An adaptive inexact Newton method

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Outline

1. Introduction

2. A posteriori error estimate

3. Stopping criteria, efficiency, and robustness

4. Applications

5. Numerical results

6. References and bibliography

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Inexact iterative linearization

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*

   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3. 1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do an algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*

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Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- What is a good stopping criterion for the linear solver?
- What is a good stopping criterion for the nonlinear solver?

Question (Error)

- How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?

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Outline

Model steady problem, discretization

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$
- example: p-Laplacian with $\boldsymbol{\sigma}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$
- f piecewise polynomial for simplicity
- weak solution: $u \in V := W_0^{1,p}(\Omega)$ such that

$$(\boldsymbol{\sigma}(u,\nabla u), \nabla v) = (f, v) \qquad \forall v \in V$$

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step k, algebraic step i
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K)\quad \forall K \in \mathcal{T}_h\} \not\subset V$

Model steady problem, discretization

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\begin{aligned}
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Intrinsic error measure

Energy error in the Laplace case

$$\|\nabla(u - u_h)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u-u_h), \nabla\varphi)^2}_{\text{dual norm of the residual}} + \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v-u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

Intrinsic error measure

$$\mathcal{J}_u(u_h^{k,i}) := \underbrace{\sup_{\varphi \in V;\, \|\nabla\varphi\|_p=1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi)}_{\text{dual norm of the residual}}$$

$$+ \underbrace{\left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}}_{\text{distance of } u_h \text{ to } V}$$

there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$

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Abstract assumptions

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ and $\rho_h^{k,i} \in L^q(\Omega)$ such that

$$\nabla \cdot \boldsymbol{\sigma}_h^{k,i} = f - \underbrace{\rho_h^{k,i}}_{\text{algebraic remainder}} .$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ such that

(i) $\boldsymbol{\sigma}_h^{k,i} = \mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i}$;

(ii) *as the linear solver converges, $\|\mathbf{a}_h^{k,i}\|_q \to 0$;*

(iii) *as the nonlinear solver converges, $\|\mathbf{l}_h^{k,i}\|_q \to 0$.*

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Estimate distinguishing error components

> ## Theorem (Estimate distinguishing different error components)
>
> *Let*
> - *$u \in V$ be the weak solution,*
> - *$u_h^{k,i} \in V(\mathcal{T}_h)$ be arbitrary,*
> - *Assumptions A and B hold.*
>
> *Then there holds*
>
> $$\mathcal{J}_u(u_h^{k,i}) \leq \eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{rem}}^{k,i}.$$

Estimators

- *discretization* estimator

$$\eta_{\text{disc},K}^{k,i} := 2^{\frac{1}{p}} \left(\|\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{ \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}} \right)$$

- *linearization* estimator

$$\eta_{\text{lin},K}^{k,i} := \|\mathbf{l}_h^{k,i}\|_{q,K}$$

- *algebraic* estimator

$$\eta_{\text{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_{q,K}$$

- *algebraic remainder* estimator

$$\eta_{\text{rem},K}^{k,i} := h_\Omega \|\rho_h^{k,i}\|_{q,K}$$

- $\eta_{\cdot}^{k,i} := \left\{ \sum_{K \in \mathcal{T}_h} \left(\eta_{\cdot,K}^{k,i} \right)^q \right\}^{1/q}$

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Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\text{rem}}^{k,i} \leq \gamma_{\text{rem}} \max\{\eta_{\text{disc}}^{k,i}, \eta_{\text{lin}}^{k,i}, \eta_{\text{alg}}^{k,i}\},$$

$$\eta_{\text{alg}}^{k,i} \leq \gamma_{\text{alg}} \max\{\eta_{\text{disc}}^{k,i}, \eta_{\text{lin}}^{k,i}\},$$

$$\eta_{\text{lin}}^{k,i} \leq \gamma_{\text{lin}} \eta_{\text{disc}}^{k,i}$$

- $\gamma_{\text{rem}}, \gamma_{\text{alg}}, \gamma_{\text{lin}} \approx 0.1$

Local stopping criteria

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$$\eta_{\mathrm{alg},K}^{k,i} \leq \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$

$$\eta_{\mathrm{lin},K}^{k,i} \leq \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Assumptions for efficiency

Assumption C (Piecewise polynomials, meshes, quadrature)

The approximation $u_h^{k,i}$ is piecewise polynomial. The meshes $\mathcal{T}_h$ are shape-regular. The quadrature error is negligible.

