Localization of global norms and robust a posteriori error control for transmission problems with sign-changing coefficients

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Inria Paris & Ecole des Ponts

Uxbridge, June 18, 2019

Outline

Localization

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, open polygon/polyhedron
- $\mathcal{T}$ simplicial mesh of $\overline{\Omega}$, $\mathcal{V}$ set of vertices, $\omega_{\mathbf{a}}$ vertex patch
- $H^1(\mathcal{T})$ broken Sobolev space, ∇_h elementwise gradient

Localization

- localization of integral norms: for all $v \in L^2(\Omega)$

$$\|v\|^2 = \sum_{K \in \mathcal{T}} \|v\|_K^2$$

- localization of **dual norms** on $H^{-1}(\Omega)$: $\mathcal{R} \in H^{-1}(\Omega)$ (orth.)

$$\|\mathcal{R}\|^2_{H^{-1}(\Omega)} \approx \sum_{\mathbf{a} \in \mathcal{V}} \|\mathcal{R}\|^2_{H^{-1}(\omega_{\mathbf{a}})}$$

- localization of **distance** to $H^1_0(\Omega)$: for all $v \in H^1(\mathcal{T})$ (jump)

$$\min_{\zeta \in H^1_0(\Omega)} \|\nabla_h(v - \zeta)\|^2 \approx \sum_{\mathbf{a} \in \mathcal{V}} \min_{\zeta \in H^1_{\partial\Omega}(\omega_{\mathbf{a}})} \|\nabla_h(v - \zeta)\|^2_{\omega_{\mathbf{a}}}$$

- $\approx$ only depends on d and shape-regularity of $\mathcal{T}$

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A posteriori error estimates (appr. $u_h \in \mathbb{P}_p(\mathcal{T})$ of u)

- identification of **norm** on $H^1(\mathcal{T})$ such that

$$|||u - u_h|||^2 \approx \sum_{\boldsymbol{a} \in \mathcal{V}} |||u - u_h|||^2_{\omega_{\boldsymbol{a}}}$$

- **guaranteed** constant-free **upper bound**:

$$|||u - u_h|||^2 \le \sum_{K \in \mathcal{T}} \eta_K(u_h)^2$$

- **local** efficiency:

$$\eta_K^2(u_h) \lesssim \sum_{\boldsymbol{a} \in \mathcal{V}_K} |||u - u_h|||^2_{\omega_{\boldsymbol{a}}} \qquad \forall K \in \mathcal{T}$$

- **data- & polynomial-degree-robustness**: $\lesssim$ only depends on space dimension d and shape-regularity of $\mathcal{T}$

Minimal regularity: $u \in H^1_0(\Omega)$.

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1. Introduction
2. Localization of global norms
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   - Localization of distances to $H_0^1(\Omega)$
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   - Non-coercive transmission problem
   - A posteriori error estimates in a unified framework
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   - Equilibrated flux reconstruction
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Localization of dual norms on $H^{-1}(\Omega)$

Partition of unity by the hat functions

$$\psi_{\boldsymbol{a}} \in \mathbb{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_0(\omega_{\boldsymbol{a}}) \text{ for } \boldsymbol{a} \in \mathcal{V}^{\rm int}, \quad \sum_{\boldsymbol{a}\in\mathcal{V}} \psi_{\boldsymbol{a}} = 1$$

Theorem (Dual norms localization, Babuška & Miller (1987), Cohen, DeVore, & Nochetto (2012), Ciarlet Jr. & V. (2018), Blechta, Málek, & V. (2018)**)**

Let $\mathcal{R} \in H^{-1}(\Omega)$, be arbitrary subject to

$$\underbrace{\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle = 0}_{\text{lowest-modes orthogonality}} \qquad \forall \boldsymbol{a} \in \mathcal{V}^{\rm int}.$$

Then

$$\underbrace{\|\mathcal{R}\|^2_{H^{-1}(\Omega)}}_{\sup_{v\in H^1_0(\Omega);\, \|\nabla v\|=1} \langle \mathcal{R}, v\rangle} \approx \sum_{\boldsymbol{a}\in\mathcal{V}} \underbrace{\|\mathcal{R}\|^2_{H^{-1}(\omega_{\boldsymbol{a}})}}_{\sup_{v\in H^1_0(\omega_{\boldsymbol{a}});\, \|\nabla v\|_{\omega_{\boldsymbol{a}}}=1} \langle \mathcal{R}, v\rangle}.$$

Condition $\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle = 0$ is only needed in the left inequality.

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Proof ($\lesssim$): partition of unity & Poincaré–Friedrichs in.

