

A priori and a posteriori error analysis in $H(\text{curl})$: localization, minimal regularity, and p -optimality

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European Research Council



Ecole des Ponts
ParisTech

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 - Commuting de Rham diagram, wishlist, and context
 - A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$
- 6 Equilibration in $\mathbf{H}(\text{curl})$
 - Patchwise equilibration
 - Main tool: stable (broken) $\mathbf{H}(\text{curl})$ polynomial extensions
- 7 Numerical illustration
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The curl-curl problem (current density $\mathbf{j} \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ with $\nabla \cdot \mathbf{j} = 0$)

The curl-curl problem

Find the magnetic vector potential $\mathbf{A} : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \mathbf{j}, & \nabla \cdot \mathbf{A} &= 0 & \text{in } \Omega, \\ \mathbf{A} \times \mathbf{n}_\Omega &= \mathbf{0}, & & & \text{on } \Gamma_D, \\ (\nabla \times \mathbf{A}) \times \mathbf{n}_\Omega &= \mathbf{0}, & \mathbf{A} \cdot \mathbf{n}_\Omega &= 0 & \text{on } \Gamma_N. \end{aligned}$$

Weak formulation

$\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$$

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Three key Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $L^2(\Omega)$,
 $H^1(\Omega) := \{\mathbf{v} \in L^2(\Omega); \nabla \mathbf{v} \in L^2(\Omega)\}$

$H(\text{curl}, \Omega)$

vector-valued $L^2(\Omega)$ functions with weak curls in $L^2(\Omega)$,
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Three key Sobolev spaces with BCs

$$H_{0,N}^1(\Omega)$$

$$H_{0,N}^1(\Omega) := \{ \mathbf{v} \in H^1(\Omega); \mathbf{v} = 0 \text{ on } \Gamma_N \}$$

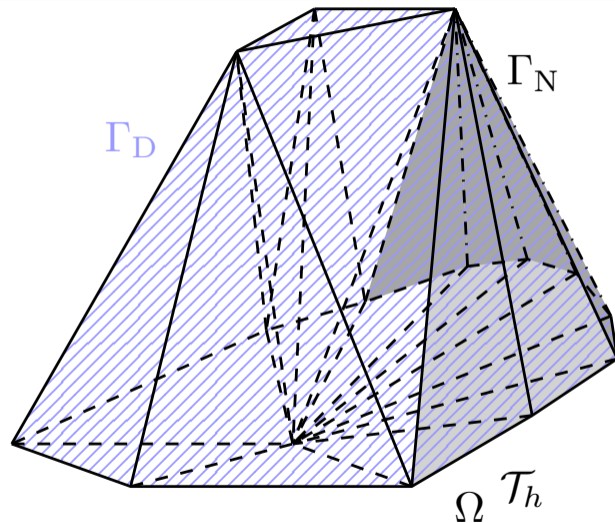
$$H_{0,N}(\text{curl}, \Omega)$$

$$H_{0,N}(\text{curl}, \Omega) := \{ \mathbf{v} \in \mathbf{H}(\text{curl}, \Omega); \mathbf{v} \times \mathbf{n}_\Omega = 0 \text{ on } \Gamma_N \text{ in appropriate sense} \}$$

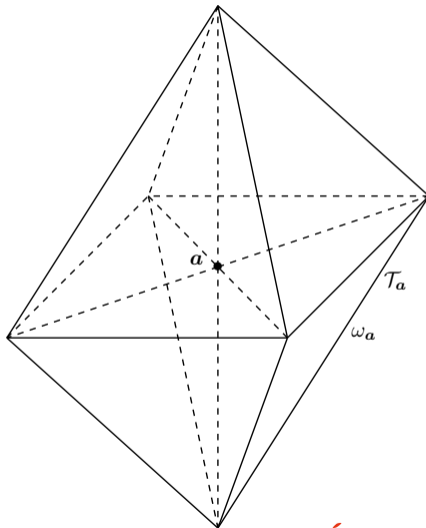
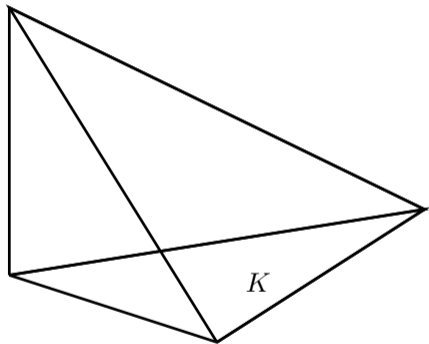
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Meshes, elements, and patches

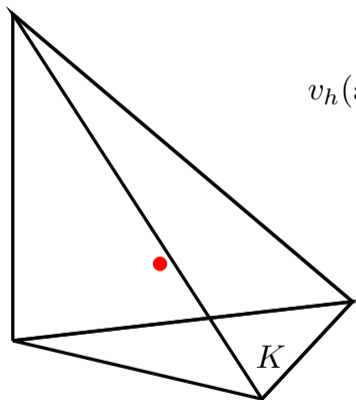


Meshes, elements, and patches



Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

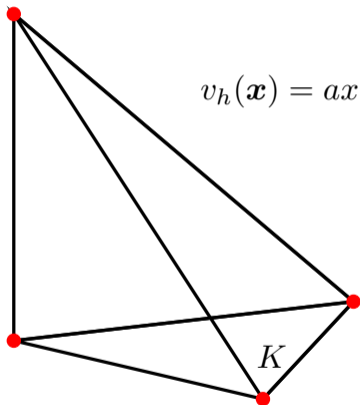
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$$v_h(\mathbf{x}) = a$$

$$p = 0$$

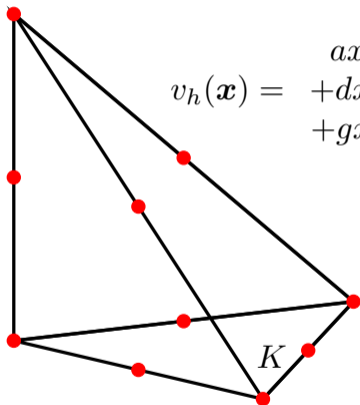
Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$



$$v_h(\mathbf{x}) = ax + by + cz + d$$

$p = 1$

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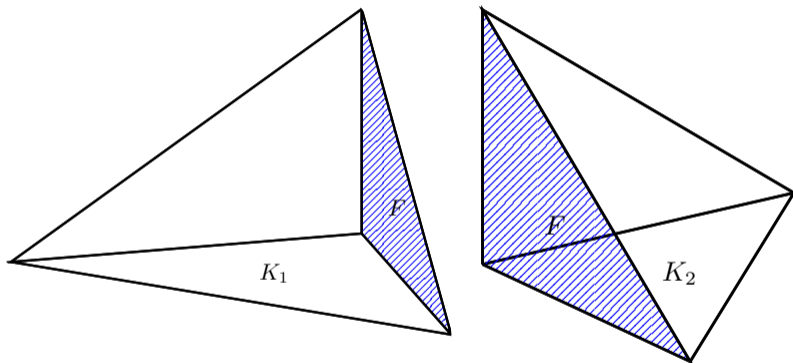


$$v_h(\mathbf{x}) = \begin{aligned} &ax^2 + by^2 + cz^2 \\ &+ dxy + eyz + fxz \\ &+ gx + hy + iz + j \end{aligned}$$

$p = 2$

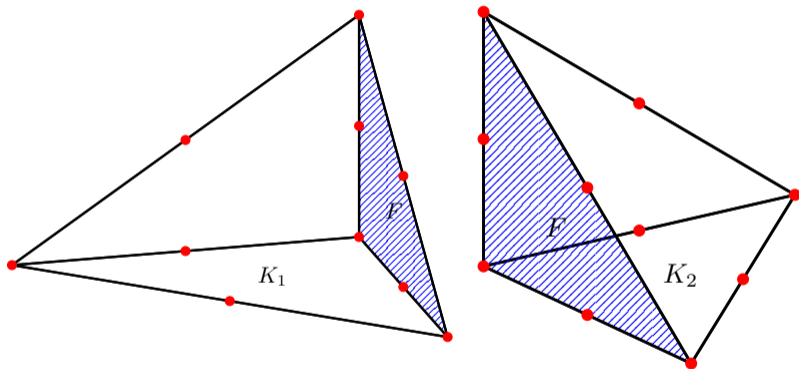
Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

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- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- \Rightarrow ensure this by putting sufficient DoFs at the face F

