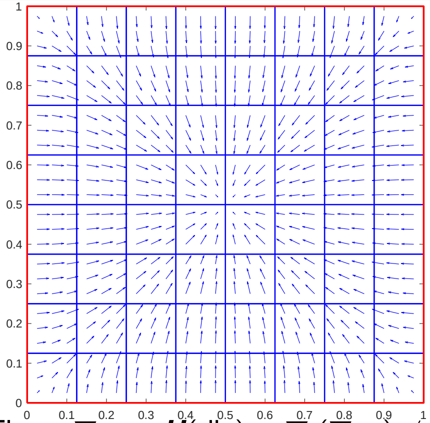

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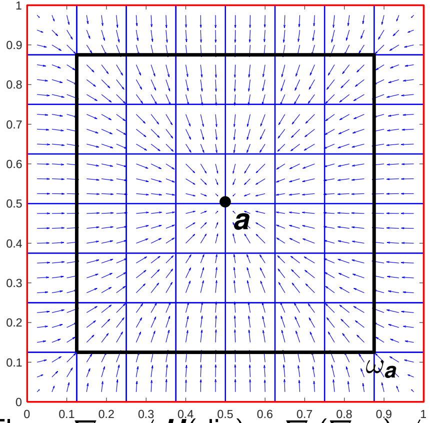

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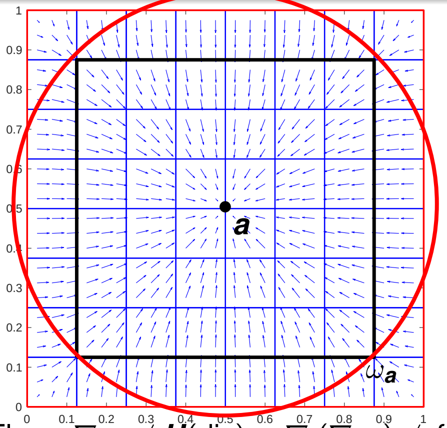

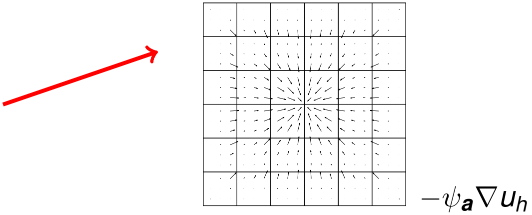

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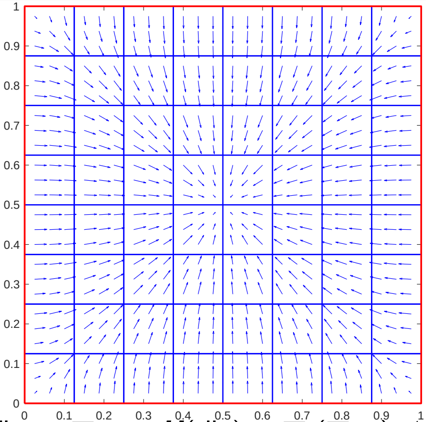

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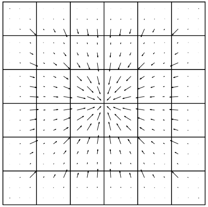

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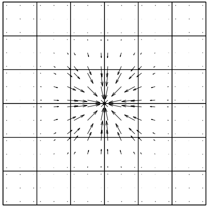

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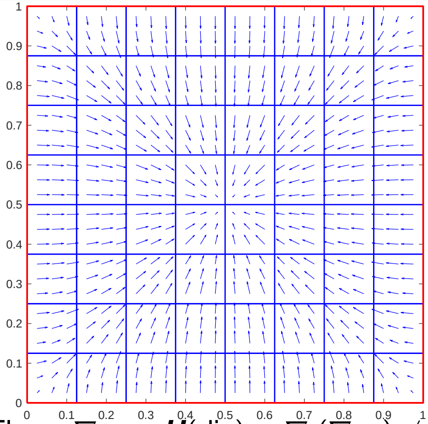

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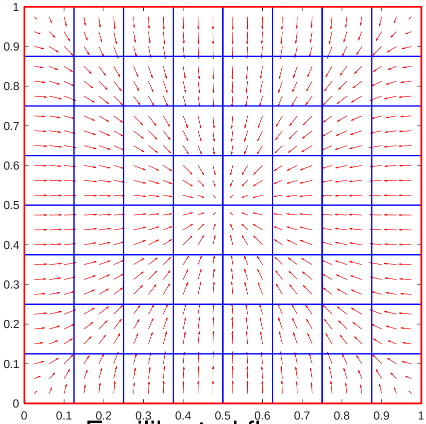

