

Full adaptivity for unsteady nonlinear problems

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Outline

- 1 Introduction
- 2 The Stefan problem
 - Dual norm a posteriori estimate and adaptivity
 - Efficiency
 - Energy error a posteriori estimate
 - Numerical results
- 3 Two-phase flow
 - A posteriori error estimate
 - Full adaptivity
 - Applications
 - Numerical results
- 4 References and bibliography

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Full adaptivity for unsteady nonlinear problems

Real (porous media) flows

- systems of PDEs
- **nonlinear** (degenerate)
- **unsteady**
- \Rightarrow difficult numerical approximation, large troublesome **systems** of **nonlinear algebraic equations**

Goals

- derive fully computable a posteriori **error upper bounds**
- distinguish different **error components**

Full adaptivity

- time step choice & mesh adaptivity
- **stopping criteria** for **regularization** and **linear** and **nonlinear** solvers

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The Stefan problem

The Stefan problem

$$\begin{aligned}\partial_t u - \Delta \beta(u) &= f && \text{in } \Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0, T)\end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β : L_β -Lipschitz continuous, $\beta(s) = 0$ in $(0, 1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$

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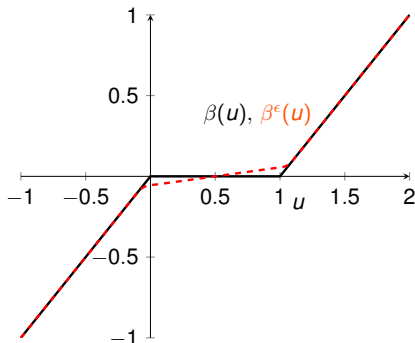
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Numerical practice: regularization

Regularization of β , parameter ϵ



- $\beta^\epsilon(\cdot)$ smooth and strictly increasing

Setting

Functional spaces

$$X := L^2(0, T; H_0^1(\Omega)), \quad Z := H^1(0, T; H^{-1}(\Omega))$$

Weak formulation

$$u \in Z \quad \text{with } \beta(u) \in X,$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

$$\langle \partial_t u, \varphi \rangle(s) + (\nabla \beta(u), \nabla \varphi)(s) = (f, \varphi)(s) \quad \forall \varphi \in H_0^1(\Omega), s \in (0, T)$$

Approximation (conforming, with linearization & regularization)

$$u_{h\tau}^{\epsilon, k} \in Z, \quad \partial_t u_{h\tau}^{\epsilon, k} \in L^2(0, T; L^2(\Omega)), \quad \beta(u_{h\tau}^{\epsilon, k}) \in X,$$

$$u_{h\tau}^{\epsilon, k}|_{I_n} \text{ is affine in time on } I_n \quad \forall 1 \leq n \leq N$$

Residual $\mathcal{R}(u_{h\tau}^{\epsilon, k}) \in X'$ and its dual norm, $\varphi \in X$

$$\langle \mathcal{R}(u_{h\tau}^{\epsilon, k}), \varphi \rangle_{X', X} := \int_0^T \left\{ \langle \partial_t(u - u_{h\tau}^{\epsilon, k}), \varphi \rangle + (\nabla(\beta(u) - \beta(u_{h\tau}^{\epsilon, k})), \nabla \varphi) \right\}(s) ds,$$

$$\|\mathcal{R}(u_{h\tau}^{\epsilon, k})\|_{X'} := \sup_{\varphi \in X, \|\varphi\|_X=1} \langle \mathcal{R}(u_{h\tau}^{\epsilon, k}), \varphi \rangle_{X', X}$$

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Time-localization of the dual norm of the residual

Time interval I_n

$$X_n := L^2(I_n; H_0^1(\Omega))$$

$$\begin{aligned} \|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X'_n} &:= \sup_{\varphi \in X_n, \|\varphi\|_{X_n}=1} \int_{I_n} \{ \langle \partial_t(u - u_{h\tau}^{\epsilon,k}), \varphi \rangle \\ &\quad + (\nabla\beta(u) - \nabla\beta(u_{h\tau}^{\epsilon,k}), \nabla\varphi) \}(\mathbf{s}) \, d\mathbf{s} \end{aligned}$$

L^2 in time:

$$\|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X'}^2 = \sum_{1 \leq n \leq N} \|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X'_n}^2$$

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Estimate distinguishing different error components

Assumption A (Equilibrated flux reconstruction)