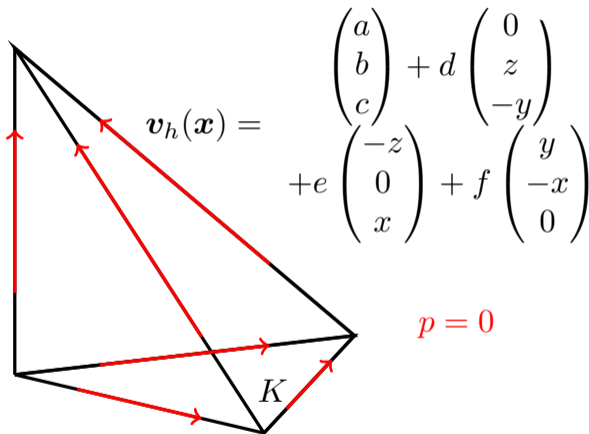
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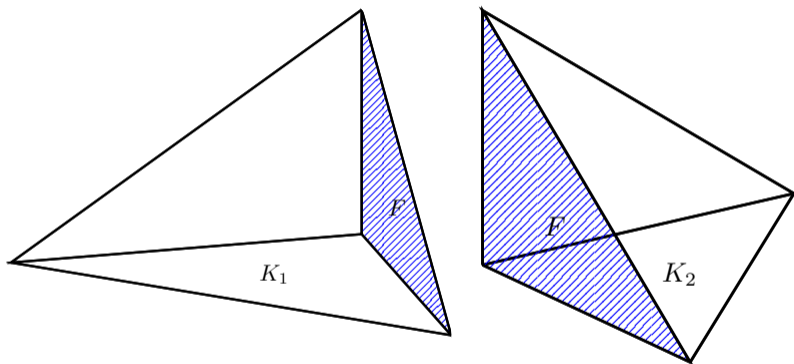
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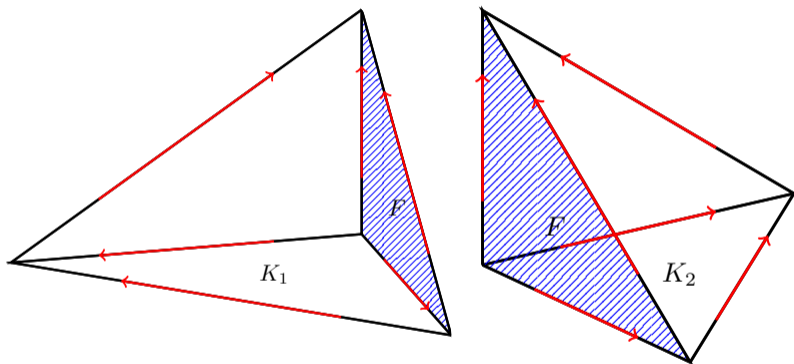


Nédélec piecewise polynomial space $\mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$, $p \geq 0$



- $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{curl}, K_2)$, and $(\mathbf{v}|_{K_1} \times \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \times \mathbf{n}_F)|_F$ in appropriate sense
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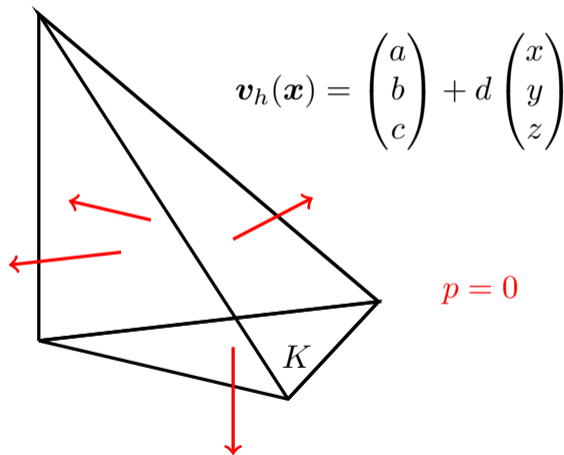
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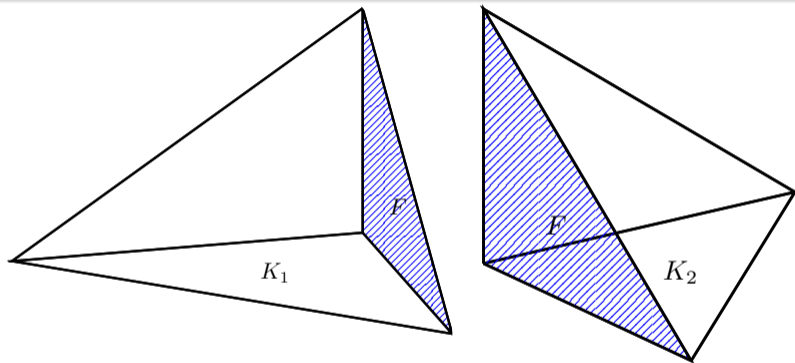
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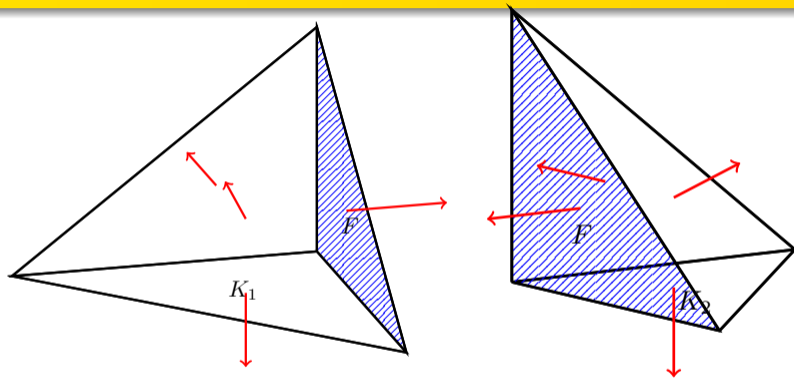


Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega)$, $p \geq 0$



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Approximation error estimates: context

h approximation estimate

Let $\mathbf{v} \in \mathbf{H}(\text{curl}, \Omega) \cap \mathbf{H}^s(\Omega)$, $s > 1/2$. Then

$$\min_{\mathbf{v}_h \in \mathcal{N}_\rho(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)} \|\mathbf{v} - \mathbf{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, s, \rho) h^{\min\{\rho+1, s\}} \|\mathbf{v}\|_{\mathbf{H}^s(\Omega)}.$$

- Nédélec (1980), Hiptmair (2002), Boffi, Brezzi, Fortin (2013)
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Approximation error estimates

Theorem (Local hp -optimal approximation under minimal Sobolev regularity)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ with

$$\mathbf{v}|_K \in \mathbf{H}^s(K), \quad (\nabla \times \mathbf{v})|_K \in \mathbf{H}^t(K) \quad \forall K \in \mathcal{T}_h$$

for $s \geq 0$ and $s \geq t \geq \max\{0, s - 1\}$. Then

$$\begin{aligned} & \min_{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{\rho + 1} \|\nabla \times (\mathbf{v} - \mathbf{v}_h)\|_K \right)^2 \right] \\ & \leq C(\kappa_{\mathcal{T}_h}, s, t) \sum_{K \in \mathcal{T}_h} \left[\left(\frac{h_K^{\min\{\rho+1, s\}}}{(\rho + 1)^s} \|\mathbf{v}\|_{\mathbf{H}^s(K)} \right)^2 + \left(\frac{h_K}{\rho + 1} \frac{h_K^{\min\{\rho+1, t\}}}{(\rho + 1)^t} \|\nabla \times \mathbf{v}\|_{\mathbf{H}^t(K)} \right)^2 \right]. \end{aligned}$$

Comments

- hp case: $\Gamma_D = \emptyset$ and convex patch subdomains ω_a for all vertices

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A posteriori error estimates ($\mathbf{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ with $\nabla \cdot \mathbf{j} = 0$)

Weak formulation

$\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$$

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Nédélec finite element discretization

$\mathbf{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,D}(\text{curl}, \Omega)$, $p \geq 0$; $\mathbf{A}_h \in \mathbf{V}_h$ satisfies

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Reliability

$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq C \underbrace{\eta}_{\text{computable estimator}}$$

Residual estimates (unknown constant C)

- Monk (1998)
- Beck, Hiptmair, Hoppe, & Wohlmuth (2000)
- Nicaise & Creusé (2003)