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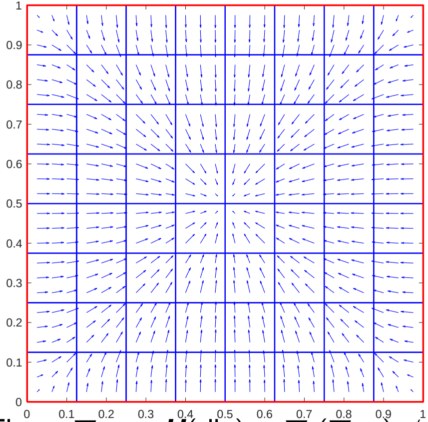
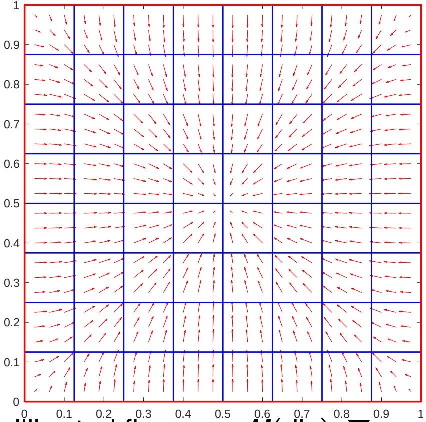
Equilibrated flux reconstruction in IGA (a first idea)



Flux $-\nabla u_h \in \boldsymbol{H}(\mathrm{div})$, $-\nabla\cdot(\nabla u_h) \neq f$



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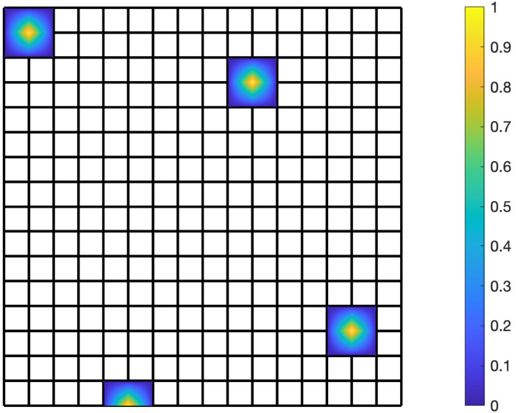
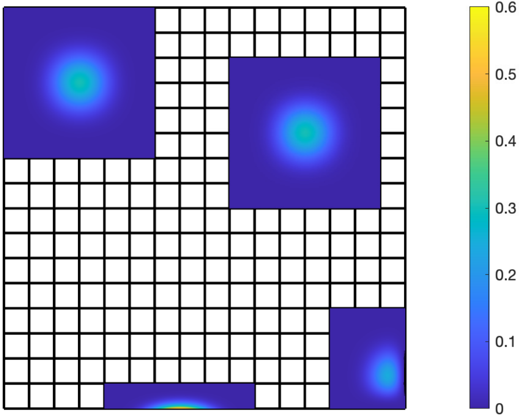
- ✗ p-robustness possibly upon extension of available tools to the large patches

Equilibration patches and partition of unity functions $\psi_{\mathbf{a}}$



$\psi_{\mathbf{a}} \in \mathbb{Q}^{1}(\mathcal{T}_h) \cap C^{0}(\Omega)$, p arbitrary

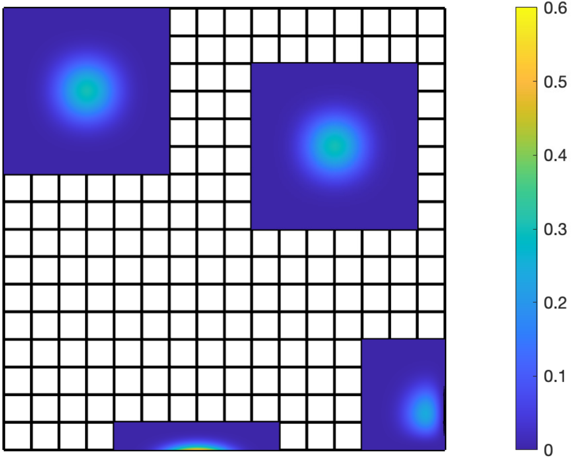


$\psi_{\mathbf{a}} \in \mathbb{Q}^{p}(\mathcal{T}_h) \cap C^{p-1}(\Omega)$, $p = 5$