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- $w \in H^1(\omega_a)$ with mean value 0 on ω_a or 0 on part of $\partial\omega_a$:

$$\begin{aligned}\|\nabla(\psi_a w)\|_{\omega_a} &= \|\nabla\psi_a w + \psi_a \nabla w\|_{\omega_a}\\ &\le \|\nabla\psi_a\|_{\infty,\omega_a}\|w\|_{\omega_a} + \|\psi_a\|_{\infty,\omega_a}\|\nabla w\|_{\omega_a}\\ &\le \underbrace{(1 + C_{\mathrm{PF},\omega_a} h_{\omega_a}\|\nabla\psi_a\|_{\infty,\omega_a})}_{\le C_{\rm cont,PF}}\|\nabla w\|_{\omega_a}\end{aligned}$$

- Cauchy–Schwarz inequality (& finite overlapping):

$$\langle \mathcal{R}, v\rangle \le C_{\rm cont,PF}\left\{\sum_{a \in \mathcal{V}} \|\mathcal{R}\|^2_{H^{-1}(\omega_a)}\right\}^{1/2}\left\{\sum_{a \in \mathcal{V}} \|\nabla v\|^2_{\omega_a}\right\}^{1/2}$$

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Proof ($\gtrsim$): local Laplacian liftings

- Laplacian lifting of $\mathcal{R}$ on each patch $\omega_{\mathbf{a}}$: $r^{\mathbf{a}} \in H_0^1(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v \rangle \qquad \forall v \in H_0^1(\omega_{\mathbf{a}})$$

- energy equality:

$$\|\nabla r^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 = (\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}} \rangle = \|\mathcal{R}\|_{H^{-1}(\omega_{\mathbf{a}})}^2$$

- setting $r := \sum_{\mathbf{a}\in\mathcal{V}} r^{\mathbf{a}} \in H_0^1(\Omega)$:

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- overlapping of the patches:

$$\|\nabla r\|^2 \le (d+1) \sum_{\mathbf{a}\in\mathcal{V}} \|\nabla r^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2$$

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$$\sum_{\boldsymbol{a} \in \mathcal{V}} \|\mathcal{R}\|^2_{H^{-1}(\omega_{\boldsymbol{a}})} = \sum_{\boldsymbol{a} \in \mathcal{V}} \langle \mathcal{R}, r^{\boldsymbol{a}} \rangle = \langle \mathcal{R}, r \rangle \leq \|\mathcal{R}\|_{H^{-1}(\Omega)} \|\nabla r\|$$

- overlapping of the patches:

$$\|\nabla r\|^2 \leq (d+1) \sum_{\boldsymbol{a} \in \mathcal{V}} \|\nabla r^{\boldsymbol{a}}\|^2_{\omega_{\boldsymbol{a}}}$$

Proof ($\gtrsim$): local Laplacian liftings

- Laplacian lifting of $\mathcal{R}$ on each patch $\omega_{\mathbf{a}}$: $r^{\mathbf{a}} \in H_0^1(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v \rangle \qquad \forall v \in H_0^1(\omega_{\mathbf{a}})$$

- energy equality:

$$\|\nabla r^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}^2 = (\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}} \rangle = \|\mathcal{R}\|_{H^{-1}(\omega_{\mathbf{a}})}^2$$

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Localization of distances to $H^1_0(\Omega)$ (zero mean jumps)

Theorem (Localization of distance to $H^1_0(\Omega)$, Ciarlet Jr. & V. (2018))

Let $v \in H^1(\mathcal{T})$ with $\langle [\![v]\!], 1 \rangle_F = 0$ for all $F \in \mathcal{F}$ be arbitrary. Then

$$\underbrace{\min_{\zeta \in H^1_0(\Omega)} \|\nabla_h (v - \zeta)\|^2}_{\text{global distance to } H^1_0(\Omega)} \approx \sum_{\mathbf{a} \in \mathcal{V}} \underbrace{\min_{\zeta \in H^1_\#(\omega_{\mathbf{a}})} \|\nabla_h (v - \zeta)\|^2_{\omega_{\mathbf{a}}}}_{\text{local distance to } H^1_\#(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\text{int}} \text{ and } H^1_{\partial\Omega}(\omega_{\mathbf{a}}) \text{ for } \mathbf{a} \in \mathcal{V}^{\text{ext}}}.$$

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Localization of distances to $H_0^1(\Omega)$ (zero mean jumps)

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Proof ($\lesssim$)

- define $s \in H^1_0(\Omega)$ by

$$s := \sum_{\mathbf{a} \in \mathcal{V}} s^{\mathbf{a}} \qquad s^{\mathbf{a}} := \arg \min_{\zeta \in H^1_0(\omega_{\mathbf{a}})} \|\nabla_h(\psi_{\mathbf{a}} v - \zeta)\|_{\omega_{\mathbf{a}}},$$

- minimum, partition of unity:

$$\min_{\zeta \in H^1_0(\Omega)} \|\nabla_h(v-\zeta)\|^2 \leq \|\nabla_h(v-s)\|^2$$
$$\leq (d+1) \sum_{\mathbf{a} \in \mathcal{V}} \|\nabla_h(\psi_{\mathbf{a}} v - s^{\mathbf{a}})\|^2_{\omega_{\mathbf{a}}}$$