A posteriori error estimates ($\mathbf{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ with $\nabla \cdot \mathbf{j} = 0$)

Nédélec finite element discretization

$\mathbf{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,D}(\text{curl}, \Omega)$, $p \geq 0$; $\mathbf{A}_h \in \mathbf{V}_h$ satisfies

$$(\nabla \times \mathbf{A}_h, \nabla \times \mathbf{v}_h) = (\mathbf{j}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Guaranteed upper bound via $\mathbf{h}_h \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ s.t. $\nabla \times \mathbf{h}_h = \mathbf{j}$

$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla \times \mathbf{A}_h - \mathbf{h}_h\|}_{\text{computable estimator}}$$

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Functional estimates (global flux construction)

- Repin (2007)
- Hannukainen (2008)
- Neittaanmäki & Repin (2010)

A posteriori error estimates ($\mathbf{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ with $\nabla \cdot \mathbf{j} = 0$)

Nédélec finite element discretization

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Guaranteed upper bound and efficiency via $\mathbf{h}_h \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ s.t. $\nabla \times \mathbf{h}_h = \mathbf{j}$

$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla \times \mathbf{A}_h - \mathbf{h}_h\|}_{\text{computable estimator}} \lesssim \underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}}$$

Equilibrated estimates (local flux construction)

- Braess & Schöberl (2008): lowest-order case $p = 0$
- Licht (2019): a conceptual discussion
- Gedicke, Geevers, & Perugia (2020): equilibrated-residual-style construction
- Gedicke, Geevers, Perugia, & Schöberl (2021): p -robust modification
- Ern, Chaumont-Frelet, Vohralík (2021): p -robust broken patchwise equil.

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A posteriori error estimates

Weak formulation

$\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$$

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$\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)$ s.t. $\nabla \times \mathbf{h}_h = \mathbf{j}$: local equilibrated flux reconstruction

Theorem (Guaranteed upper bound, efficiency, and p -robustness)

$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla \times \mathbf{A}_h - \mathbf{h}_h\|}_{\text{computable estimator}} \leq C(\kappa_{\mathcal{T}_h}) \underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}}$$

A posteriori error estimates

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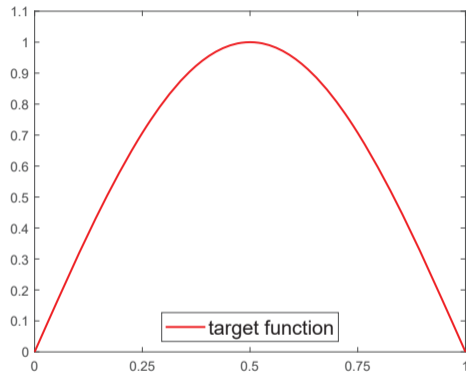
Outline

- 1 Introduction
- 2 Approximation error estimates
- 3 A posteriori error estimates
- 4 Local-best–global-best equivalence**
 - Context
 - Equivalence
- 5 A stable local commuting projector
 - Commuting de Rham diagram, wishlist, and context
 - A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$
- 6 Equilibration in $\mathbf{H}(\text{curl})$
 - Patchwise equilibration
 - Main tool: stable (broken) $\mathbf{H}(\text{curl})$ polynomial extensions
- 7 Numerical illustration
- 8 Conclusions

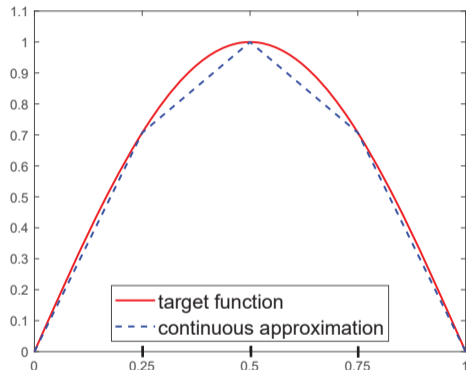
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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

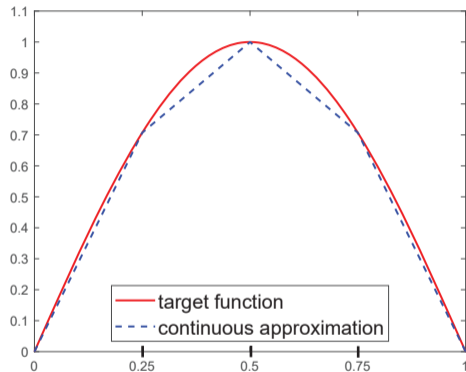


Target function in $H_0^1(\Omega)$

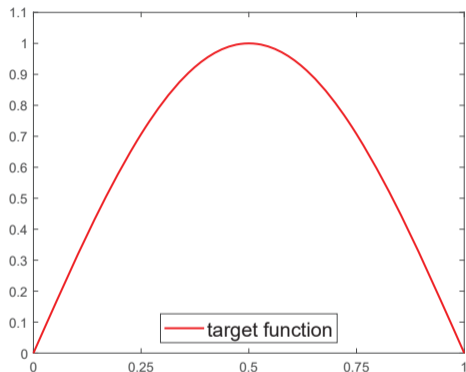
Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

Approximation by **continuous**
piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

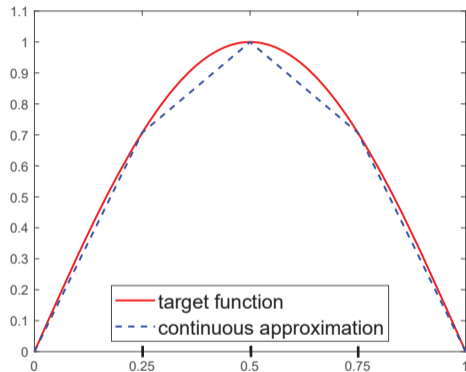


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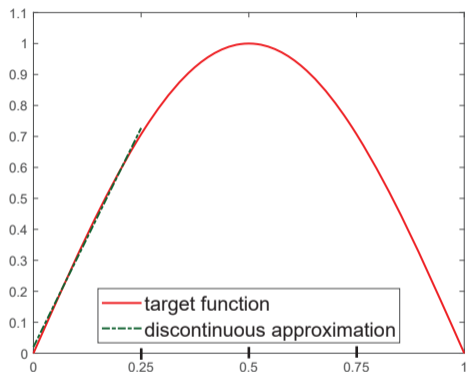


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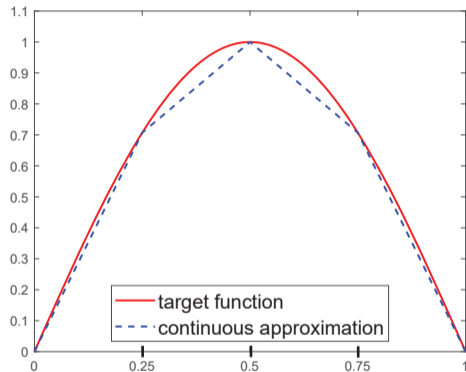


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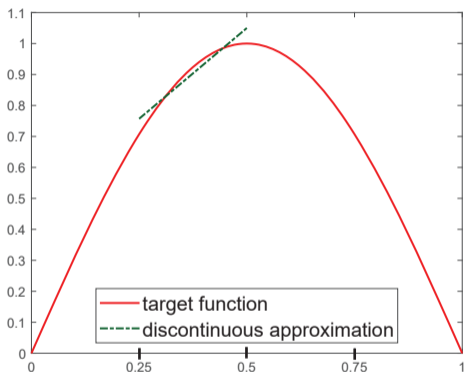


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

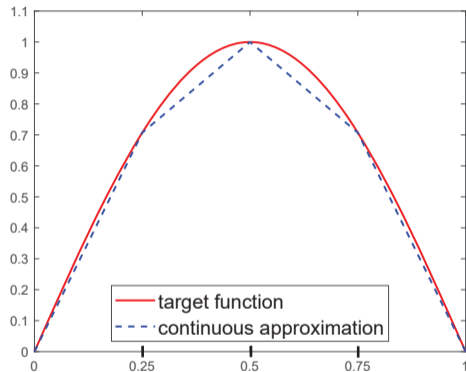


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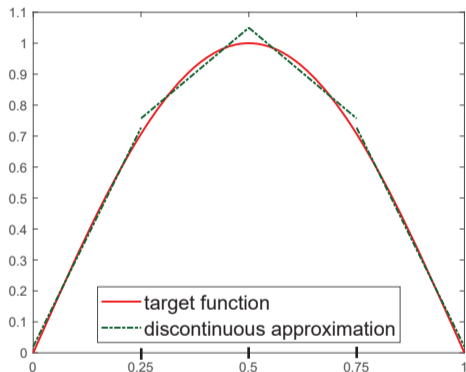