Outline

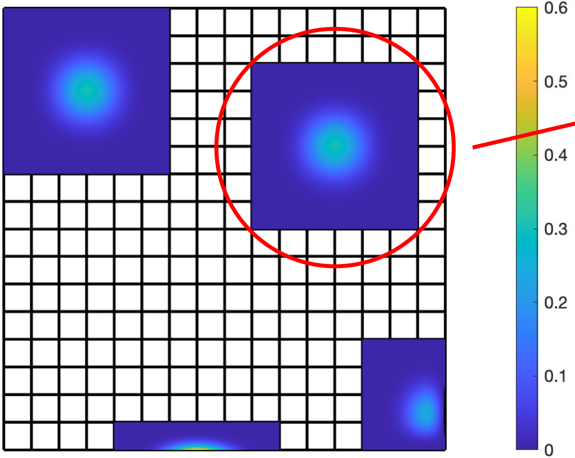
Outline

Breaking the large patch problems



1) consider the large patches (supports of $\psi_{\boldsymbol{a}}$)

Breaking the large patch problems



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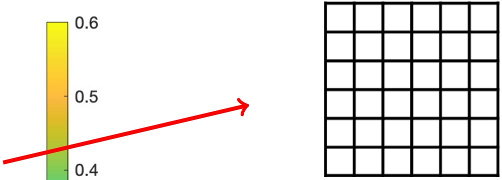
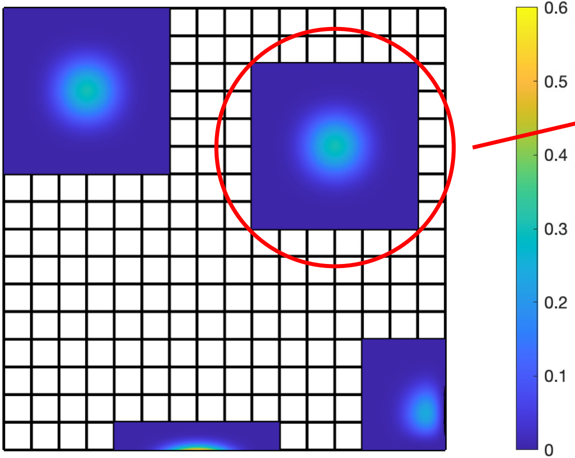


2) extract the submeshes $\mathcal{T}_{\boldsymbol{a}}$

Breaking the large patch problems



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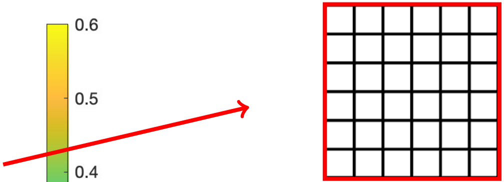

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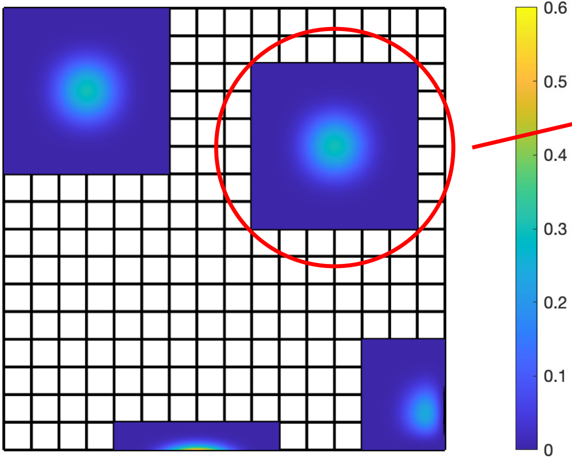
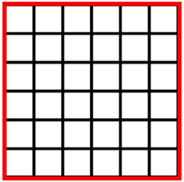
Breaking the large patch problems





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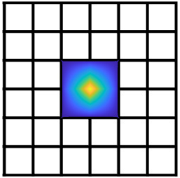