For all $n \geq 1$, $k \geq 1$, and $\epsilon > 0$, there exists $\sigma_h^{n,\epsilon,k} \in \mathbf{H}(\text{div}; \Omega)$ s.t.

$$(\nabla \cdot \sigma_h^{n,\epsilon,k}, 1)_K = (f|_{I_n}, 1)_K - (\partial_t u_{h\tau}^{\epsilon,k}|_{I_n}, 1)_K \quad \forall K \in \mathcal{T}^n.$$

Theorem (An estimate distinguishing the error components)

Let *Assumption A* hold. Then, for any $n \geq 1$, $k \geq 1$, and $\epsilon > 0$,

$$\|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X'_h} \leq \eta_{\text{sp}}^{n,\epsilon,k} + \eta_{\text{tm}}^{n,\epsilon,k} + \eta_{\text{reg}}^{n,\epsilon,k} + \eta_{\text{lin}}^{n,\epsilon,k}.$$

$$(\eta_{\text{sp}}^{n,\epsilon,k})^2 := \tau^n \sum_{K \in \mathcal{T}^n} \left(\eta_{E,K}^{n,\epsilon,k} + \|\nabla \beta^{\epsilon,k-1}(u_h^{n,\epsilon,k}) + \sigma_h^{n,\epsilon,k}\|_K \right)^2,$$

$$(\eta_{\text{tm}}^{n,\epsilon,k})^2 := \int_{I_n} \sum_{K \in \mathcal{T}^n} \|\nabla(\beta(u_{h\tau}^{\epsilon,k})(t) - \beta(u_h^{n,\epsilon,k}))\|_K^2 dt,$$

$$(\eta_{\text{reg}}^{n,\epsilon,k})^2 := \tau^n \sum_{K \in \mathcal{T}^n} \|\nabla(\beta(u_h^{n,\epsilon,k}) - \beta^\epsilon(u_h^{n,\epsilon,k}))\|_K^2,$$

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Efficiency assumptions

Assumption B (Piecewise polynomials, data, and meshes)

The approximations and the data f and u_0 are *piecewise polynomial*. The meshes are *shape-regular*.

Residual estimators

$$\left(\eta_{\text{res},1}^{n,\epsilon,k}\right)^2 := \tau^n \sum_{K \in \mathcal{T}^{n-1,n}} h_K^2 \|f|_{I_n} - \partial_t u_{h\tau}^{\epsilon,k}|_{I_n} + \Delta \beta^{\epsilon,k-1}(u_h^{n,\epsilon,k})\|_K^2,$$

$$\left(\eta_{\text{res},2}^{n,\epsilon,k}\right)^2 := \tau^n \sum_{e \in \mathcal{E}^{\text{int},n-1,n}} h_e \|[\![\nabla \beta^{\epsilon,k-1}(u_h^{n,\epsilon,k}) \cdot \mathbf{n}_e]\!] \|_e^2$$

Assumption C (Approximation property)

For all $1 \leq n \leq N$, there holds

$$\tau^n \sum_{K \in \mathcal{T}^{n-1,n}} \|\nabla \beta^{\epsilon,k-1}(u_h^{n,\epsilon,k}) + \sigma_h^{n,\epsilon,k}\|_K^2 \leq C \left(\left(\eta_{\text{res},1}^{n,\epsilon,k}\right)^2 + \left(\eta_{\text{res},2}^{n,\epsilon,k}\right)^2 \right).$$

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Efficiency assumptions

Theorem (Efficiency)

Let, for all $1 \leq n \leq N$, the *stopping* and *balancing criteria* be satisfied with the parameters *small enough*. Let *Assumptions B* and *C* hold. Then

$$\eta_{\text{sp}}^{n,\epsilon,k} + \eta_{\text{tm}}^{n,\epsilon,k} + \eta_{\text{reg}}^{n,\epsilon,k} + \eta_{\text{lin}}^{n,\epsilon,k} \leq C \|\mathcal{R}(u_h^{n,\epsilon,k})\|_{X'_n}.$$

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Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

$$\begin{aligned} & \frac{L_\beta}{2} \|u - u_{h_T}\|_{X'}^2 + \frac{L_\beta}{2} \|(u - u_{h_T})(\cdot, T)\|_{H^{-1}(\Omega)}^2 + \|\beta(u) - \beta(u_{h_T})\|_{Q_T}^2 \\ & \leq \frac{L_\beta}{2} (2e^T - 1) \left(\|\mathcal{R}(u_{h_T})\|_{X'}^2 + \|(u - u_{h_T})(\cdot, 0)\|_{H^{-1}(\Omega)}^2 \right) \end{aligned}$$