Assumption D (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\sigma(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} \leq C \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f + \nabla \cdot \sigma(u_h^{k,i}, \nabla u_h^{k,i})\|_{q,K'}^q \right.$$
$$+ \sum_{e \in \mathfrak{C}_K^{\text{int}}} h_e \|[\![\sigma(u_h^{k,i}, \nabla u_h^{k,i}) \cdot \mathbf{n}_e]\!]\|_{q,e}^q$$
$$\left. + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

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$$\left. + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

Global efficiency

Theorem (Global efficiency)

Let the Assumptions C and D be satisfied. Let the global stopping criteria hold. Then,

$$\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} \leq C \mathcal{J}_u(u_h^{k,i}),$$

where C is independent of σ and q.

Theorem (Local efficiency)

*Let the Assumptions C and D be satisfied. Let the **local** stopping criteria hold. Then*

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \leq C \mathcal{J}_{u,\mathfrak{T}_K}(u_h^{k,i}) \qquad \forall K \in \mathcal{T}_h.$$

- **robustness** with respect to the **nonlinearity** thanks to the choice of $\mathcal{J}_u$ as error measure

Global efficiency

Theorem (Global efficiency)

Let the Assumptions C and D be satisfied. Let the global stopping criteria hold. Then,

$$\eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{rem}}^{k,i} \leq C\mathcal{J}_u(u_h^{k,i}),$$

where C is independent of σ and q.

Theorem (Local efficiency)

Let the Assumptions C and D be satisfied. Let the local stopping criteria hold. Then

$$\eta_{\text{disc},K}^{k,i} + \eta_{\text{lin},K}^{k,i} + \eta_{\text{alg},K}^{k,i} + \eta_{\text{rem},K}^{k,i} \leq C\mathcal{J}_{u,\mathfrak{T}_K}(u_h^{k,i}) \qquad \forall K \in \mathcal{T}_h.$$

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Outline

Nonconforming finite elements for the *p*-Laplacian

Discretization
Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- $V_h \not\subset V$ the Crouzeix–Raviart space
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

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Linearization

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Find $u_h^k \in V_h$ such that

$$(\sigma^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi$$

- Newton linearization

$$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi + (p-2)|\nabla u_h^{k-1}|^{p-4}$$
$$(\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\xi - \nabla u_h^{k-1})$$

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Algebraic solution

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Find $u_h^{k,i} \in V_h$ such that

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- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\text{int}}}$
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Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,
$$(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e},$$
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Definition (Construction of $\mathbf{a}_h^{k,i}$)

Set $\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ for (adaptively chosen) $\nu > 0$ additional algebraic solvers steps; $R^{k,i+\nu} \rightsquigarrow \rho_h^{k,i}$.

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Verification of the assumptions

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$

Lemma (Assumptions C and D)

Assumptions C and D hold.

