Localization of dual norms, local stopping criteria, and fully adaptive solvers

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in collaboration with

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Outline

1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace
2. Localization of dual norms
   - Local–global equivalence
   - Numerical illustration
3. Fully adaptive solvers
   - Setting
   - Guaranteed reliability
   - Local stopping criteria, local efficiency, and robustness
   - Numerical results
4. Conclusions and ongoing work

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Residual and its dual norm for Laplacian

The Laplace problem (polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$)

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ **of** $u_h \in H_0^1(\Omega)$

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \qquad v \in H_0^1(\Omega)$$

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European Research Council

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$$\|\mathcal{R}(u_h)\|_{-1} := \sup_{v \in H_0^1(\Omega),\, \|\nabla v\| = 1} \langle \mathcal{R}(u_h), v \rangle$$

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Remark (Equivalence energy error–dual norm of the residual)

Let $u_h \in H_0^1(\Omega)$. Then

$$\|\mathcal{R}(u_h)\|_{-1} = \|\nabla(u - u_h)\| = \overbrace{\left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(u - u_h)\|_K^2 \right\}^{\frac{1}{2}}}^{\text{localization}}.$$

Outline

Nonlinear Laplacian

Quasi-linear elliptic problem

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\begin{aligned}
-\nabla\cdot\boldsymbol{\sigma}(u, \nabla u) &= f \qquad \text{in } \Omega,\\
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\end{aligned}
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- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$
- example: p-Laplacian with $\boldsymbol{\sigma}(u, \nabla u) = |\nabla u|^{p-2}\nabla u$

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The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{W_0^{1,p}(\Omega)'} \approx \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}(u_h)\|^q_{W_0^{1,p}(\omega_{\mathbf{a}})'} \right\}^{\frac{1}{q}} ?$$

- $\mathcal{V}_h$ vertices, $\omega_{\mathbf{a}}$ patches of elements of a partition $\mathcal{T}_h$ of Ω;
- the constant hidden in $\approx$ must not depend on p, Ω, u_h, the mesh size h, the regularity of u...

How to give tight and robust **computable bounds** on $\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'}$ on each Newton step k and algebraic step i?

How to **steer adaptively** (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to predict **error distribution** = refine at the right place?

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Eisenstat and Walker (1994), Deuflhard (1996), Chaillou and Suri (2006, 2007), Kim (2007)

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Localization dual norms

Setting

- $V := W_0^{1,p}(\Omega)$, $p > 1$, bounded linear functional $\mathcal{R} \in V'$
- localized energy space $V^{\mathbf{a}} := W_0^{1,p}(\omega_{\mathbf{a}})$ for $\mathbf{a} \in \mathcal{V}_h$
- restriction of $\mathcal{R}$ to $(V^{\mathbf{a}})'$ (zero extension of $v \in V^{\mathbf{a}}$),

$$\langle \mathcal{R}, v \rangle_{(V^{\mathbf{a}})', V^{\mathbf{a}}} := \langle \mathcal{R}, v \rangle_{V', V} \qquad v \in V^{\mathbf{a}},$$

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Theorem (Localization of $\|\mathcal{R}\|_{V'}$)

There holds

$$\|\mathcal{R}\|_{V'} \leq (d+1) C_{\mathrm{cont,PF}} \left\{ \frac{1}{(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}} \quad \text{if } \langle \mathcal{R}, \psi_{\mathbf{a}} \rangle = 0 \ \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}},$$

$$\left\{ \frac{1}{(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}} \leq \|\mathcal{R}\|_{V'}.$$

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Localization of the dual residual norm

Upper bound (needs vanishing lowest modes).

- partition of unity, the linearity of $\mathcal{R}$, orthogonality wrt $\psi_{\mathbf{a}}$:

$$\langle \mathcal{R}, v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h^{\rm int}} \langle \mathcal{R}, \psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v)\rangle + \sum_{\mathbf{a}\in\mathcal{V}_h^{\rm ext}} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle$$

- stability (Poincaré–Friedrichs):

$$\|\nabla(\psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v))\|_{p,\omega_{\mathbf{a}}} \le C_{\rm cont,PF} \|\nabla v\|_{p,\omega_{\mathbf{a}}}$$

- Hölder inequality:

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- overlapping of the patches:

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} = \sum_{K\in\mathcal{T}_h} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla v\|^p_{p,K} \le (d+1) \overbrace{\sum_{K\in\mathcal{T}_h} \|\nabla v\|^p_{p,K}}^{\|\nabla v\|^p_p}$$

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Localization of the dual residual norm

Lower bound (unconditioned).