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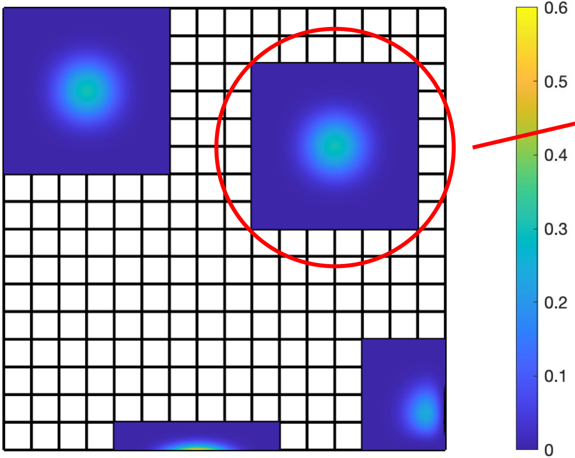
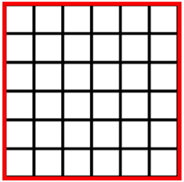
Breaking the large patch problems





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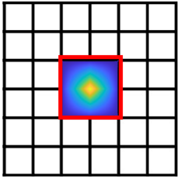

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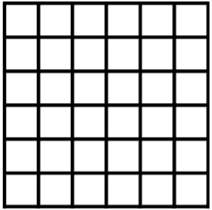
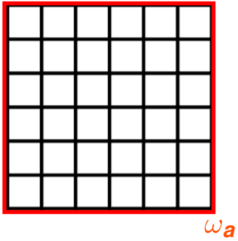
Breaking the large patch problems



Breaking the large patch problems



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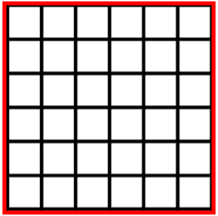
Breaking the large patch problems



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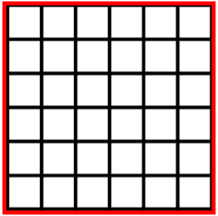
Breaking the large patch problems



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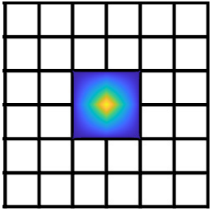
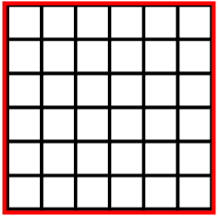


Breaking the large patch problems



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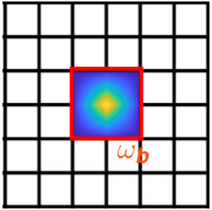

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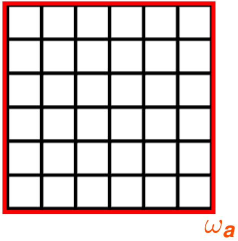
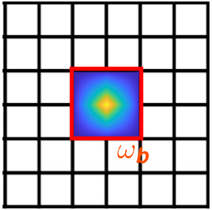
$$\boxed{\boldsymbol{\sigma}_h^{\boldsymbol{a},\boldsymbol{b}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{2p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Upsilon_{Q_h^{\boldsymbol{a},\boldsymbol{b}}}(f \psi_{\boldsymbol{a}} \psi_{\boldsymbol{b}} - \nabla u_h \cdot \nabla(\psi_{\boldsymbol{a}} \psi_{\boldsymbol{b}}) - \nabla r_h^{\boldsymbol{a}} \cdot \nabla \psi_{\boldsymbol{b}})}} \|\psi_{\boldsymbol{b}}(\psi_{\boldsymbol{a}} \nabla u_h + \nabla r_h^{\boldsymbol{a}}) + \boldsymbol{v}_h\|_{\omega_{\boldsymbol{b}}}^2}$$

Breaking the large patch problems





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6) **combine**:

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \sum_{\boldsymbol{b} \in \mathcal{V}_h^{\boldsymbol{a}}} \boldsymbol{\sigma}_h^{\boldsymbol{a},\boldsymbol{b}}, \qquad \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$

Breaking the large patch problems

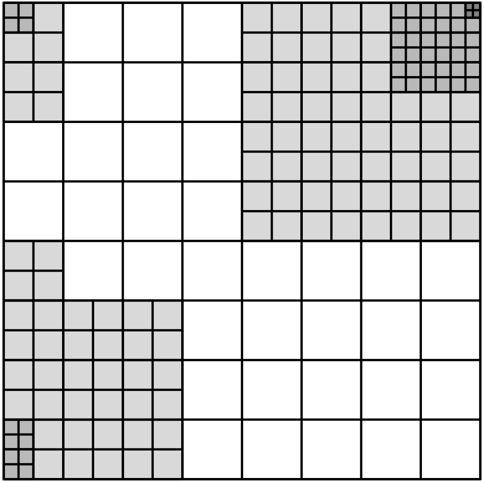
Same building principles

Additive Schwarz smoother/preconditiner Schöberl, Melenk, Pechstein, & Zaglmayr (2008): only $\mathbb{P}_1$ global problem, then high-order patch remainders