Comments

- quadrature error is zero
- $\mathbf{d}_h^{k,i}$ is close to $\sigma(\nabla u_h^{k,i})$: approximation properties of the Raviart–Thomas–Nédélec spaces

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Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

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Outline

Numerical experiment I

Model problem

- p-Laplacian

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

$$u(x,y) = -\frac{p-1}{p}\left((x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\tfrac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- Crouzeix–Raviart nonconforming finite elements

Analytical and approximate solutions
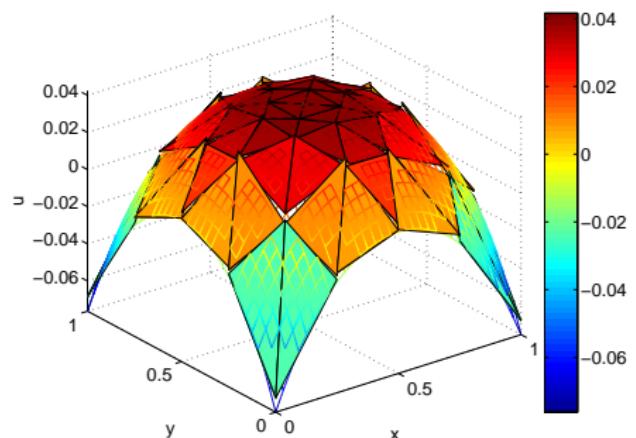
Case $p = 1.5$
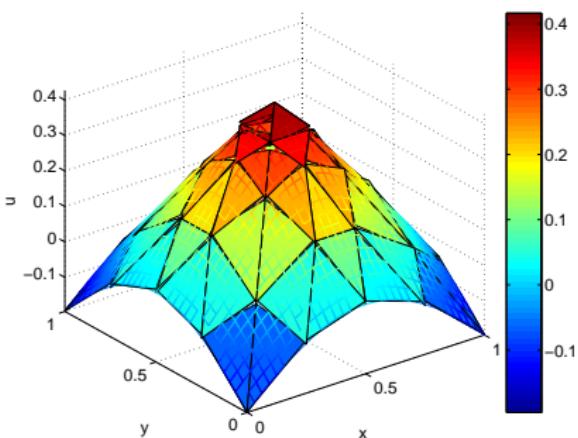
Case $p = 10$

Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.
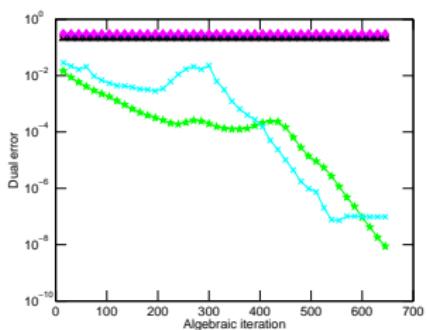
Newton
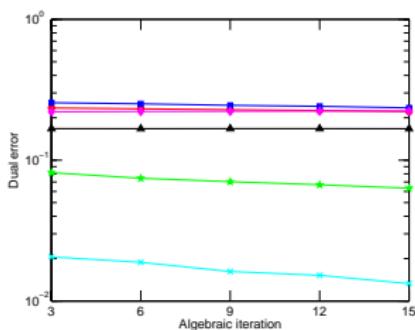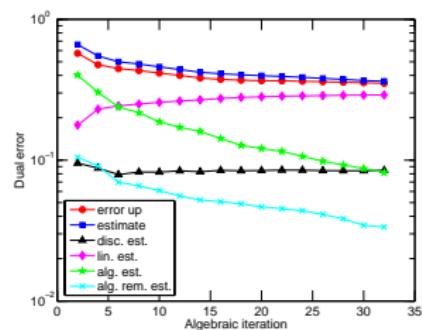
inexact Newton

ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh

Newton   inexact Newton   ad. inexact Newton
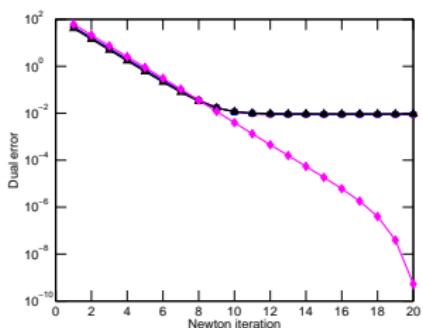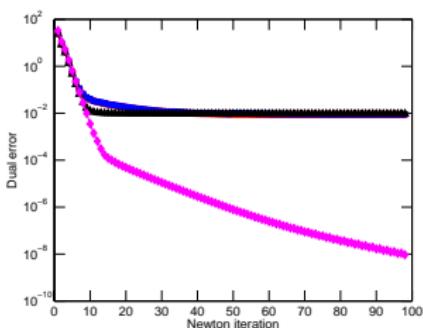
Error and estimators, $p = 10$
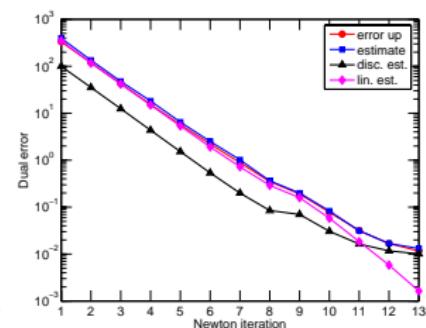

Newton   inexact Newton   ad. inexact Newton
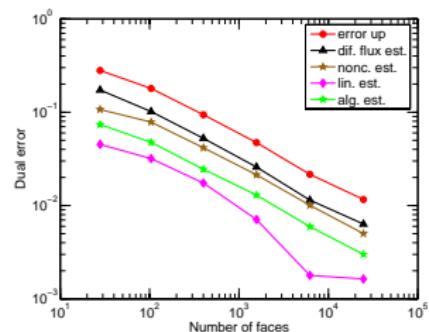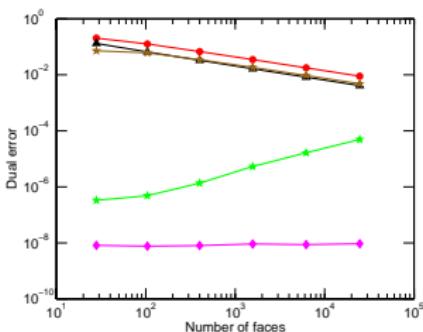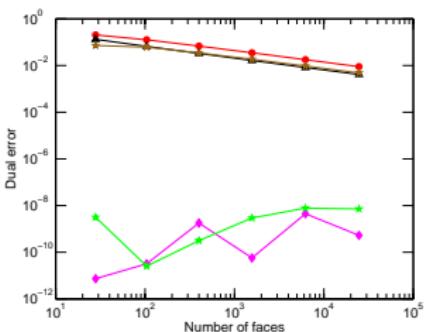
# Effectivity indices, $p = 10$
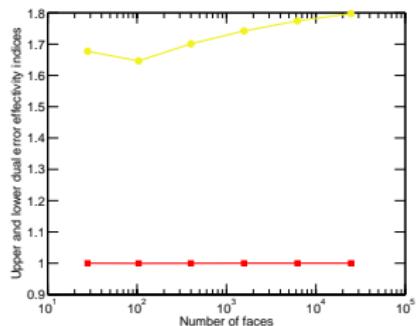
Newton
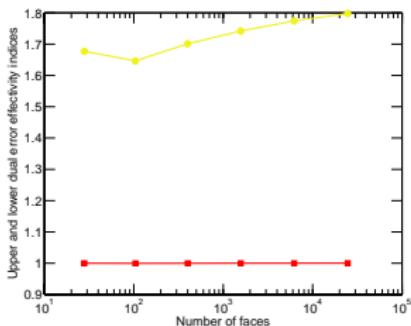
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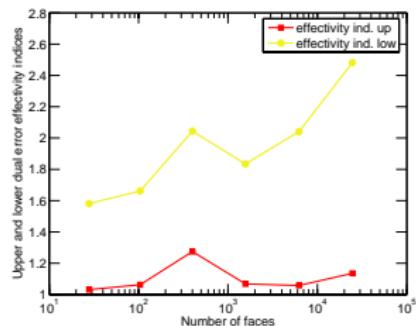
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Error distribution, $p = 10$
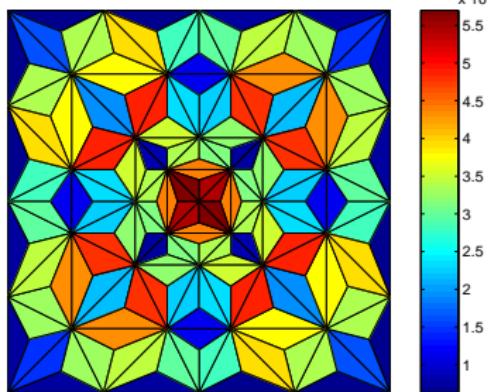
Estimated error distribution
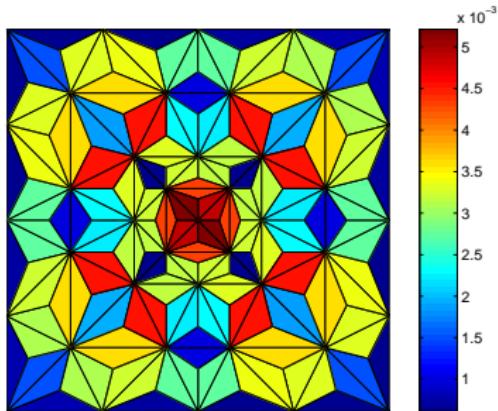
Exact error distribution