- p-Laplacian lifting of the residual on the patch $\omega_{\mathbf{a}}$:
  $r^{\mathbf{a}} \in V^{\mathbf{a}} = W_0^{1,p}(\omega_{\mathbf{a}})$ such that
  $$(|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v \rangle \qquad \forall v \in V^{\mathbf{a}}$$
- energy equality:
  $$\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = (|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}} \rangle = \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$
- setting $r := \sum_{\mathbf{a}\in\mathcal{V}_h} r^{\mathbf{a}} \in V$:
  $$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, r^{\mathbf{a}} \rangle = \langle \mathcal{R}, r \rangle \le \|\mathcal{R}\|_{V'} \|\nabla r\|_p$$
- overlapping of the patches:
  $$\|\nabla r\|_p^p \le (d+1)^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p$$

Localization of the dual residual norm

Lower bound (unconditioned).

- p-Laplacian lifting of the residual on the patch $\omega_{\mathbf{a}}$:
  $r^{\mathbf{a}} \in V^{\mathbf{a}} = W_0^{1,p}(\omega_{\mathbf{a}})$ such that
  $$(|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v\rangle \qquad \forall v \in V^{\mathbf{a}}$$
- energy equality:
  $$\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = (|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}}\rangle = \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$
- setting $r := \sum_{\mathbf{a}\in\mathcal{V}_h} r^{\mathbf{a}} \in V$:
  $$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, r^{\mathbf{a}}\rangle = \langle \mathcal{R}, r\rangle \le \|\mathcal{R}\|_{V'}\|\nabla r\|_p$$
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  $$\|\nabla r\|_p^p \le (d+1)^{p-1}\sum_{\mathbf{a}\in\mathcal{V}_h}\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p$$

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  $\mathscr{r}^{\mathbf{a}} \in V^{\mathbf{a}} = W_0^{1,p}(\omega_{\mathbf{a}})$ such that

$$(|\nabla \mathscr{r}^{\mathbf{a}}|^{p-2} \nabla \mathscr{r}^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v \rangle \qquad \forall v \in V^{\mathbf{a}}$$

- energy equality:

$$\|\nabla \mathscr{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = (|\nabla \mathscr{r}^{\mathbf{a}}|^{p-2} \nabla \mathscr{r}^{\mathbf{a}}, \nabla \mathscr{r}^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, \mathscr{r}^{\mathbf{a}} \rangle = \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$

- setting $\mathscr{r} := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathscr{r}^{\mathbf{a}} \in V$:

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- overlapping of the patches:

$$\|\nabla \mathscr{r}\|_p^p \leq (d+1)^{p-1} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla \mathscr{r}^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p$$

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$$\|\nabla r\|^p_p \le (d+1)^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}\|^p_{p,\omega_{\mathbf{a}}}$$

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Outline

1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace
2. Localization of dual norms
   - Local–global equivalence
   - Numerical illustration
3. Fully adaptive solvers
   - Setting
   - Guaranteed reliability
   - Local stopping criteria, local efficiency, and robustness
   - Numerical results
4. Conclusions and ongoing work

Numerical results

Model problems

- p-Laplacian

$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \partial\Omega \end{aligned}$$

- $\Omega = (0,1)\times(0,1)$ and, for $p = 1.5$ and 10,

$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- $\Omega = (-1,1)\times(-1,1)\setminus[0,1]\times[-1,0]$ and, for $p = 4$,