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D

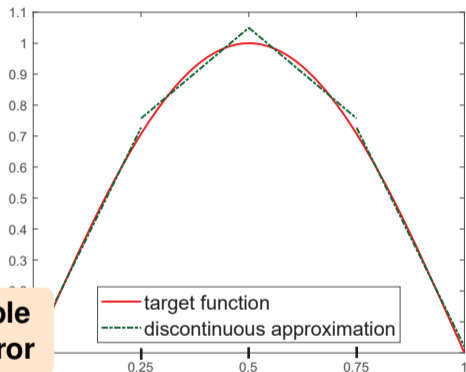
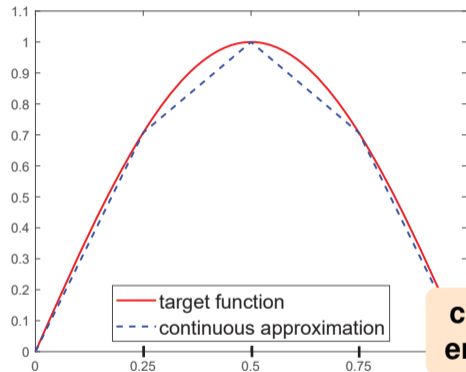


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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



comparable energy error

Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veiser (2016)

bigger \approx smaller

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1 , Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veerer (2016)

$$\min_{\text{smaller space}} \approx \min_{\text{bigger space}}$$

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$$\min_{CG \text{ space}} \approx \min_{DG \text{ space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

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Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(T_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in T_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in T_h \\ \text{no trace-continuity constraint}}}$$

- \approx_p : up to a generic constant that only depends on the shape-regularity of the mesh \mathcal{K}_{T_h} and the polynomial degree p

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Global-best approximation \approx local-best approximation in $H(\text{curl})$

Theorem (Constrained equivalence in $H(\text{curl})$)

bigger \approx smaller

Global-best approximation \approx local-best approximation in $H(\text{curl})$

Theorem (Constrained equivalence in $H(\text{curl})$)

$$\min_{\text{smaller space with curl constraints}} \approx \min_{\text{bigger space without curl constraints}}$$

Global-best approximation \approx local-best approximation in $H(\text{curl})$

Theorem (Constrained equivalence in $H(\text{curl})$)

$$\min_{\text{conforming Nédélec space with curl constraints}} \approx \min_{\text{broken Nédélec space without curl constraints}}$$

Global-best approximation \approx local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) \\ \nabla \times \mathbf{v}_h = \mathbf{P}_h^{p, \text{div}}(\nabla \times \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{RT}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \text{curl constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\left[\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{RT}^p(\nabla \times \mathbf{v})\|_K \right)^2 \right]}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}}$$

- \approx_p : only depends on shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

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$$\approx_p \sum_{K \in \mathcal{T}_h} \left[\underbrace{\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{RT}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}} \right].$$

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Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

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$$\approx_p \sum_{K \in \mathcal{T}_h} \left[\underbrace{\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}} \right].$$

- \approx_p : only depends on shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Global-best approximation \approx local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) \\ \nabla \times \mathbf{v}_h = \mathbf{P}_h^{p, \text{div}}(\nabla \times \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\text{global-best on } \Omega}$$

*global-best on Ω
tangential-trace-continuity constraint
curl constraint*

$$\approx_p \sum_{K \in \mathcal{T}_h} \left[\underbrace{\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\text{local-best on each } K \in \mathcal{T}_h} \right].$$

*local-best on each $K \in \mathcal{T}_h$
no tangential-trace-continuity constraint
no curl constraint*

- \approx_p : only depends on shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

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 - A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$
- 6 Equilibration in $\mathbf{H}(\text{curl})$
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Commuting de Rham diagram and wishlist for $\mathbf{P}_h^{p,\text{curl}}$

Commuting de Rham diagram

$$\begin{array}{ccccccc}
 H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & L_*^2(\Omega) \\
 \downarrow \mathbf{P}_h^{p,\text{grad}} & & \downarrow \mathbf{P}_h^{p,\text{curl}} & & \downarrow \mathbf{P}_h^{p,\text{div}} & & \downarrow \Pi_h^p \\
 \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
 \end{array}$$

Commuting de Rham diagram and wishlist for $\mathbf{P}_h^{p,\text{curl}}$

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 \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
 \end{array}$$

Commuting de Rham diagram and wishlist for $\mathbf{P}_h^{p,\text{curl}}$

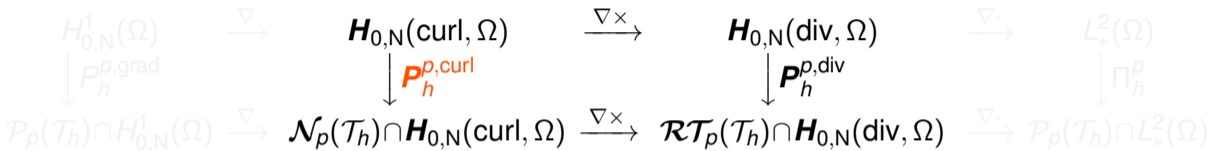
Commuting de Rham diagram

$$\begin{array}{ccccccc}
 H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & L_*^2(\Omega) \\
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 \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
 \end{array}$$

- $\mathbf{P}_h^{p,\text{div}}$: Ern, Gudi, Smears, Vohralík (2022)

Commuting de Rham diagram and wishlist for $\mathbf{P}_h^{p,\text{curl}}$

Commuting de Rham diagram



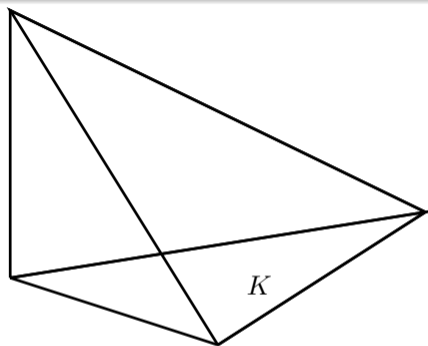
Requirements on $\mathbf{P}_h^{p,\text{curl}}$

- 1 be defined over the **entire** infinite-dimensional space $\mathbf{H}_{0,N}(\text{curl}, \Omega)$
- 2 be defined **locally** (in neighborhood of mesh elements)
- 3 be defined **simply** (starting from elementwise polynomial projections)
- 4 have **optimal approximation properties**, that of **elementwise curl-unconstrained L^2 -orthogonal projector** (local-global equivalence)
- 5 be **stable in $L^2(\Omega)$** (up to data oscillation)
- 6 satisfy the **commuting properties** expressed by the arrows
- 7 be **projector**, i.e., leave intact piecewise polynomials

Stable local commuting projectors defined on $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$

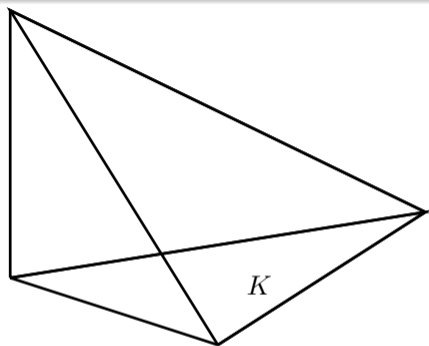
- Schöberl (2001, 2005): **not local**
- Christiansen and Winther (2008): **not local**
- Bespalov and Heuer (2011): low regularity but still **not $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$**
- Falk and Winther (2014): **local** and **$\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$ -stable** but **not L^2 -stable**
- Ern and Guermond (2016): **not local**
- Ern and Guermond (2017): **$\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$ regularity** but **not commuting**
- Licht (2019): **essential boundary conditions** on part of $\partial\Omega$
- Arnold and Guzmán (2021): **L^2 -stable**