Theorem (Temperature and enthalpy errors, tight Gronwall)

Let $u_{h_T} \in Z$ such that $\beta(u_{h_T}) \in X$ be arbitrary. There holds

$$\begin{aligned} & \frac{L_\beta}{2} \|u - u_{h_T}\|_{X'}^2 + \frac{L_\beta}{2} \|(u - u_{h_T})(\cdot, T)\|_{H^{-1}(\Omega)}^2 + \|\beta(u) - \beta(u_{h_T})\|_{Q_T}^2 \\ & + 2 \int_0^T \left(\|\beta(u) - \beta(u_{h_T})\|_{Q_t}^2 + \int_0^t \|\beta(u) - \beta(u_{h_T})\|_{Q_s}^2 e^{t-s} ds \right) dt \\ & \leq \frac{L_\beta}{2} \left\{ (2e^T - 1) \|(u - u_{h_T})(\cdot, 0)\|_{H^{-1}(\Omega)}^2 + \|\mathcal{R}(u_{h_T})\|_{X'}^2 \right. \\ & \left. + 2 \int_0^T \left(\|\mathcal{R}(u_{h_T})\|_{X'_t}^2 + \int_0^t \|\mathcal{R}(u_{h_T})\|_{X'_s}^2 e^{t-s} ds \right) dt \right\}. \end{aligned}$$

Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

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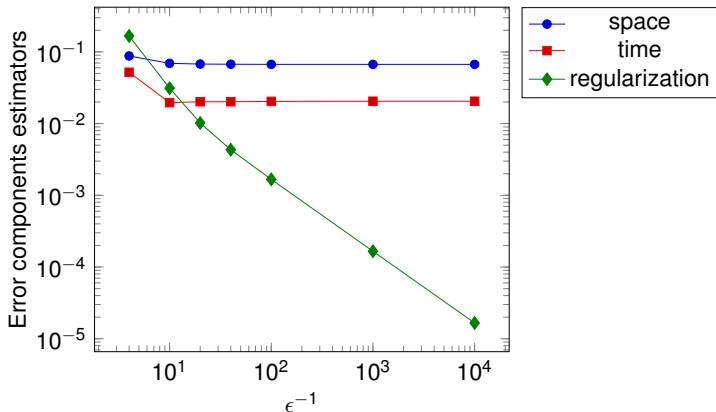
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Regularization stopping criterion

Regularization stopping criterion

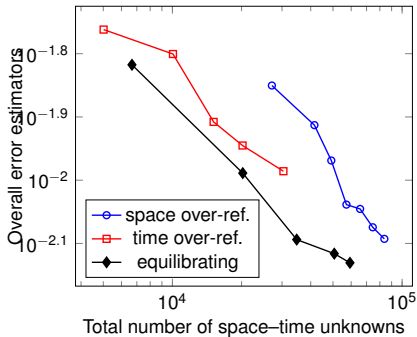
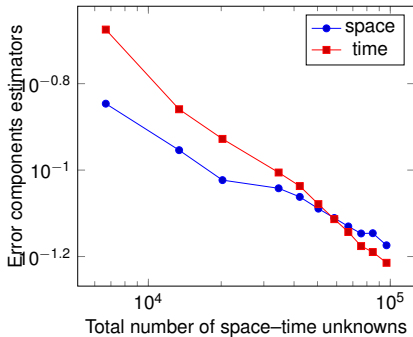
$$\eta_{\text{reg}}^{n,\epsilon,k} \leq \Gamma_{\text{reg}} (\eta_{\text{sp}}^{n,\epsilon,k} + \eta_{\text{tm}}^{n,\epsilon,k})$$