- $\psi_{\mathbf{a}}\zeta \in H^1_0(\omega_{\mathbf{a}})$ for any $\zeta \in H^1_\#(\omega_{\mathbf{a}})$, definition of $s^{\mathbf{a}}$:

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$$\inf_{\zeta \in H^1_\#(\omega_{\mathbf{a}})} \|\nabla_h(\psi_{\mathbf{a}}(v-\zeta))\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{cont,bPF}} \min_{\zeta \in H^1_\#(\omega_{\mathbf{a}})} \|\nabla_h(v-\zeta)\|_{\omega_{\mathbf{a}}}$$

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Theorem (Localization of distance to $H^1_0(\Omega)$, Ciarlet Jr. & V. (2018))

Let $v \in H^1(\mathcal{T})$ be arbitrary. Then

$$\underbrace{\min_{\zeta \in H^1_0(\Omega)} \|\nabla_\theta (v-\zeta)\|^2 + \sum_{F\in\mathcal{F}} h_F^{-1} \|\Pi_F^0 [\![v]\!]\|_F^2}_{\text{global distance to } H^1_0(\Omega)}$$

$$\approx \sum_{\mathbf{a}\in\mathcal{V}} \Bigg\{ \underbrace{\min_{\zeta \in H^1_\#(\omega_{\mathbf{a}})} \|\nabla_\theta (v-\zeta)\|^2_{\omega_{\mathbf{a}}} + \sum_{F\in\mathcal{F},\, \mathbf{a}\in F} h_F^{-1} \|\Pi_F^0 [\![v]\!]\|_F^2}_{\text{local distance to } H^1_\#(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}) \text{ for } \mathbf{a}\in\mathcal{V}^{\rm int} \text{ and } H^1_{\partial\Omega}(\omega_{\mathbf{a}}) \text{ for } \mathbf{a}\in\mathcal{V}^{\rm ext}} \Bigg\},$$

where, for $\theta \in \{-1, 0, 1\}$,

$$\underbrace{\nabla_\theta v}_{\text{discrete gradient}} := \nabla_h v - \theta \sum_{F\in\mathcal{F}} \underbrace{\iota_F([\![v]\!])}_{\text{lifting of the jumps}}.$$

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Outline

A transmission problem with sign-changing coefficients

Model problem

$$-\nabla\cdot(\underline{\boldsymbol{\Sigma}}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\underline{\boldsymbol{\Sigma}}$ **not positive definite** (and symmetric)
- example: $\Omega = \Omega_+ \cup \Omega_-$, $\sigma_+ > 0$ and $\sigma_- < 0$,

$$\underline{\boldsymbol{\Sigma}}|_{\Omega_+} = \sigma_+ \underline{\boldsymbol{I}}, \quad \underline{\boldsymbol{\Sigma}}|_{\Omega_-} = \sigma_- \underline{\boldsymbol{I}}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\underline{\boldsymbol{\Sigma}}\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

- well-posed (T-coercivity) Bonnet-Ben Dhia, Chesnel, Ciarlet Jr. (2012), numerical discretization following e.g. Chesnel and Ciarlet Jr. (2013)

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Intrinsic norm and its localization

Energy norm $\|v\|_{\mathrm{en}}^2 := (\underline{\Sigma}\nabla v, \nabla v)$ for $v \in H_0^1(\Omega)$

- not well-defined: $(\underline{\Sigma}\nabla v, \nabla v) < 0$ may happen

Broken H^1 seminorm when $\underline{\Sigma} = \underline{I}$, $v \in H^1(\mathcal{T})$

$$\|\nabla_\theta v\|^2 = \underbrace{\max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla_\theta v, \nabla\varphi)^2}_{\text{dual norm}} + \underbrace{\min_{\zeta \in H_0^1(\Omega)} \|\nabla_\theta (v - \zeta)\|^2}_{\text{distance to } H_0^1(\Omega)}$$

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$$|||u-u_h|||^2 = \max_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} \overbrace{(\underline{\Sigma}\nabla_\theta(u-u_h), \nabla\varphi)^2}^{(\mathcal{R},\varphi)^2} + \min_{\zeta \in H_0^1(\Omega)} \|\nabla_\theta(u_h-\zeta)\|^2$$

$$+ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_F^0 \underbrace{[\![u-u_h]\!]}_{[\![-u_h]\!]}\|_F^2 \qquad u_h \in H^1(\mathcal{T})$$

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Localized $\underline{\Sigma}$- and p-robust a posteriori error estimates

Theorem ($\underline{\Sigma}$- and p-robust a posteriori estimate Ciarlet Jr. & V. (2018))

- *Let* $\underline{\Sigma} \in [\mathbb{P}_0(\mathcal{T})]^{d\times d}$ *and* $f \in \mathbb{P}_{p-1}(\mathcal{T})$, $p \geq 1$, *for simplicity;*
- *let* $u \in H^1_0(\Omega)$ *be the weak solution;*
- $u_h \in \mathbb{P}_p(\mathcal{T})$ *be arbitrary subject to*

$$(\underline{\Sigma}\nabla_\theta u_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = (f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}^{\rm int};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H^1_0(\Omega)$ potential reconstruction;
- $\xi_h := -\underline{\Sigma}\nabla_\theta u_h, f$: $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ flux reconstruction.