H^{-1} problems and parabolic time stepping Ern, Smears, & Vohralík (2017): arbitrary coarsening

Outline

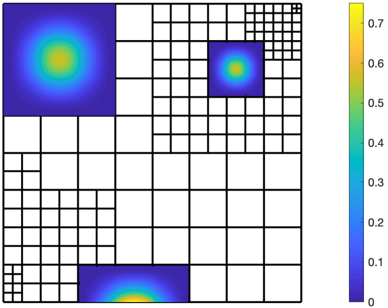
Hierarchical mesh



Hierarchical mesh $\mathcal{T}_h$ with levels 0 to 3 highlighted in white, light gray, gray, and dark gray

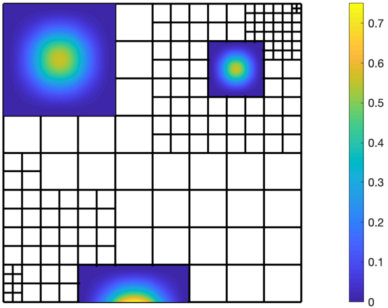
Outline

Hierarchical B-splines



Hierarchical B-splines of level 0 (left and bottom) and hierarchical B-spline of level 1 (right)

Hierarchical B-splines



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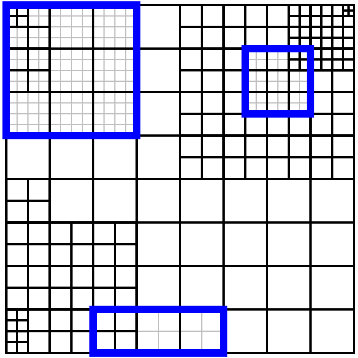

(Sub)meshes of hierarchical B-splines of level 0 (left and bottom) and hierarchical B-spline of level 1 (right)

Outline

Bi-Lipschitz mapping $\boldsymbol{F}$

Assumption (Parametrization)

Ω can be parametrized over $\widehat{\Omega} := (0,1)^d$ via a bi-Lipschitz mapping $\boldsymbol{F} : \widehat{\Omega} \to \Omega$ with positive Jacobian determinant.

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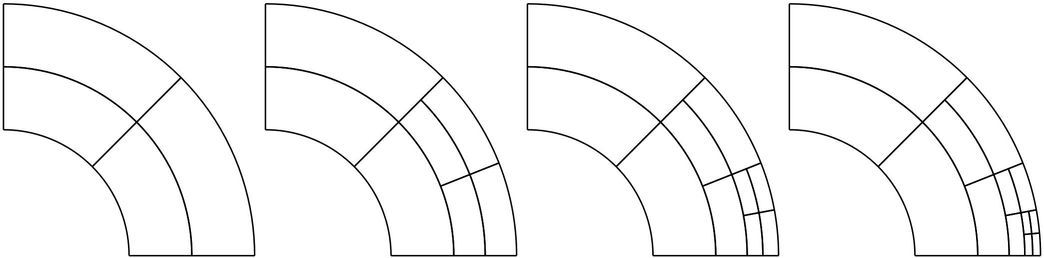

Physical domain Ω and physical hierarchical mesh $\mathcal{T}_h$

Outline

Theoretical results

Theorem (Guaranteed upper bound)

There holds

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega + \textit{osc}.$$

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Here C_{eff} only depends on space dimension d, the mapping $\boldsymbol{F}$ via $\max\{\|\boldsymbol{DF}\|_{\infty,\widehat{\Omega}}, \|(\boldsymbol{DF})^{-1}\|_{\infty,\widehat{\Omega}}\}$, .