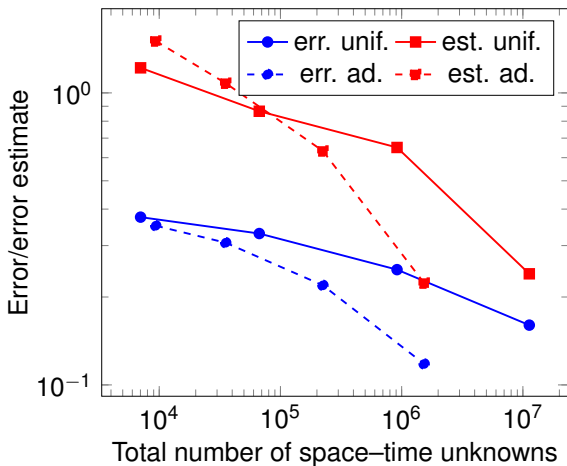
Equilibrating time and space errors

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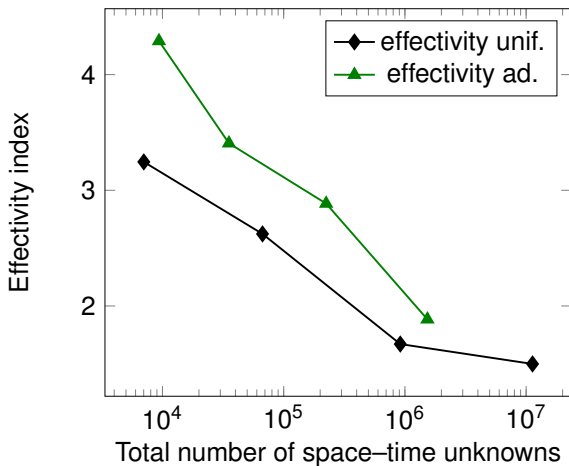
$$\gamma_{tm} \eta_{sp}^{n,\epsilon,k} \leq \eta_{tm}^{n,\epsilon,k} \leq \Gamma_{tm} \eta_{sp}^{n,\epsilon,k}$$



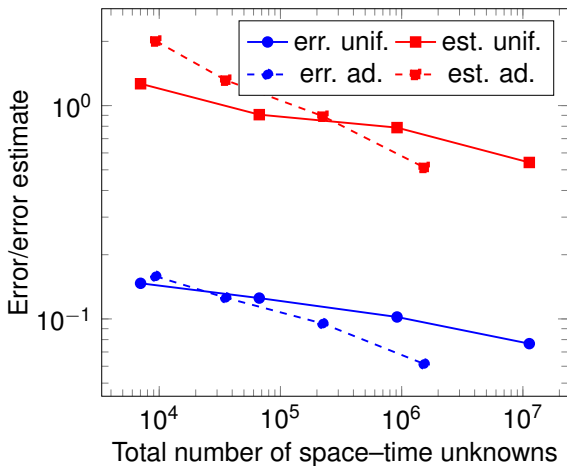
Error and estimate (dual norm of the residual)



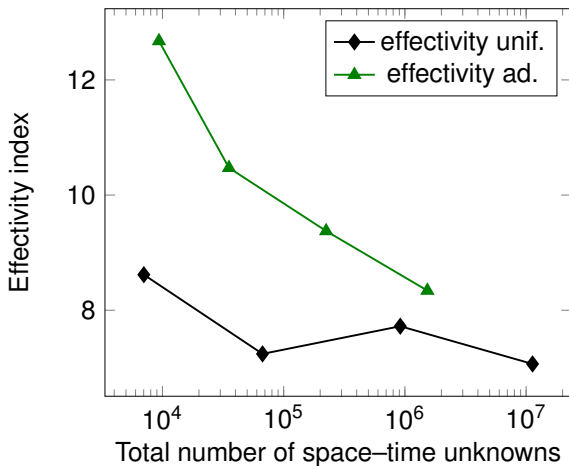
Effectivity indices (dual norm of the residual)



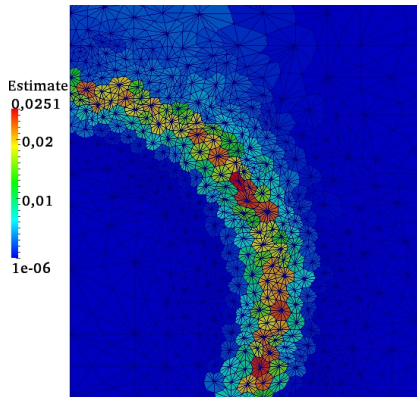
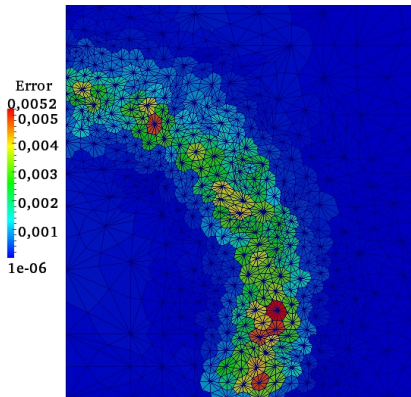
Error and estimate (energy norm)