Classical elementwise interpolation



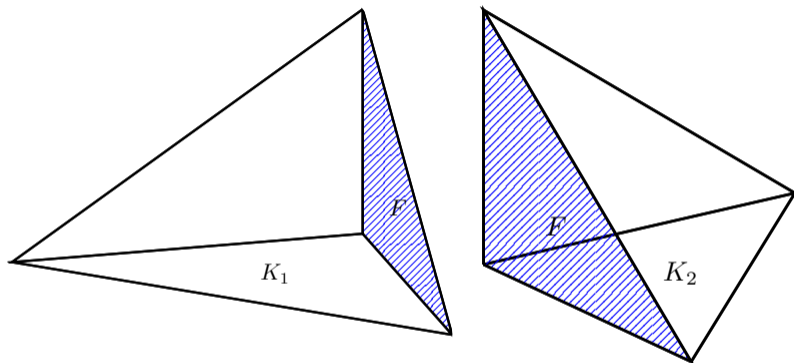
- $\|\mathbf{v} - \mathbf{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{v} - \mathbf{v}_h\|_K^2$
- $\mathbf{v} \in \mathbf{H}(\text{curl}, \Omega) \Rightarrow \mathbf{v}|_K \in \mathbf{H}(\text{curl}, K) \Rightarrow$ so interpolate $\mathbf{v}|_K$

Classical elementwise interpolation



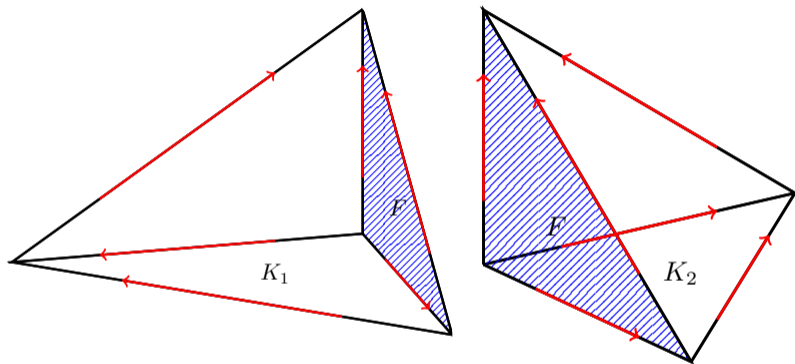
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Classical elementwise interpolation: conformity enforcement



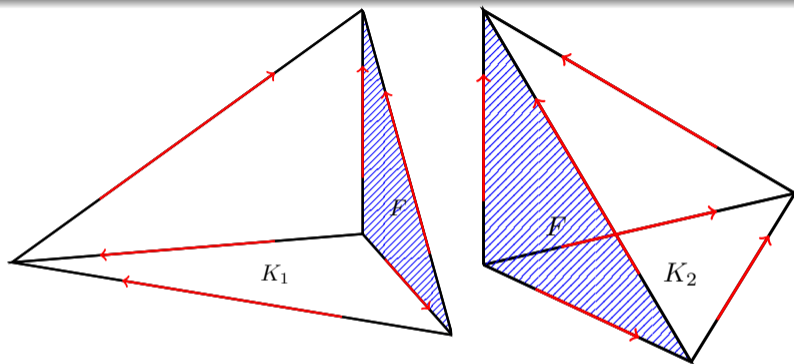
- $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{curl}, K_2)$, and $(\mathbf{v}|_{K_1} \times \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \times \mathbf{n}_F)|_F$ in appropriate sense ($p = 0$)

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Classical elementwise interpolation: conformity enforcement

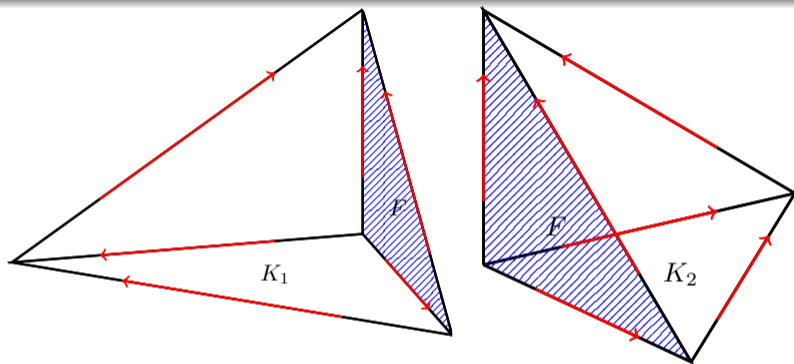


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Clash

Edge integrals not available in $\mathbf{H}(\text{curl})$.

Classical elementwise interpolation: conformity enforcement

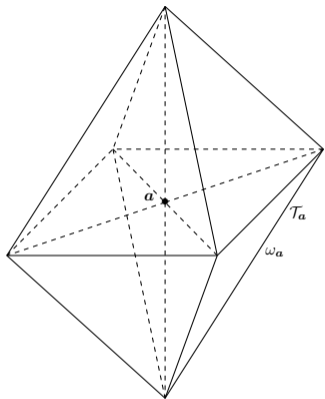


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Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\mathbf{v} \in \mathbf{H}(\text{curl}, \Omega)$ requested.

Classical patchwise interpolation (Clément)

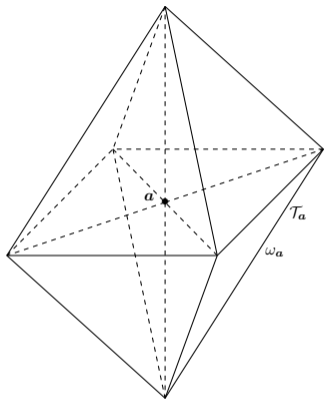


- some local-best polynomial approximation on ω_a
- values on ω_a as coefficients for basis functions supported on ω_a

Conclusion

Allows the **minimal regularity** but breaks the **projection property**, the **elementwise structure**, and the **commuting diagram**.

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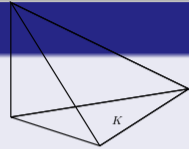
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A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$

Definition (A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ be given (minimal regularity).



- 1 For each $K \in \mathcal{T}_h$, prepare the datum $\tau_h|_K$

$$\tau_h|_K := \arg \min_{\substack{\mathbf{w}_h \in \mathcal{RT}_p(K) \\ \nabla \cdot \mathbf{w}_h = 0}} \|\nabla \times \mathbf{v} - \mathbf{w}_h\|_K$$

and define $\iota_h|_K$ by the **elementwise constrained projection**

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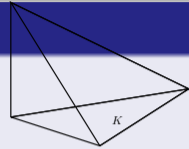
(discrete, tangential trace discontinuous).

- 2 Obtain $\mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)$ by applying the **flux equilibration procedure** to ι_h ; in particular, $\mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) := \mathbf{h}_h := \sum_{a \in \mathcal{V}_h} \mathbf{h}_h^a$, where \mathbf{h}_h^a are obtained by **local energy minimizations** on the patch subdomains ω_a .

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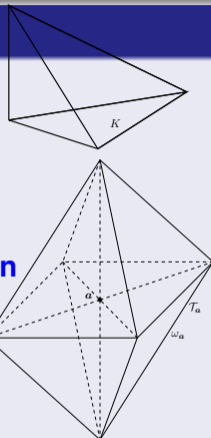
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A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$

Theorem (A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$)

$\mathbf{P}_h^{p,\text{curl}}$ is a **commuting projector** since

$$\nabla \times \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) = \mathbf{P}_h^{p,\text{div}}(\nabla \times \mathbf{v})$$

$$\forall \mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega),$$

$$\mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) = \mathbf{v}$$

$$\forall \mathbf{v} \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega).$$

Moreover, it has **local-best approximation properties** and is **L^2 stable** up to data oscillation, since, for all $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ and $K \in \mathcal{T}_h$,

$$\|\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times (\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}))\|_K \right)^2$$

$$\lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \min_{\mathbf{v}_h \in \mathcal{N}_p(K')} \|\mathbf{v} - \mathbf{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\},$$

$$\|\mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 \lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \|\mathbf{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\}.$$

A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$

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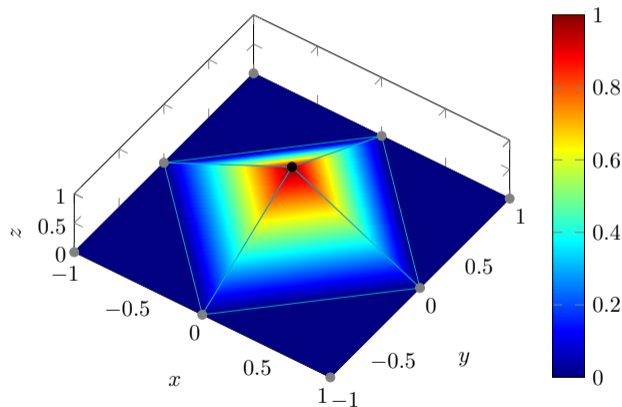
$$\begin{aligned} & \|\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times (\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}))\|_K \right)^2 \\ & \lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \min_{\mathbf{v}_h \in \mathcal{N}_p(K')} \|\mathbf{v} - \mathbf{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\}, \\ & \|\mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 \lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \|\mathbf{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\}. \end{aligned}$$