Newton and algebraic iterations, $p = 10$
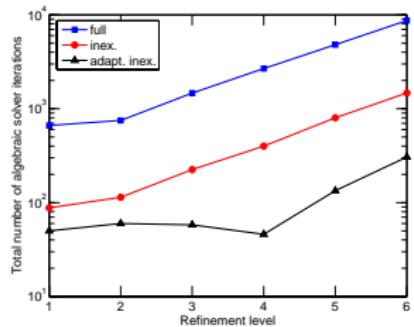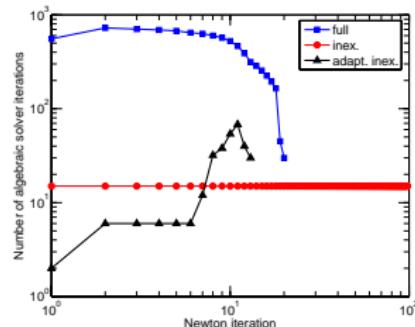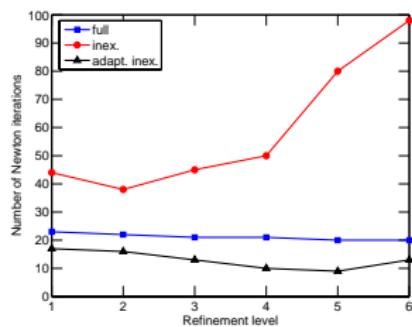
Newton it. / refinement   alg. it. / Newton step   alg. it. / refinement

Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.
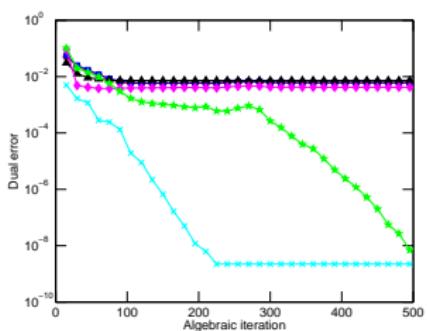
Newton   inexact Newton   ad. inexact Newton
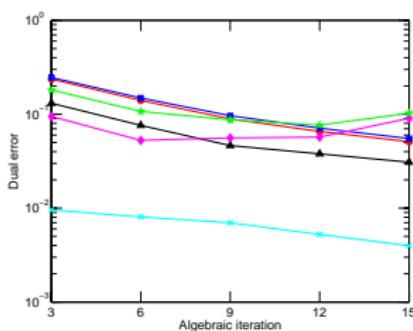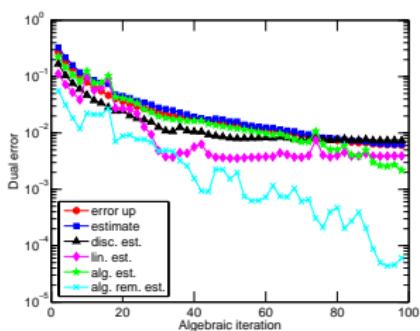
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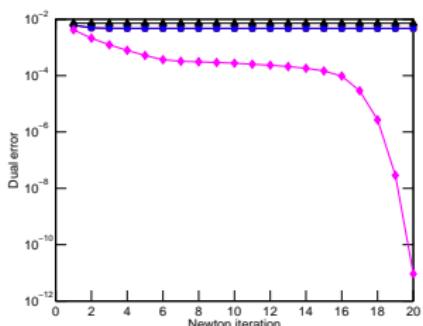
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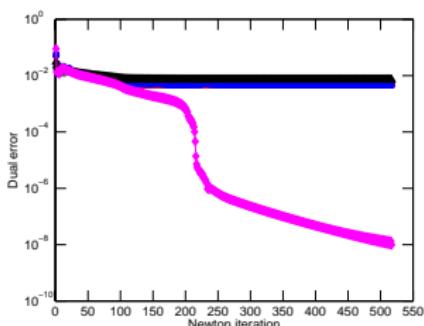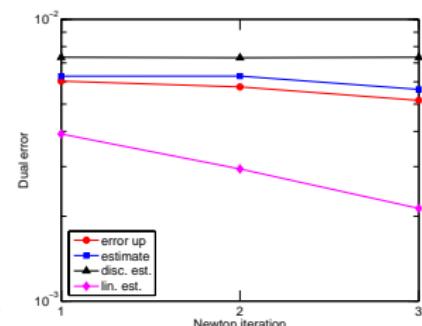
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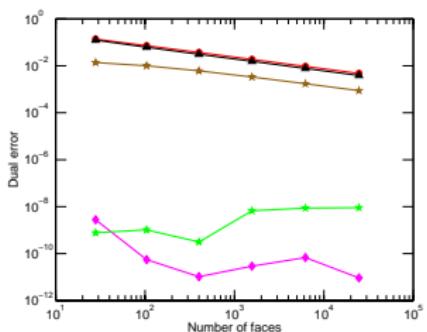
Newton   inexact Newton   ad. inexact Newton
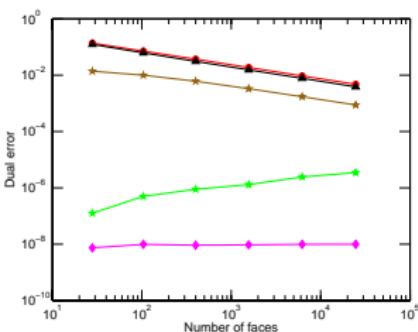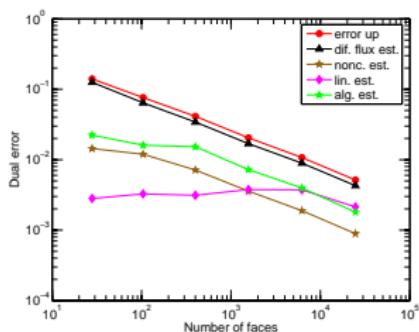
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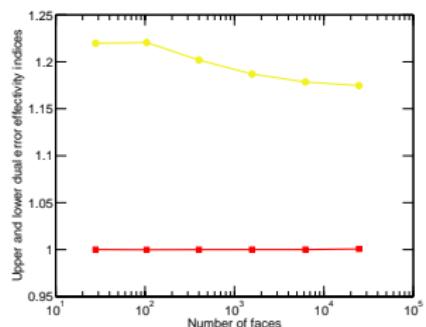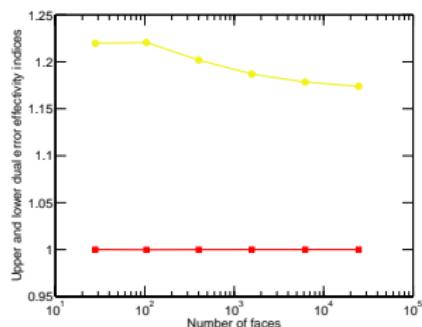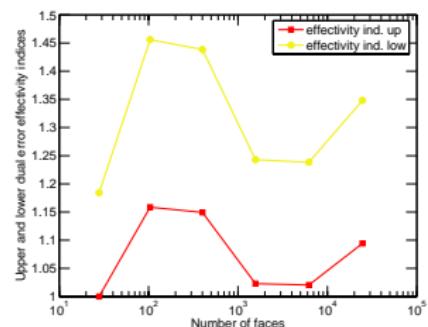
Newton   inexact Newton   ad. inexact Newton