Then, $\underline{\Sigma}$*- and* p*-robust* **localized** *equivalence holds:*

$$|||u - u_h|||^2 \leq \sum_{K\in\mathcal{T}} \left[\|\underline{\Sigma}\nabla_\theta u_h + \boldsymbol{\sigma}_h\|_K^2 + \|\nabla_\theta(u_h - s_h)\|_K^2\right] + \sum_{F\in\mathcal{F}} h_F^{-1}\|\Pi_F^0[\![u_h]\!]\|_F^2.$$

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Localized $\underline{\Sigma}$- and p-robust a posteriori error estimates

Theorem ($\underline{\Sigma}$- and p-robust a posteriori estimate Ciarlet Jr. & V. (2018))

- *Let* $\underline{\Sigma} \in [\mathbb{P}_0(\mathcal{T})]^{d\times d}$ *and* $f \in \mathbb{P}_{p-1}(\mathcal{T})$, $p \geq 1$, *for simplicity;*
- *let* $u \in H^1_0(\Omega)$ *be the weak solution;*
- $u_h \in \mathbb{P}_p(\mathcal{T})$ *be arbitrary subject to*

$$(\underline{\Sigma}\nabla_\theta u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H^1_0(\Omega)$ ‣ potential reconstruction;
- $\xi_h := -\underline{\Sigma}\nabla_\theta u_h, f$: $\sigma_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\mathrm{div}, \Omega)$ ‣ flux reconstruction.

Then, $\underline{\Sigma}$*- and* p*-***robust localized** *equivalence holds:*

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$$\|\underline{\Sigma}\nabla_\theta u_h + \boldsymbol{\sigma}_h\|^2_K + \|\nabla_\theta(u_h - s_h)\|^2_K \lesssim \sum_{\boldsymbol{a}\in\mathcal{V}_K} |||u-u_h|||^2_{\omega_{\boldsymbol{a}}} \quad \forall K \in \mathcal{T}.$$

Localized $\underline{\Sigma}$- and p-robust a posteriori error estimates

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Applications

Unified framework for all classical discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ mixed finite elements
- ✓ various finite volumes

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Transmission problem: regular solution

Data

- $\Omega := (-1, 1) \times (-1, 1)$
- $\Omega_+ := (0, 1) \times (-1, 1)$, $\Omega_- := (-1, 0) \times (-1, 1)$
- $\sigma_+ = 1$, $\sigma_- < 0$

Exact solution

$$u(x, y) = \sigma_- x(x+1)(x-1)(y+1)(y-1) \text{ for } (x, y) \in \Omega_+,$$
$$u(x, y) = x(x+1)(x-1)(y+1)(y-1) \text{ for } (x, y) \in \Omega_-$$

Discretization

- conforming finite elements with $p = 1$: $u_h \in H_0^1(\Omega)$
- unstructured triangular grids
- uniform h refinement
- effectivity index $= \eta / |||u - u_h|||$

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- unstructured triangular grids
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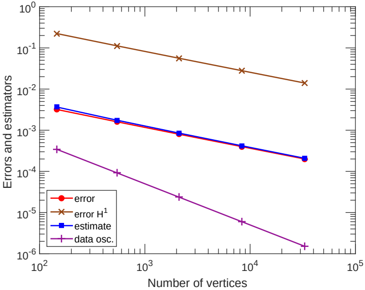
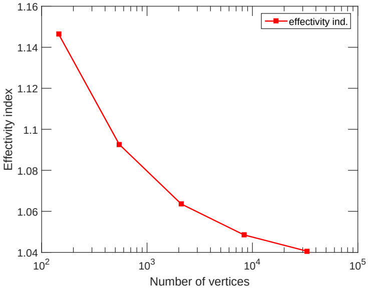
- effectivity index = $\eta / |||u - u_h|||$

Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -0.01$



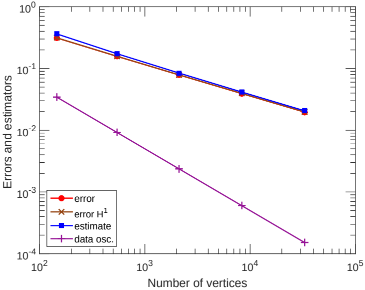
Error $|||u - u_h|||$ and estimate



Effectivity index

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -0.99$



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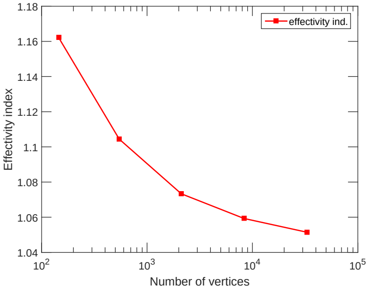
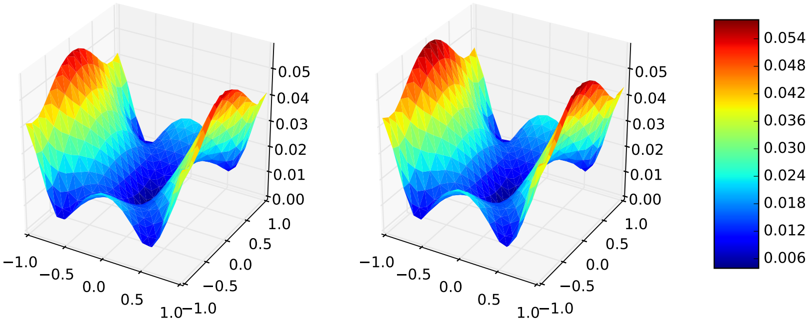


Effectivity index

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

Error localization: exact and prediction: $\sigma_- = -1/3$



Exact (left) and estimated (right) error distribution

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

Transmission problem: singular solution

Data

- $\Omega := (-1,1) \times (-1,1)$
- $\Omega_+ := (0,1) \times (0,1)$, $\Omega_- := \Omega \setminus \overline{\Omega_+}$
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Exact solution

$$u(x,y) = r^\lambda(c_1 \sin(\lambda\phi) + c_2 \sin(\lambda(\pi/2 - \phi))) \text{ for } (x,y) \in \Omega_+,$$

$$u(x,y) = r^\lambda(d_1 \sin(\lambda(\phi - \pi/2)) + d_2 \sin(\lambda(2\pi - \phi)) \text{ for } (x,y) \in \Omega_-$$

- $u \in H^{1+\lambda}(\Omega)$
- $\sigma_- = -5$: $\lambda \approx 0.4601069123$
- $\sigma_- = -3.1$: $\lambda \approx 0.1391989493$

Discretization

- conforming finite elements with $p = 1$: $u_h \in H^1_0(\Omega)$
- unstructured triangular grids
- adaptive h refinement
- effectivity index $= \eta / |||u - u_h|||$

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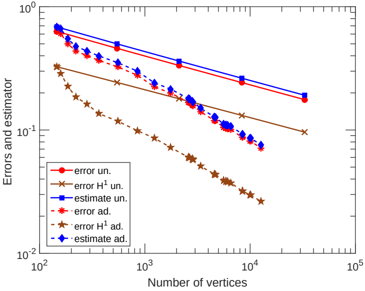
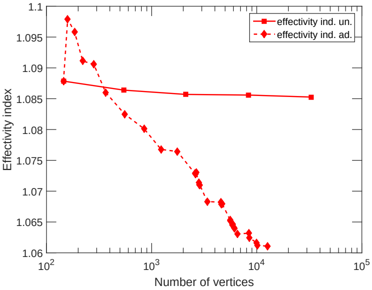
Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -5$



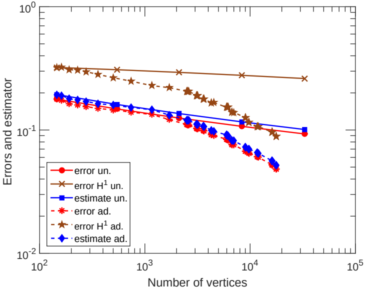
Error $|||u - u_h|||$ and estimate



Effectivity index

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

Robustness with respect to $\underline{\Sigma}$: $\sigma_- = -3.1$



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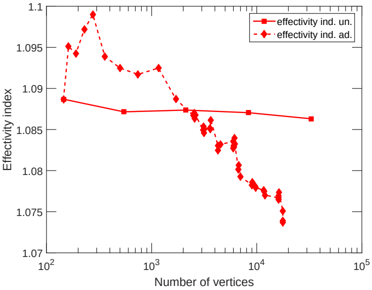
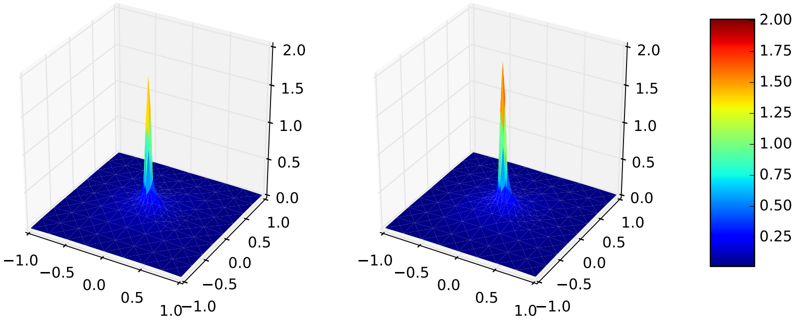