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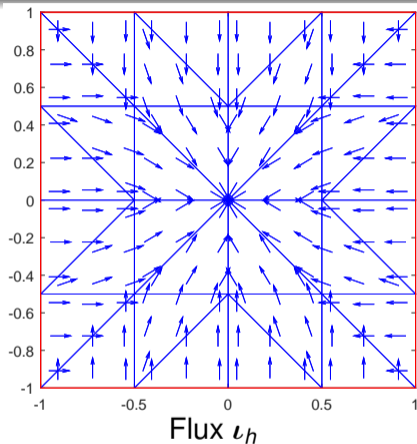
Partition of unity

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi^{\mathbf{a}} = 1$$



Hat basis function $\psi^{\mathbf{a}}$

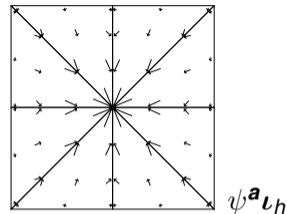
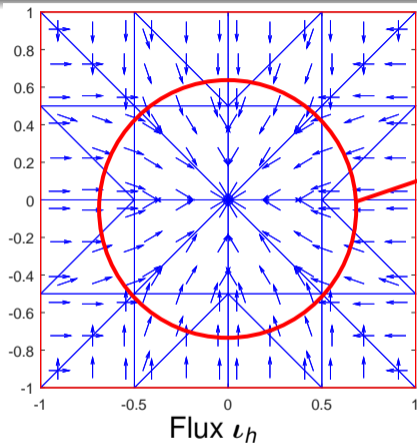
Equilibrated flux reconstruction in $H(\text{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



$$\underbrace{\iota_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f, \psi^a)_{\omega_a} + (\iota_h, \nabla \psi^a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}}$$

Equilibrated flux reconstruction in $H(\text{div})$

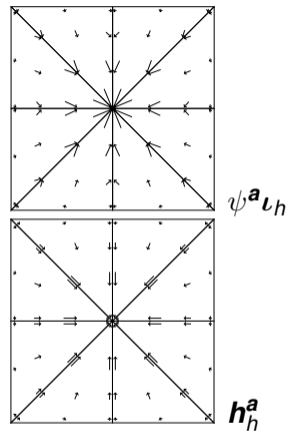
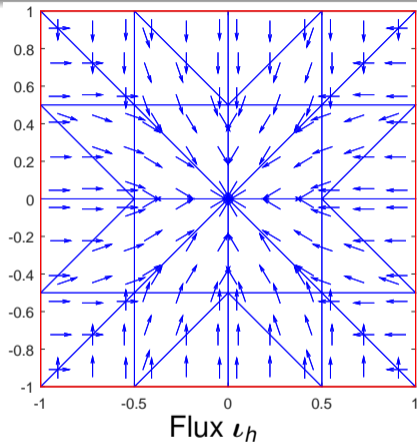
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Equilibrated flux reconstruction in $H(\text{div})$

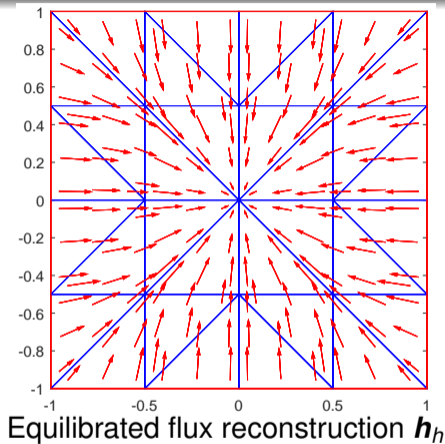
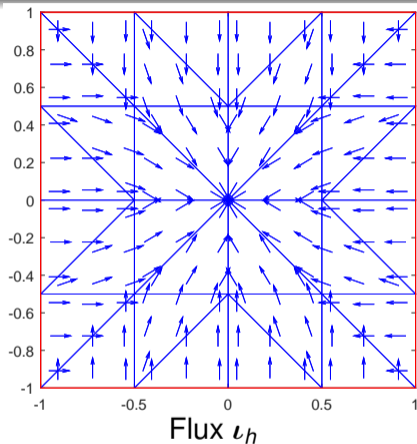
Destuynder and Métivet (1998), Braess & Schöberl (2008)



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$$\underbrace{\iota_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\iota_h, \nabla \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}} \rightarrow \mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^{\mathbf{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{div}), \nabla \cdot \mathbf{h}_h = f$$

Equilibration – the bottom line

$H(\text{div})$ -case

- When there exists $\mathbf{v}_h \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)$ such that $\nabla \cdot \mathbf{v}_h = j_h^a$?
- When $j_h^a \in \mathcal{P}_{p+1}(\mathcal{T}_a)$ and $(j_h^a, \mathbf{1})_{\omega_a} = 0$ if $\mathbf{a} \notin \overline{\Gamma_D}$.

$H(\text{curl})$ -case

- When there exists $\mathbf{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{curl}, \omega_a)$ such that $\nabla \times \mathbf{v}_h = \mathbf{j}_h^a$?
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Outline

- 1 Introduction
- 2 Approximation error estimates
- 3 A posteriori error estimates
- 4 Local-best–global-best equivalence
 - Context
 - Equivalence
- 5 A stable local commuting projector
 - Commuting de Rham diagram, wishlist, and context
 - A stable local commuting projector $P_h^{p,\text{curl}}$
- 6 **Equilibration in $H(\text{curl})$**
 - **Patchwise equilibration**
 - Main tool: stable (broken) $H(\text{curl})$ polynomial extensions
- 7 Numerical illustration
- 8 Conclusions

Patchwise equilibrated fluxes

Continuous level

- $\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies
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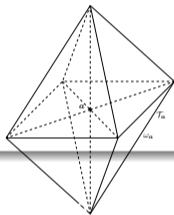
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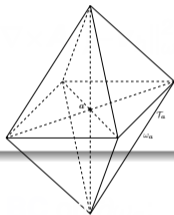
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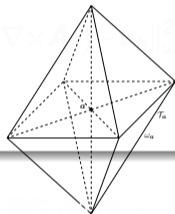
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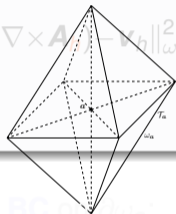
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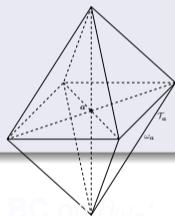
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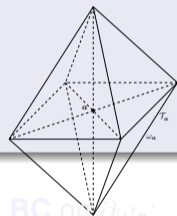
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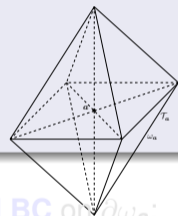
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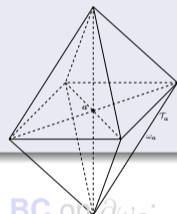
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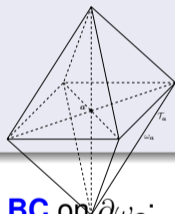
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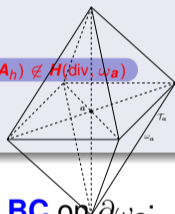
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► $\psi^a \mathbf{j} \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)$ but $\nabla \psi^a \times (\nabla \times \mathbf{A}_h) \notin \mathbf{H}(\text{div}, \omega_a)$

$$\mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^a.$$



Key points

- **homogeneous tangential BC** on $\partial \omega_a$:

$$\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$$

- **global equilibrium** $\nabla \times \mathbf{h}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^a$

$$= \sum_{\mathbf{a} \in \mathcal{V}_h} (\psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}_h)) = \mathbf{j}$$

Patchwise equilibrated fluxes

Continuous level

- $\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies
 $(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$

- Thus $\nabla \times \mathbf{A} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ with
 $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{j}.$