Effectivity indices (energy norm)



Actual and estimated error distribution



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Two-phase flow in porous media

Two-phase flow in porous media

$$\begin{aligned} \partial_t(\phi \mathbf{s}_\alpha) + \nabla \cdot \mathbf{u}_\alpha &= q_\alpha, & \alpha \in \{\mathbf{n}, \mathbf{w}\}, \\ -\lambda_\alpha(\mathbf{s}_w) \underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha \mathbf{g} \nabla z) &= \mathbf{u}_\alpha, & \alpha \in \{\mathbf{n}, \mathbf{w}\}, \\ \mathbf{s}_n + \mathbf{s}_w &= 1, \\ \rho_n - \rho_w &= \rho_c(\mathbf{s}_w) \end{aligned}$$

+ boundary & initial conditions

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–degenerate parabolic type
- dominant advection

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- coupled system
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Global and complementary pressures

Global pressure

$$p(s_w, p_w) := p_w + \int_0^{s_w} \frac{\lambda_n(a)}{\lambda_w(a) + \lambda_n(a)} p'_c(a) da$$

Complementary pressure

$$q(s_w) := - \int_0^{s_w} \frac{\lambda_w(a)\lambda_n(a)}{\lambda_w(a) + \lambda_n(a)} p'_c(a) da$$

Comments

- necessary for the correct definition of the weak solution
- equivalent Darcy velocities expressions

$$\mathbf{u}_w(s_w, p_w) := - \underline{\mathbf{K}}(\lambda_w(s_w) \nabla p(s_w, p_w) + \nabla q(s_w) + \lambda_w(s_w) \rho_w g \nabla Z),$$

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Weak formulation

Energy space

$$X := L^2((0, T); H_D^1(\Omega))$$

Definition (Weak solution (Arbogast 1992, Chen 2001))

Find (s_w, p_w) such that, with $s_n := 1 - s_w$,

$$s_w \in C([0, T]; L^2(\Omega)), \quad s_w(\cdot, 0) = s_w^0,$$

$$\partial_t s_w \in L^2((0, T); (H_D^1(\Omega))'),$$

$$p(s_w, p_w) \in X,$$

$$q(s_w) \in X,$$

$$\int_0^T \{ \langle \partial_t(\phi s_\alpha), \varphi \rangle - (\mathbf{u}_\alpha(s_w, p_w), \nabla \varphi) - (q_\alpha, \varphi) \} dt = 0$$

$$\forall \varphi \in X, \alpha \in \{n, w\}.$$

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Link energy-type error – dual norm of the residual

Dual norm of the residual on the time interval I_n

$$\mathcal{J}_{\mathbf{s}_w, \rho_w}^n(\mathbf{s}_w, h_\tau, \rho_w, h_\tau) := \left\{ \sum_{\alpha \in \{n, w\}} \left\{ \sup_{\varphi \in X_n, \|\varphi\|_{X_n} = 1} \int_{I_n} \left\{ \langle \partial_t(\phi \mathbf{s}_\alpha) - \partial_t(\phi \mathbf{s}_{\alpha, h_\tau}), \varphi \rangle - (\mathbf{u}_\alpha(\mathbf{s}_w, \rho_w) - \mathbf{u}_\alpha(\mathbf{s}_w, h_\tau, \rho_w, h_\tau), \nabla \varphi) \right\} dt \right\}^2 \right\}^{\frac{1}{2}}$$

Theorem (Link energy-type error – dual norm of the residual)

Let (\mathbf{s}_w, ρ_w) be the *weak solution*. Let $(\mathbf{s}_w, h_\tau, \rho_w, h_\tau)$ be *arbitrary* such that $\mathbf{p}(\mathbf{s}_w, h_\tau, \rho_w, h_\tau) \in X$ and $\mathbf{q}(\mathbf{s}_w, h_\tau) \in X$ (and satisfying the initial and boundary conditions for simplicity). Then

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Distinguishing the error components

Theorem (Distinguishing the error components)

Consider a vertex-centered finite volume / backward Euler approximation and Newton linearization. Let

- n be the *time* step,
- k be the *linearization* step,
- i be the *algebraic solver* step,

with the approximations $(s_{w,h_T}^{n,k,i}, p_{w,h_T}^{n,k,i})$. Then

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Error components

- $\eta_{sp}^{n,k,i}$: spatial discretization
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- only a **necessary number** of **algebraic/linearization solver iterations**
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- **“online decisions”**: algebraic step / linearization step / space mesh refinement / time step modification
- important computational savings
- guaranteed and robust a posteriori error estimates