Effectivity index

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

Error localization: exact and prediction: $\sigma_- = -3.1$



Exact (left) and estimated (right) error distribution

P. Ciarlet Jr., M. Vohralík, M2AN. Mathematical Modelling and Numerical Analysis (2018)

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Laplace problem: asymptotic exactness in h and p

h	p	$\eta(u_h)$	rel. estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\text{eff}} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	**1.17**
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	**1.09**
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	**1.06**
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	**1.04**
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	**1.06**
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	**1.04**
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	**1.03**
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	**1.01**
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	**1.02**
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	**1.01**

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

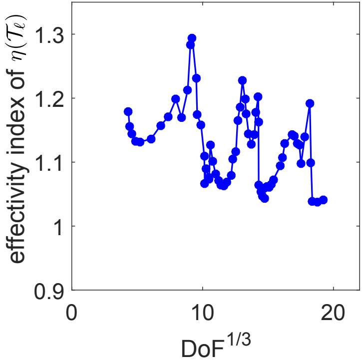
Smooth exact solution

$$u(x, y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty dG method: $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
- uniform h and p refinement

Laplace problem: *hp* refinement



P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Singular exact solution

$$u(r,\phi) = r^{2/3}\sin(2\phi/3)$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- adaptive *hp* refinement

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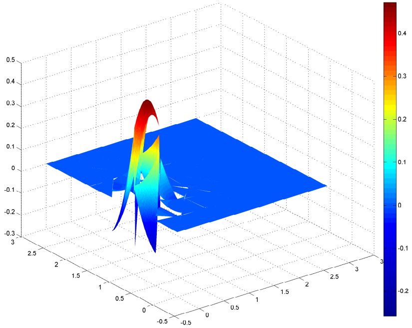
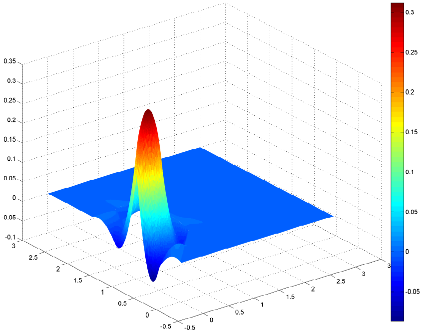
Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \to s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}} \xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}} s_h^{\boldsymbol{a}}.$$

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h (\psi_{\boldsymbol{a}} \xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_{\boldsymbol{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013))

For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **local minimization problem**

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
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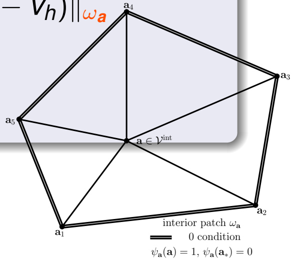

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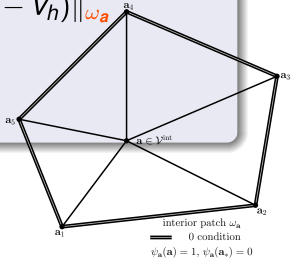

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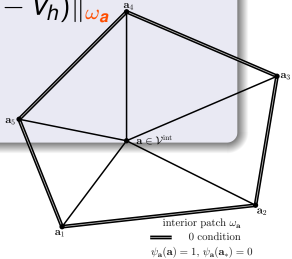

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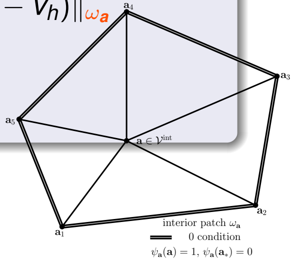

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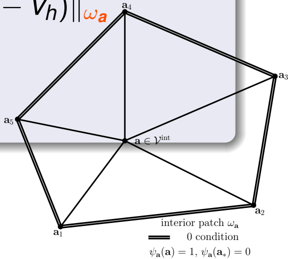

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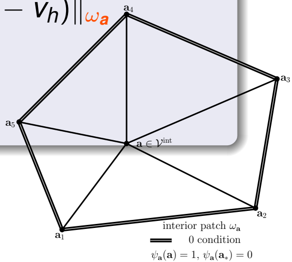