- Take $\mathbf{h}^a := \psi^a(\nabla \times \mathbf{A}) \in \mathbf{H}_0(\text{curl}, \omega_a)$
 and note that $\sum_{a \in \mathcal{V}_h} \mathbf{h}^a = \nabla \times \mathbf{A}.$

- **Rewritten implicitly,**

$$\mathbf{h}^a = \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v} = \mathbf{j}^a}} \|\psi^a(\nabla \times \mathbf{A}) - \mathbf{v}\|_{\omega_a}^2$$

with

$$\mathbf{j}^a := \psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}).$$

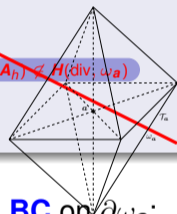
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Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)$ to a Raviart–Thomas space

For all vertices $\mathbf{a} \in \mathcal{V}_h$, consider p' := min{p, 1}-degree patchwise minimizations:

$$\theta_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = -\nabla\psi^{\mathbf{a}} \cdot \mathbf{j}}} \|\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_a}^2.$$

$(\mathbf{v}_h, \mathbf{r}_h)_K = (\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h), \mathbf{r}_h)_K \quad \forall \mathbf{r}_h \in [P_0(K)]^3, \forall K \in \mathcal{T}_a$

Comments

- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)$
- remainder $\delta_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}}$
 - should be zero (\sim partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_h} \{\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A})\} = 0$), but is not
 - $\delta_h \in \mathcal{RT}_{p'}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ and $\nabla \cdot \delta_h = 0$
- additional orthogonality constraint
 - crucial for stage 2 below
 - only possible thanks to the lowest-order Galerkin orthogonality of \mathbf{A}_h
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Stage 2: divergence-free decomposition of the given divergence-free Raviart-Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p + 1)$ -degree elementwise minimizations:

$$\delta_h^a|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_1(K) \\ \nabla \cdot \mathbf{v}_h = 0 \\ \mathbf{v}_h \cdot \mathbf{n}_K = l_{\mathcal{RT}}^1(\psi^a \delta_h) \cdot \mathbf{n}_K \text{ on } \partial K}} \|\mathbf{v}_h - l_{\mathcal{RT}}^1(\psi^a \delta_h)\|_K^2 \quad \forall \mathbf{a} \in \mathcal{V}_K \text{ when } p = 0,$$

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Comments

- patchwise contributions

$$\delta_h^a \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \quad \text{and} \quad \nabla \cdot \delta_h^a = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h$$

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• δ_h^a form a divergence-free decomposition of δ_h , $\delta_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \delta_h^a$



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Stage 2: divergence-free decomposition of the given divergence-free current density \mathbf{j}

Divergence-free decomposition of the current density \mathbf{j}

Set

$$\mathbf{j}_h^{\mathbf{a}} := \psi^{\mathbf{a}} \mathbf{j} + \boldsymbol{\theta}_h^{\mathbf{a}} - \boldsymbol{\delta}_h^{\mathbf{a}}.$$

Then

$$\begin{aligned} \mathbf{j}_h^{\mathbf{a}} &\in \mathcal{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{a}}), \\ \nabla \cdot \mathbf{j}_h^{\mathbf{a}} &= 0, \\ \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{j}_h^{\mathbf{a}} &= \mathbf{j}. \end{aligned}$$

Stage 3: discrete patchwise equilibrated fluxes

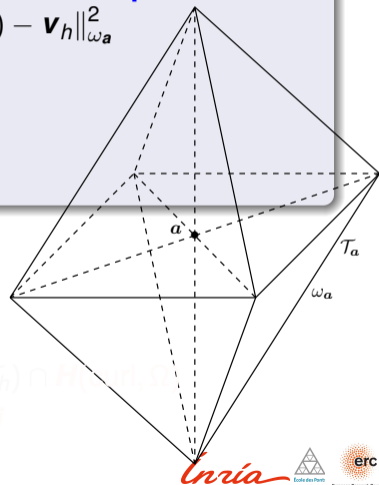
Definition (Chaumont-Frelet, Vohralík (2021))

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and combine

$$\mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^{\mathbf{a}}$$



Key points

- homogeneous tangential BC on $\partial\omega_a$: $\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl}, \Omega)$
- global equilibrium $\nabla \times \mathbf{h}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^{\mathbf{a}} = \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{J}_h^{\mathbf{a}} = \mathbf{J}$

Stage 3: discrete patchwise equilibrated fluxes

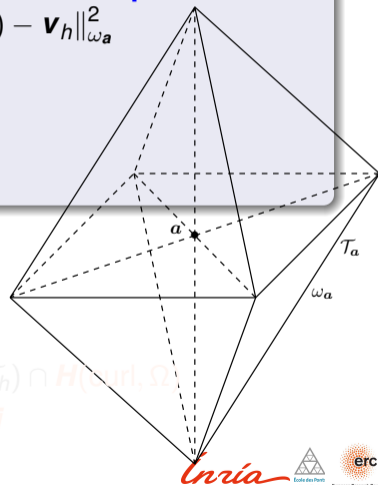
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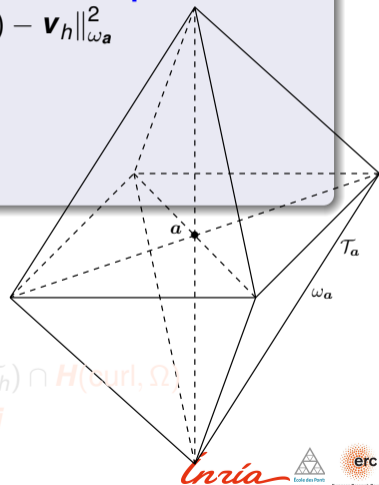
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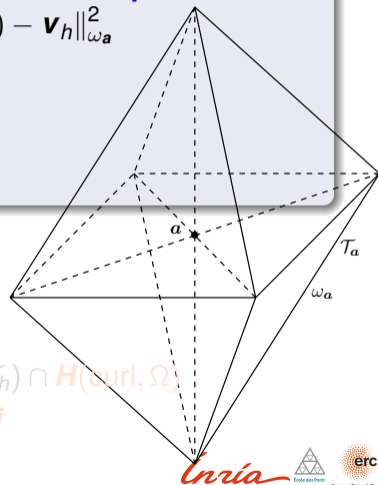
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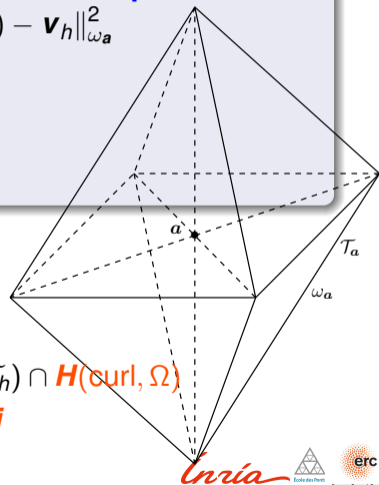
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Outline

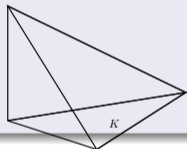
- 1 Introduction
- 2 Approximation error estimates
- 3 A posteriori error estimates
- 4 Local-best–global-best equivalence
 - Context
 - Equivalence
- 5 A stable local commuting projector
 - Commuting de Rham diagram, wishlist, and context
 - A stable local commuting projector $P_h^{p,\text{curl}}$
- 6 Equilibration in $\mathbf{H}(\text{curl})$
 - Patchwise equilibration
 - Main tool: stable (broken) $\mathbf{H}(\text{curl})$ polynomial extensions
- 7 Numerical illustration
- 8 Conclusions

$H(\text{curl})$ polynomial extensions on a tetrahedron

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Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K . Then, for every polynomial degree $p \geq 0$, for all $\mathbf{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla \cdot \mathbf{r}_K = 0$, and for all $\mathbf{r}_{\mathcal{F}} \in \mathcal{N}_p^{\tau}(\Gamma_{\mathcal{F}})$ such that $\mathbf{r}_K \cdot \mathbf{n}_F = \text{curl}_F(\mathbf{r}_{\mathcal{F}})$ for all $F \in \mathcal{F}$, there holds



$$\min_{\substack{\mathbf{v}_p \in \mathcal{N}_p(K) \\ \nabla \times \mathbf{v}_p = \mathbf{r}_K \\ \mathbf{v}_p|_{\mathcal{F}} = \mathbf{r}_{\mathcal{F}}} } \|\mathbf{v}_p\|_K \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{curl}, K) \\ \nabla \times \mathbf{v} = \mathbf{r}_K \\ \mathbf{v}|_{\mathcal{F}} = \mathbf{r}_{\mathcal{F}}} } \|\mathbf{v}\|_K.$$