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Outline

1. Introduction
   - The Poisson model problem and its Galerkin approximation
   - State of the art & goals
   - Equilibration in finite elements
   - Equilibration in IGA: a first idea
2. Inexpensive equilibration in IGA
   - Main idea
   - Hierarchical mesh in the parameter domain
   - Hierarchical B-splines in the parameter domain
   - Bi-Lipschitz mapping $\boldsymbol{F}$ and the physical domain Ω
3. Theoretical results
4. Numerical experiments
5. Conclusions

Numerical experiments

Setting

- Ω: quarter ring $\Omega := \{r(\cos(\varphi), \sin(\varphi)) \,:\, r \in (1/2, 1) \text{ and } \varphi \in (0, \pi/2)\}$
- NURBS parametrization $\boldsymbol{F}$
- exact solution $u(x, y) = xy \sin(4\pi(x^2 + y^2))$
- polynomial degrees $p \in \{1, \dots, 5\}$, multiplicities $m \in \{1, p\}$
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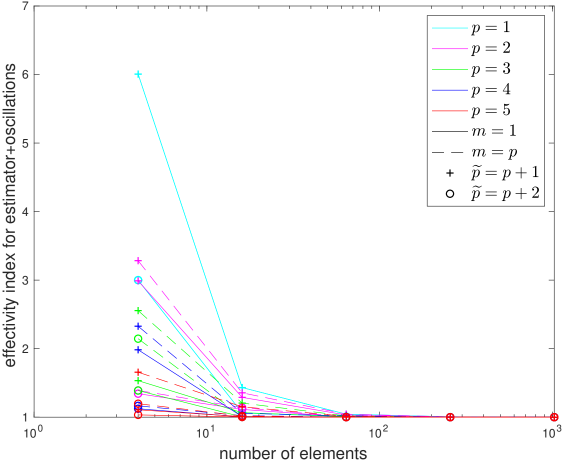
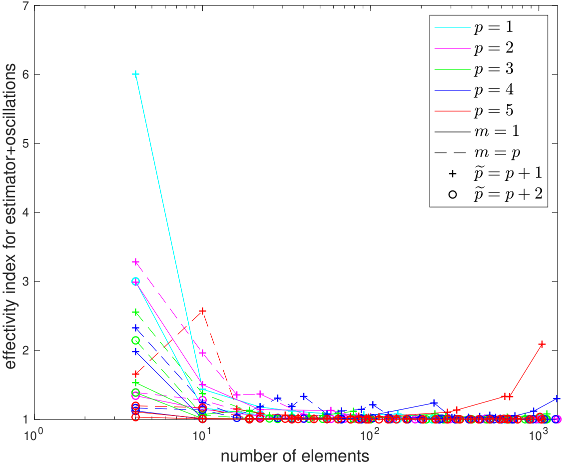
- four different mesh refinements
  - uniform refinement
  - adaptive refinement with Dörfler marking
  - artificial refinement enforcing an arbitrary number of hanging nodes
  - artificial refinement enforcing an arbitrary number of overlapping patches

How **large** is the error? (effectivity indices)

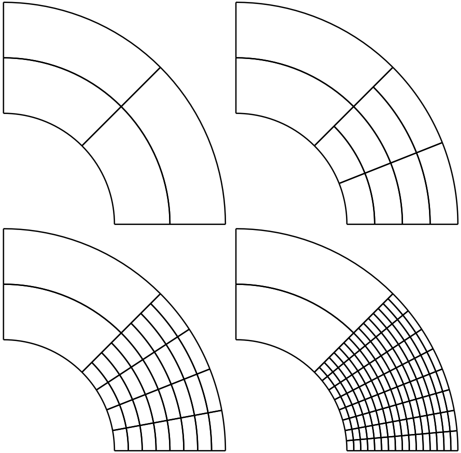


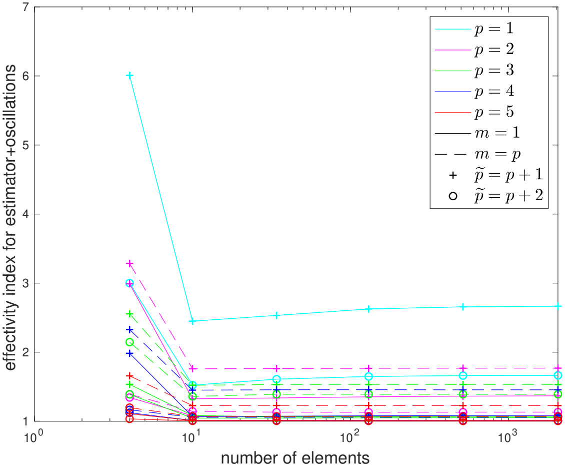
$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.})/\|\nabla(u-u_h)\|_\Omega$
(uniform mesh refinement)