$$u(r,\theta) = r^{\frac{7}{8}}\sin(\theta\tfrac{7}{8})$$

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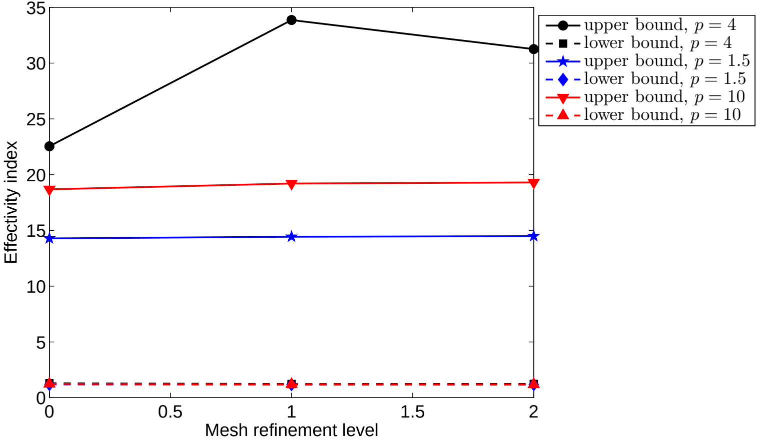
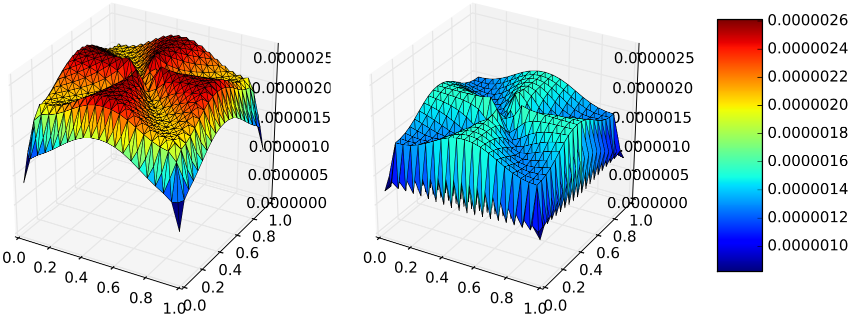
Effectivity indices of the localization bounds



Global and local residual distributions, $p = 1.5$



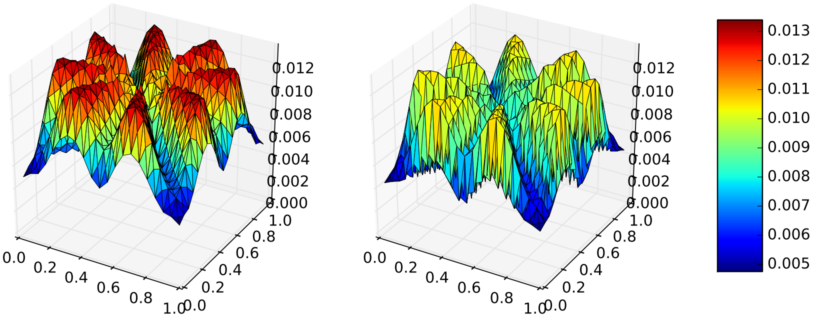
(global error)^q:

$$\|\nabla r\|_p^p = \|\mathcal{R}\|_{V'}^q$$

(localized error)^q:

$$\frac{1}{d+1}\sum_{\mathbf{a}\in\mathcal{V}_h}\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = \frac{1}{d+1}\sum_{\mathbf{a}\in\mathcal{V}_h}\|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$

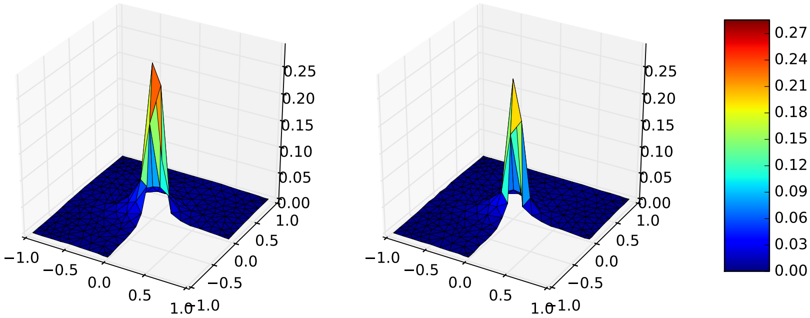
Global and local residual distributions, $p = 10$



Global

Local

Global and local residual distributions, $p = 4$



Global

Local

Outline

Outline

Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step k, algebraic step i
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\} \not\subset V$

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \boldsymbol{\sigma}_h^{k,i} = f_h - \underbrace{\rho_h^{k,i}}_{\text{algebraic remainder}}.$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\boldsymbol{\sigma}_h^{k,i} = \mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