Not treated for the moment

- convergence and optimality

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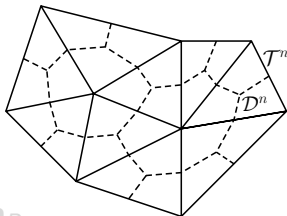
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Iteratively coupled vertex-centered finite volumes

Vertex-centered finite volumes

- simplicial meshes \mathcal{T}^n , dual meshes \mathcal{D}^n
- saturations & pressures continuous and pw affine on \mathcal{T}^n



Implicit pressure equation on step k

$$\begin{aligned}
 & -((\lambda_w(s_{w,h}^{n,k-1}) + \lambda_n(s_{w,h}^{n,k-1})) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k} \cdot \mathbf{n}_D \\
 & + \lambda_n(s_{w,h}^{n,k-1}) \underline{\mathbf{K}} \nabla \bar{p}_c(s_{w,h}^{n,k-1}) \cdot \mathbf{n}_D, 1)_{\partial D \setminus \partial \Omega} = 0 \quad \forall D \in \mathcal{D}^{\text{int},n}
 \end{aligned}$$

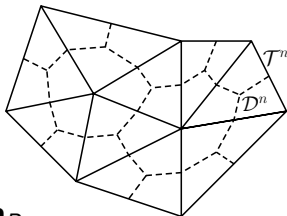
Explicit saturation equation on step k

$$s_{w,D}^{n,k} := \frac{\tau^n}{\phi |D|} (\lambda_w(s_{w,h}^{n,k-1}) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k} \cdot \mathbf{n}_D, 1)_{\partial D \setminus \partial \Omega} + s_{w,D}^{n-1} \quad \forall D \in \mathcal{D}^{\text{int},n}$$

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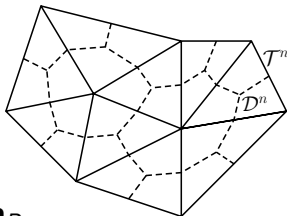
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Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$\begin{aligned}
 & -((\lambda_w(\mathbf{s}_{w,h}^{n,k-1}) + \lambda_n(\mathbf{s}_{w,h}^{n,k-1})) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k,i} \cdot \mathbf{n}_D \\
 & + \lambda_n(\mathbf{s}_{w,h}^{n,k-1}) \underline{\mathbf{K}} \nabla \bar{p}_c(\mathbf{s}_{w,h}^{n,k-1}) \cdot \mathbf{n}_D, 1)_{\partial D \setminus \partial \Omega} = -R_{t,D}^{n,k,i} \quad \forall D \in \mathcal{D}^{\text{int},n}
 \end{aligned}$$

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Flux reconstructions

Total velocities reconstructions

$$\begin{aligned}
 (\mathbf{d}_{t,h}^{n,k,i} \cdot \mathbf{n}_D, 1)_e &:= - \left((\lambda_w(s_{w,h}^{n,k,i}) + \lambda_n(s_{w,h}^{n,k,i})) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k,i} \cdot \mathbf{n}_D \right. \\
 &\quad \left. + \lambda_n(s_{w,h}^{n,k,i}) \underline{\mathbf{K}} \nabla \bar{p}_c(s_{w,h}^{n,k,i}) \cdot \mathbf{n}_D, 1 \right)_e, \\
 ((\mathbf{d}_{t,h}^{n,k,i} + \mathbf{l}_{t,h}^{n,k,i}) \cdot \mathbf{n}_D, 1)_e &:= - \left((\lambda_w(s_{w,h}^{n,k-1}) + \lambda_n(s_{w,h}^{n,k-1})) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k,i} \cdot \mathbf{n}_D \right. \\
 &\quad \left. + \lambda_n(s_{w,h}^{n,k-1}) \underline{\mathbf{K}} \nabla \bar{p}_c(s_{w,h}^{n,k-1}) \cdot \mathbf{n}_D, 1 \right)_e, \\
 \mathbf{a}_{t,h}^{n,k,i} &:= \mathbf{d}_{t,h}^{n,k,i+\nu} + \mathbf{l}_{t,h}^{n,k,i+\nu} - (\mathbf{d}_{t,h}^{n,k,i} + \mathbf{l}_{t,h}^{n,k,i})
 \end{aligned}$$