$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.})/\|\nabla(u-u_h)\|_\Omega$
(adaptive mesh refinement)

How **large** is the error? (effectivity indices)





$$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.}) / \|\nabla(u - u_h)\|_\Omega$$

(arbitrary number of hanging nodes)

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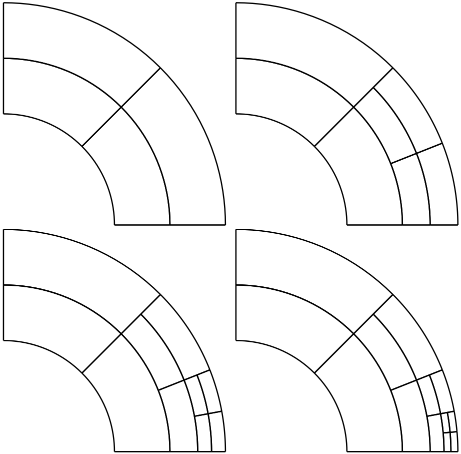
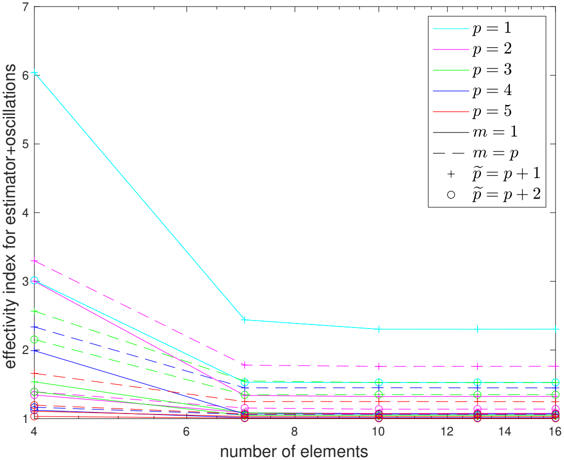
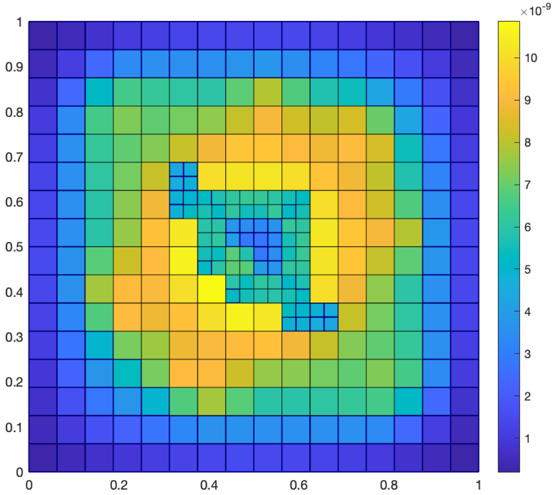




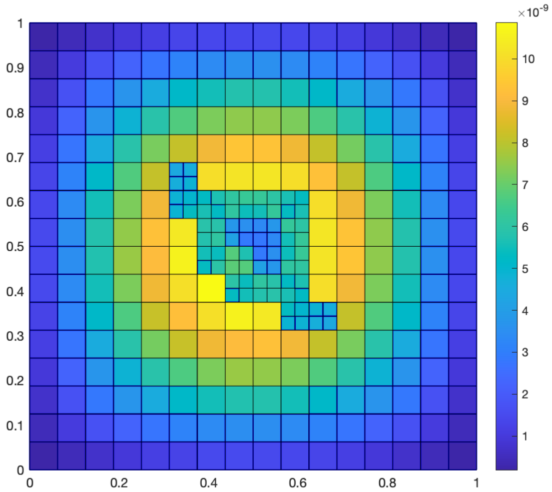
$$(\|\nabla u_h + \boldsymbol{\sigma}_h\|_\Omega + \text{osc.}) / \|\nabla(u - u_h)\|_\Omega$$

(arbitrary number of overlapping patches)

Where is the error **localized**?



Estimator distribution $\eta_K(u_h) = \|\nabla u_h + \boldsymbol{\sigma}_h\|_K$



Error distribution $\|\nabla(u - u_h)\|_K$

Outline

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Thank you for your attention!