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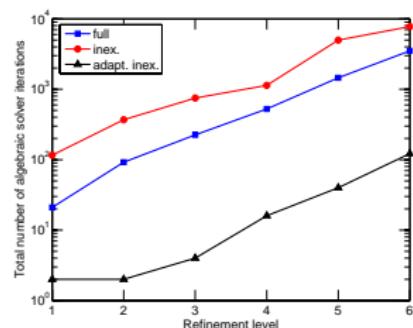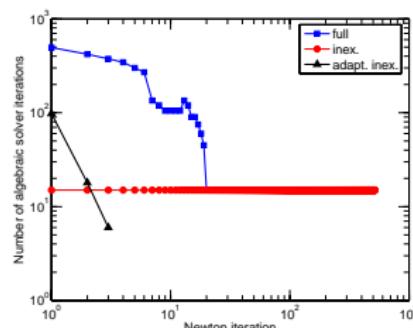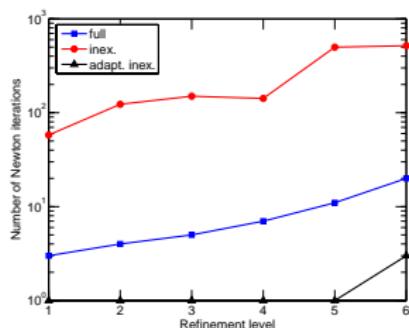
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Numerical experiment II

Model problem

- *p*-Laplacian

$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

$$u(r,\theta) = r^{\frac{7}{8}}\sin(\theta\tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- Crouzeix–Raviart nonconforming finite elements