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Outline

Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} + \mathrm{NC} \leq \eta_{\mathrm{disc}}^{k,i} + \underbrace{\eta_{\mathrm{lin}}^{k,i}}_{\|\mathbf{l}_h^{k,i}\|_q} + \underbrace{\eta_{\mathrm{alg}}^{k,i}}_{\|\mathbf{a}_h^{k,i}\|_q} + \underbrace{\eta_{\mathrm{rem}}^{k,i}}_{h_\Omega \|\rho_h^{k,i}\|_q} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}},$$

with $\eta_{\cdot}^{k,i} := \left\{\sum_{K\in\mathcal{T}_h} (\eta_{\cdot,K}^{k,i})^q\right\}^{1/q}$ *and*

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{\frac{1}{p}}\left(\|\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{\sum_{e\in\mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q\right\}^{\frac{1}{q}}\right).$$

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Outline

Assumptions for efficiency

Assumption C (Piecewise polynomials, meshes, quadrature)

The approximation $u_h^{k,i}$ is piecewise polynomial. The meshes $\mathcal{T}_h$ are shape-regular. The quadrature error is negligible.

Assumption D (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} \leq C \left\{ \sum_{K' \in \mathcal{T}_K} h_{K'}^q \|f + \nabla \cdot \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_{q,K'}^q + \sum_{e \in E_K^{\mathrm{int}}} h_e \|[\![\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \cdot \mathbf{n}_e]\!]\|_{q,e}^q + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

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Stopping criteria and efficiency

Local stopping criteria ($\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$)

$$\eta_{\mathrm{rem},K}^{k,i} \le \gamma_{\mathrm{rem},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}, \eta_{\mathrm{alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$

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Theorem (Local efficiency)

Let the Assumptions C and D be satisfied. Let the local stopping criteria hold. Then, for all $K \in \mathcal{T}_h$,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \le C \sum_{\mathbf{a} \in \mathcal{V}_K} \left(\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\omega_{\mathbf{a}})'} + \mathrm{NC}_{\omega_{\mathbf{a}}} \right),$$

where C is independent of σ and q.

- **robustness** with respect to the **nonlinearity**
- local stopping criteria & localizable error measure $\Rightarrow$ **local efficiency**

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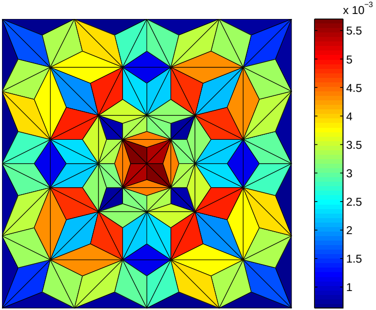
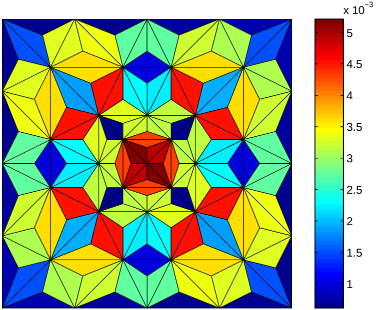
Outline

Error distribution, $p = 10$, Crouzeix–Raviart NCFE

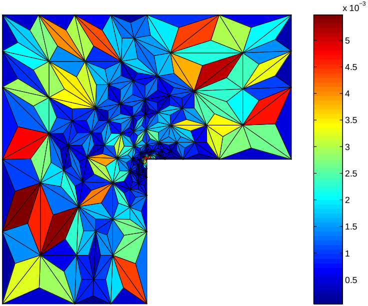


Estimated error distribution

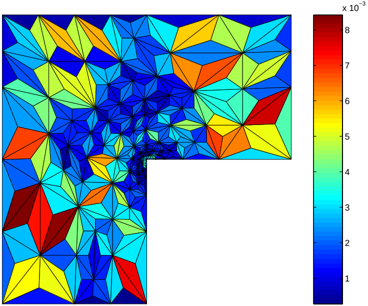


Exact error distribution

Error distribution, adaptively refined mesh, Crouzeix–Raviart NCFE



Estimated error distribution



Exact error distribution

Outline

1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace
2. Localization of dual norms
   - Local–global equivalence
   - Numerical illustration
3. Fully adaptive solvers
   - Setting
   - Guaranteed reliability
   - Local stopping criteria, local efficiency, and robustness
   - Numerical results
4. Conclusions and ongoing work

Conclusions and future directions

Conclusions

- dual residual norms are localizable
- local stopping criteria then lead to local efficiency
- concept of full adaptivity (linear solver, nonlinear solver, mesh)

Ongoing work

- multigrid as a linear solver
- convergence and optimality

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Thank you for your attention!