Comments

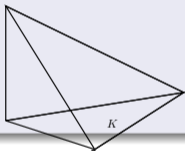
- C_{st} only depends on the **shape-regularity** of K
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- extension to an **edge patch**: Chaumont-Frelet, Ern, & Vohralík (2021)
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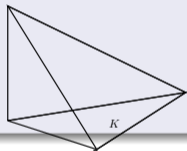
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$H(\text{curl})$ polynomial extensions on a tetrahedron and on patches

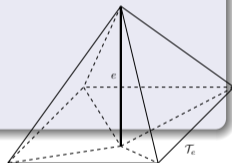
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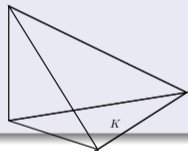
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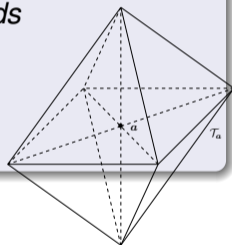
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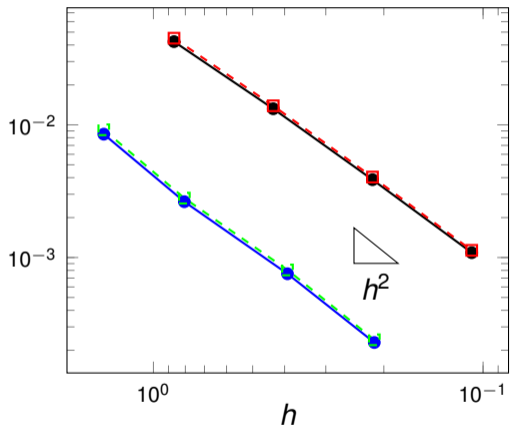
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- 2 Approximation error estimates
- 3 A posteriori error estimates
- 4 Local-best–global-best equivalence
 - Context
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- 5 A stable local commuting projector
 - Commuting de Rham diagram, wishlist, and context
 - A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$
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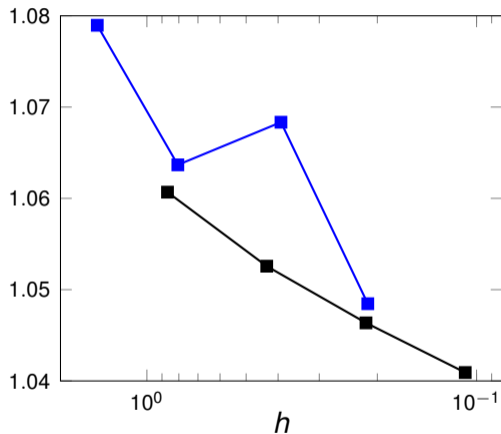
Patchwise equilibration, H^3 solution, h -refinement

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|$$



- error - -□- - estimate, $p = 1$
- error - -□- - estimate, $p = 2$

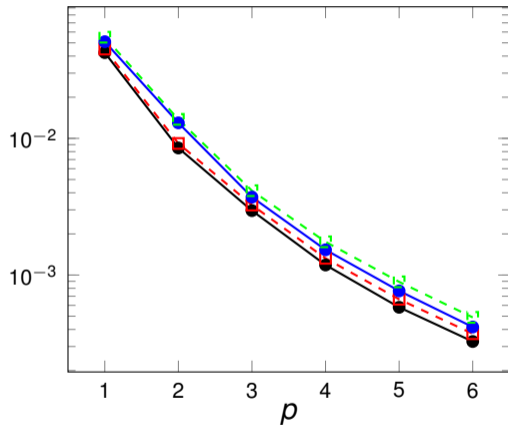
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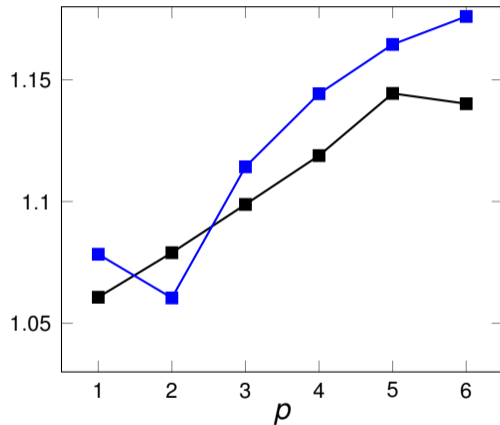
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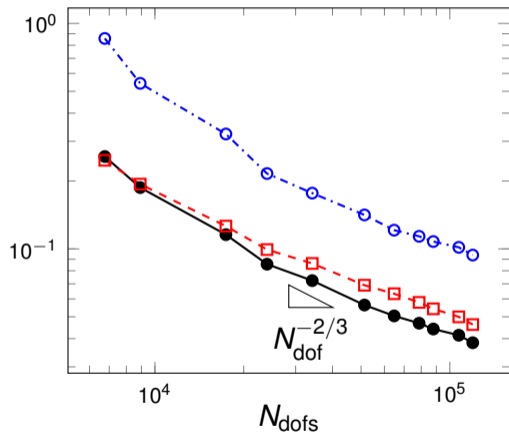


- error - -□- - estimate, struct. mesh
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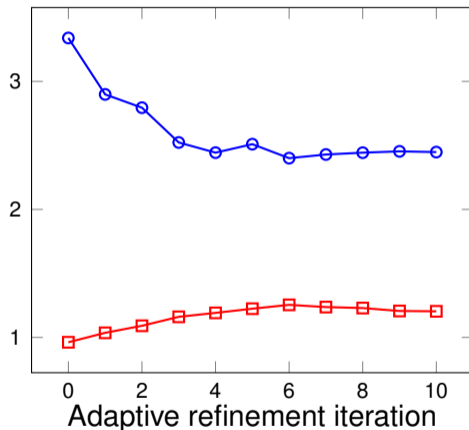
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Patchwise equilibration, **singular solution, adap.** refinement ($p = 2$)

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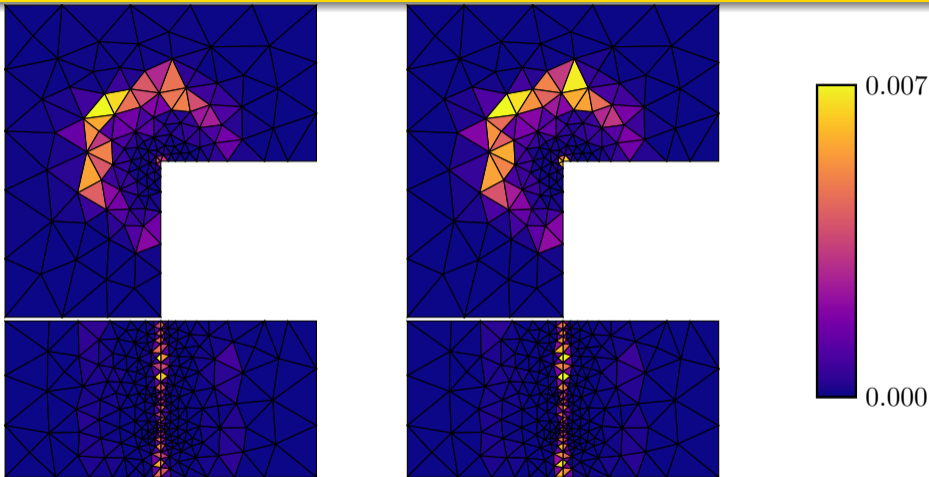


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error
 estimate
 est.+d. osc.
 no d. osc.
 d. osc.

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Estimators (left) and actual error (right), adaptive mesh refinement iteration #10.
 Top view (top) and side view (bottom)

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Thank you for your attention!

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Outline

9 Primal and dual approximations

The grad–grad problem: primal and dual approximations

Problem (source $f \in L^2(\Omega)$)

Find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

$u \in H_0^1(\Omega)$ (primal variable), $\sigma := -\nabla u \in H(\text{div}, \Omega)$ with $\nabla \cdot \sigma = f$ (dual variable)

Primal approximation

Find $u_h \in V_h := \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Dual approximation

$$\sigma_h := \arg \min_{\substack{v_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap H(\text{div}, \Omega) \\ \nabla \cdot v_h = \Pi_p f}} \|v_h\|^2$$

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