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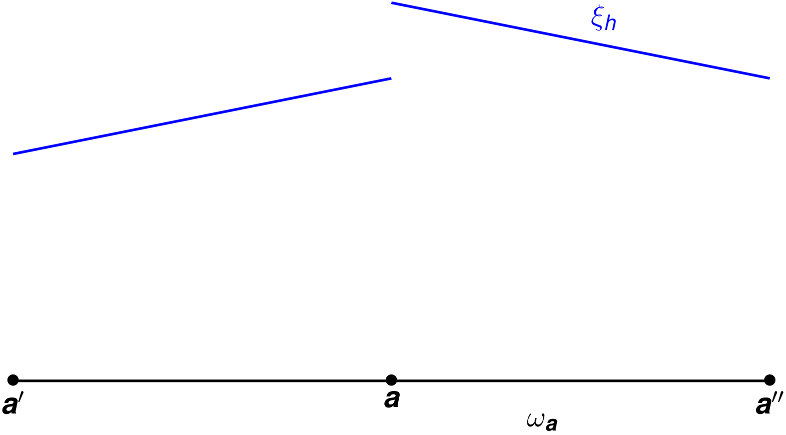
Potential reconstruction in 1D, $p = 1$, $p' = 2$



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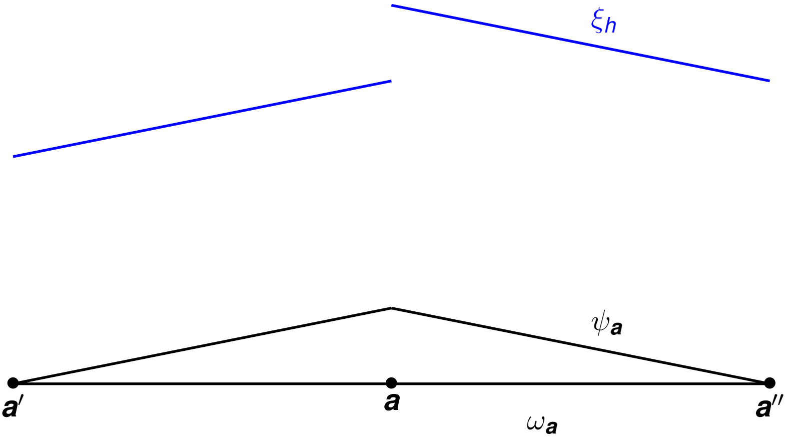

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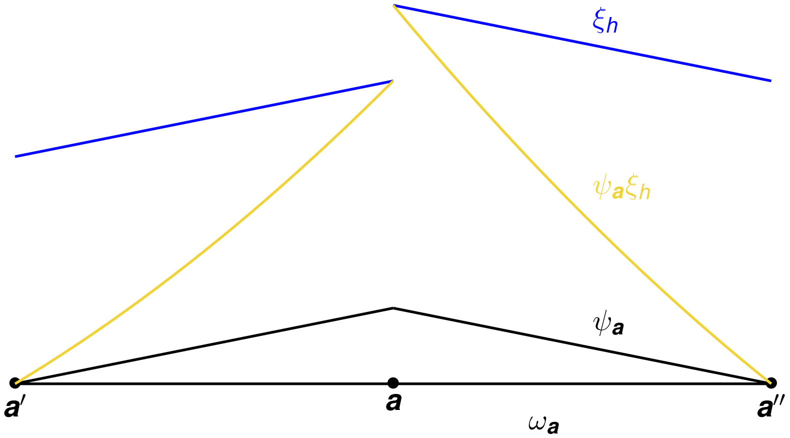

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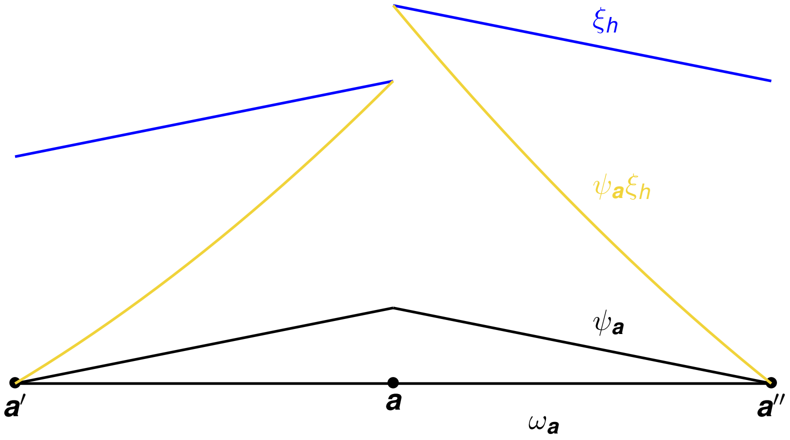
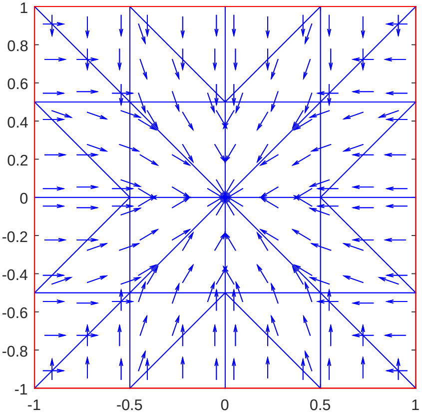
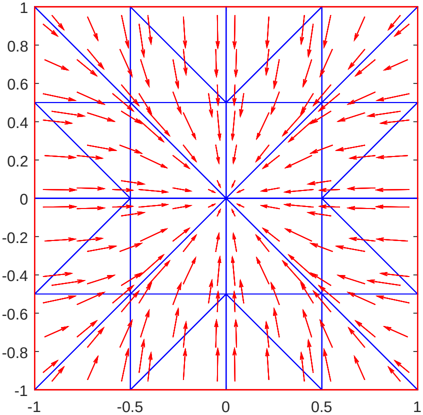