# Error distribution on an adaptively refined mesh


Estimated error distribution

Exact error distribution
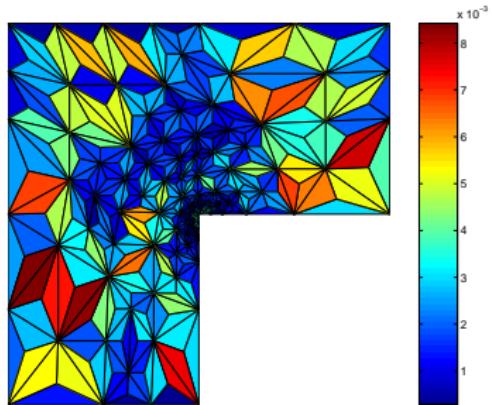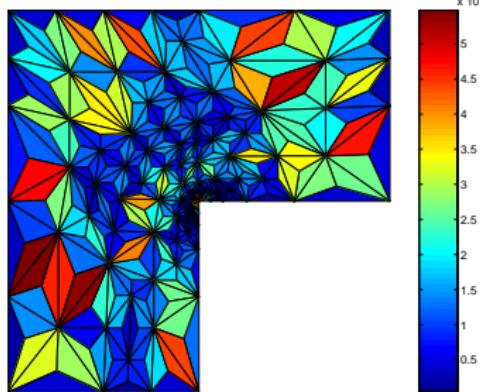
# Estimated and actual errors and the effectivity index
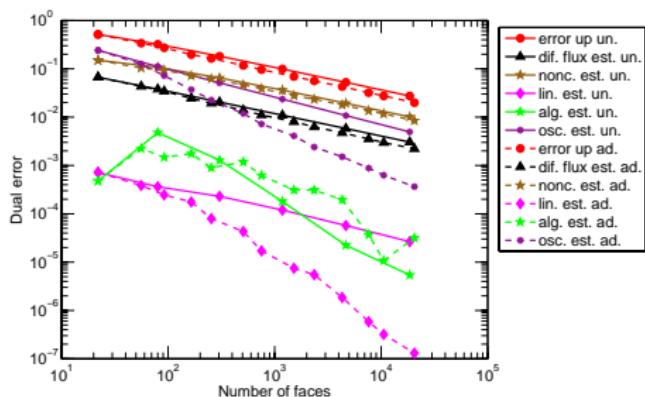
Estimated and actual errors
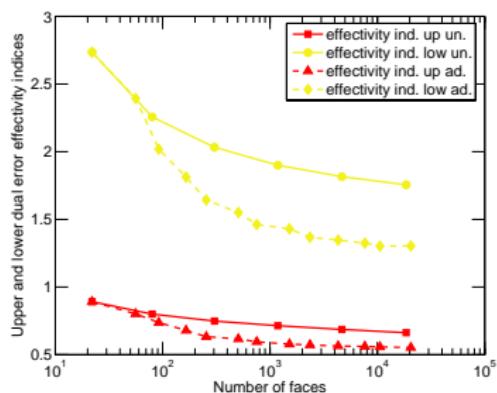
Effectivity index

Energy error and overall performance
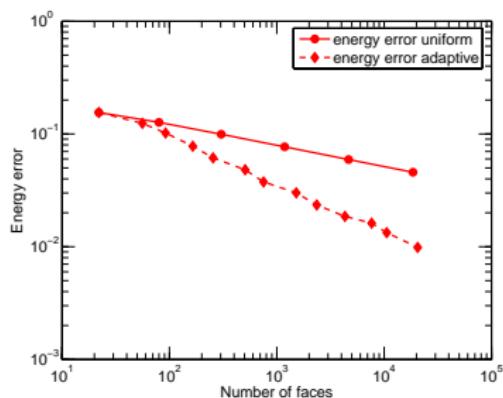
Energy error
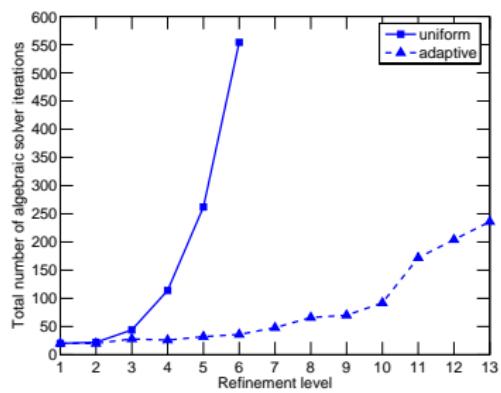
Overall performance

Outline

Previous results

Inexact Newton method

- Eisenstat and Walker (1990's) (conception, convergence, a priori error estimates)
- Moret (1989) (discrete a posteriori error estimates)

Adaptive inexact Newton method

- Bank and Rose (1982), combination with multigrid
- Hackbusch and Reusken (1989), damping and multigrid
- Deuflhard (1990's, 2004 book), adaptivity

Stopping criteria for algebraic solvers

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid stopping criterion
- Arioli (2000's), comparison of the algebraic and discretization errors

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Previous results

A posteriori error estimates for numerical discretizations of nonlinear problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors
- Kim (2007), guaranteed estimates, locally conservative methods

Bibliography

Bibliography

- ERN A., VOHRALÍK M., Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, *SIAM J. Sci. Comput.* **35** (2013), A1761–A1791.

Thank you for your attention!