Phases velocities reconstructions

$$\begin{aligned}
 (\mathbf{d}_{w,h}^{n,k,i} \cdot \mathbf{n}_D, 1)_e &:= - (\lambda_w(s_{w,h}^{n,k,i}) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k,i} \cdot \mathbf{n}_D, 1)_e, \\
 ((\mathbf{d}_{w,h}^{n,k,i} + \mathbf{l}_{w,h}^{n,k,i}) \cdot \mathbf{n}_D, 1)_e &:= - (\lambda_w(s_{w,h}^{n,k-1}) \underline{\mathbf{K}} \nabla p_{w,h}^{n,k,i} \cdot \mathbf{n}_D, 1)_e, \\
 \mathbf{a}_{w,h}^{n,k,i} &:= 0
 \end{aligned}$$

$$\mathbf{d}_{n,h}^{n,k,i} := \mathbf{d}_{t,h}^{n,k,i} - \mathbf{d}_{w,h}^{n,k,i}, \quad \mathbf{l}_{n,h}^{n,k,i} := \mathbf{l}_{t,h}^{n,k,i} - \mathbf{l}_{w,h}^{n,k,i}, \quad \mathbf{a}_{n,h}^{n,k,i} := \mathbf{a}_{t,h}^{n,k,i} - \mathbf{a}_{w,h}^{n,k,i}$$

Equilibrated fluxes

$$\sigma_{\cdot,h}^{n,k,i} := \mathbf{d}_{\cdot,h}^{n,k,i} + \mathbf{l}_{\cdot,h}^{n,k,i} + \mathbf{a}_{\cdot,h}^{n,k,i}$$

Flux reconstructions

Total velocities reconstructions

$$(\mathbf{d}_{t,h}^{n,k,i} \cdot \mathbf{n}_D, 1)_e := - \left((\lambda_w(s_{w,h}^{n,k,i}) + \lambda_n(s_{w,h}^{n,k,i})) \underline{\mathbf{K}} \nabla \rho_{w,h}^{n,k,i} \cdot \mathbf{n}_D + \lambda_n(s_{w,h}^{n,k,i}) \underline{\mathbf{K}} \nabla \bar{\rho}_c(s_{w,h}^{n,k,i}) \cdot \mathbf{n}_D, 1 \right)_e,$$

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Equilibrated fluxes

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Model problem

Horizontal flow

$$\partial_t(\phi \mathbf{s}_\alpha) - \nabla \cdot \left(\frac{k_{r,\alpha}(s_w)}{\mu_\alpha} \underline{\mathbf{K}} \nabla p_\alpha \right) = 0,$$

$$s_n + s_w = 1,$$

$$p_n - p_w = p_c(s_w)$$

Brooks–Corey model

- relative permeabilities

$$k_{r,w}(s_w) = s_e^4, \quad k_{r,n}(s_w) = (1 - s_e)^2(1 - s_e^2)$$

- capillary pressure

$$p_c(s_w) = p_d s_e^{-\frac{1}{2}}$$

-

$$s_e := \frac{s_w - s_{rw}}{1 - s_{rw} - s_{rn}}$$

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$$s_e := \frac{s_w - s_{rw}}{1 - s_{rw} - s_{rn}}$$

Data from Klieber & Rivière (2006)

Data

$$\Omega = (0, 300)\text{m} \times (0, 300)\text{m}, \quad T = 4 \cdot 10^6 \text{s},$$

$$\phi = 0.2, \quad \mathbf{K} = 10^{-11} \mathbf{I} \text{m}^2,$$

$$\mu_w = 5 \cdot 10^{-4} \text{kg m}^{-1} \text{s}^{-1}, \quad \mu_n = 2 \cdot 10^{-3} \text{kg m}^{-1} \text{s}^{-1},$$

$$s_{rw} = s_{rn} = 0, \quad \rho_d = 5 \cdot 10^3 \text{kg m}^{-1} \text{s}^{-2}$$

Initial condition (\tilde{K} 18m \times 18m lower left corner block)

$$s_w^0 = 0.2 \text{ on } K \in \mathcal{T}_h, K \notin \tilde{K},$$

$$s_w^0 = 0.95 \text{ on } K \in \mathcal{T}_h, K \in \tilde{K}$$

Boundary conditions (\hat{K} 18m \times 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\tilde{K} \cap \partial\Omega$ and $\partial\hat{K} \cap \partial\Omega$
- \tilde{K} – injection well: $s_w = 0.95$, $\rho_w = 3.45 \cdot 10^6 \text{kg m}^{-1} \text{s}^{-2}$
- \hat{K} – production well: $s_w = 0.2$, $\rho_w = 2.41 \cdot 10^6 \text{kg m}^{-1} \text{s}^{-2}$