Outline

Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathbf{RTN}_p(\mathcal{T}),\ f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+(\boldsymbol{\xi}_h,\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\ \forall \mathbf{a}\in\mathcal{V}^{\text{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathbf{RTN}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap \mathbf{H}(\text{div},\Omega),\ \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}}.$$

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder & Métivet (1999), Braess & Schöberl (2008))

For each $\mathbf{a} \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}} := \mathbf{RTN}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f\psi_{\mathbf{a}} + \xi_h \cdot \nabla \psi_{\mathbf{a}})}} \| (\psi_{\mathbf{a}} \xi_h) - \mathbf{v}_h \|_{\omega_{\mathbf{a}}}$$

and combine

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}} \boldsymbol{\sigma}_h^{\mathbf{a}}.$$

Key points

- hom. Neumann BC on $\partial \omega_{\mathbf{a}}$: $\boldsymbol{\sigma}_h \in \mathbf{RTN}_{p'}(\mathcal{T}) \cap \mathbf{H}(\mathrm{div}, \Omega)$
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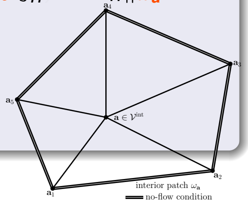

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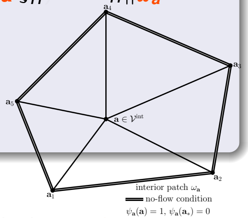

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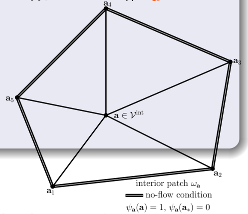

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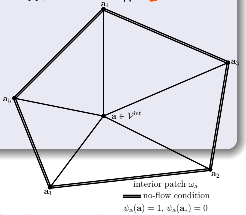

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- **equilibrium** $\nabla \cdot \boldsymbol{\sigma}_h = \sum_{\mathbf{a} \in \mathcal{V}} \nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}} = \sum_{\mathbf{a} \in \mathcal{V}} \Pi_{p'}(f \psi_{\mathbf{a}} + \xi_h \cdot \nabla \psi_{\mathbf{a}}) = \Pi_{p'} f$
- $p' = p + 1$ or $p' = p$

Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}}.$$

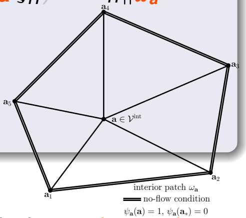
Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder & Métivet (1999), Braess & Schöberl (2008))

For each $\mathbf{a} \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}} := \mathbf{RTN}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_{\mathbf{a}} + \xi_h \cdot \nabla \psi_{\mathbf{a}})}} \| \mathbf{I}_{p'}(\psi_{\mathbf{a}} \xi_h) - \mathbf{v}_h \|_{\omega_{\mathbf{a}}}$$

and combine

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}} \boldsymbol{\sigma}_h^{\mathbf{a}}.$$



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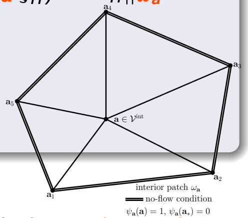

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- hom. Neumann BC on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{\sigma}_h \in \boldsymbol{RTN}_{p'}(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
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Outline

Conclusions and outlook

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- localization of dual and distance norms
- locally efficient a posteriori error estimates
- intrinsic norm for transmission problems: Σ-robustness
- broken polynomial extension operators: p-robustness
- unified framework for all classical numerical schemes

Ongoing work

- extensions to other settings

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- localization of dual and distance norms
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References

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Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials: patch

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2016))

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\rm int}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\rm int})$. Suppose the compatibility

$$r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} = 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int},$$

$$\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e = 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.$$

Then

$$\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext} \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}}) \\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext} \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.$$

Potentials: stability

Theorem (Local stability Ern & V. (2015, 2016), using Tools**)**

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

Potentials: stability

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

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Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. *Let* $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, *satisfying* $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ *if* $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

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Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

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Set $\boldsymbol{\varphi}_K := -\nabla \zeta_K$.

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

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Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: patch

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2016; 3D)**)**

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Fluxes: stability

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern, V. (2016; 3D), using ‣ Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{p'}(f \psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \Pi_{p'}(f \psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

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Corollary (Global stability; $p' = p + 1$)

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

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