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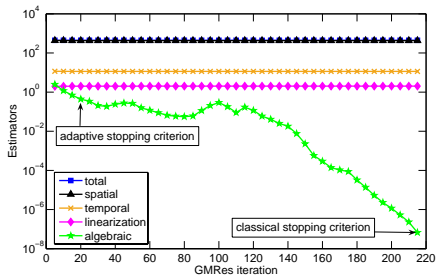
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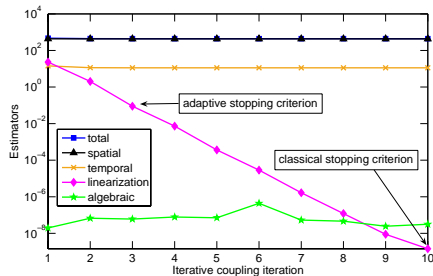
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Estimators and stopping criteria

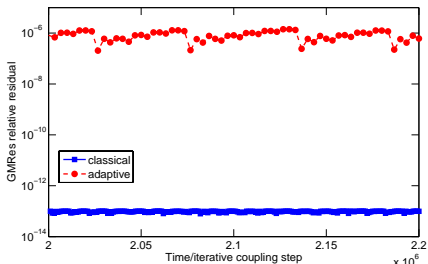


Estimators in function of GMRes iterations

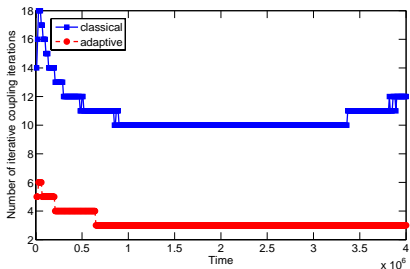


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

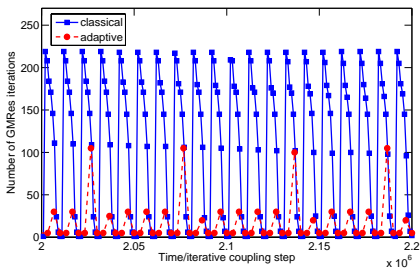


GMRes relative residual

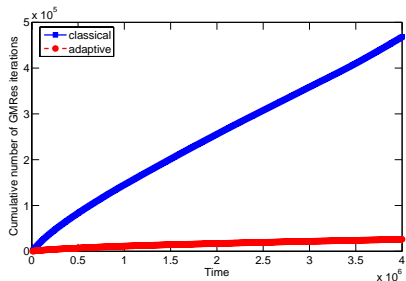


Iterative coupling iterations

GMRes iterations

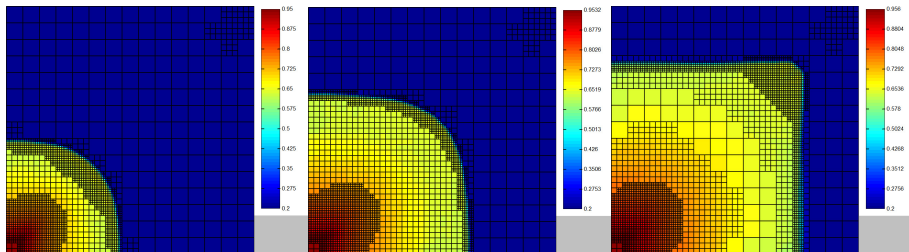


Per time and iterative coupling step



Cumulated

Space/time/nonlinear solver/linear solver adaptivity



Fully adaptive computation

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Nonlinear unsteady problems

- Eriksson and Johnson (1995), $L^\infty(0, T; L^2(\Omega))$ estimates exploiting stability of the adjoint problem
- Gallimard, Ladevèze, Pelle (1997), const. rel. estimates
- Verfürth (1998), framework for energy control, efficiency
- Ohlberger (2001), non-energy estimates, hyperbolic limit
- Akrivis, Makridakis, and Nochetto (2006), higher-order temporal discretizations

Degenerate parabolic problems

- Nochetto, Schmidt, Verdi (2000), Stefan problem
- Dolejší, Ern, Vohralík (2013), Richards problem (advection-dominated), robustness in a space–time dual mesh-dependent norm

Two-phase flows

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Thank you for your attention!