

Potential and flux reconstructions for optimal *a priori* and *a posteriori* error estimates

Alexandre Ern, Thirupathi Gudi, Iain Smears, **Martin Vohralík**

Inria Paris & Ecole des Ponts

Bad Honnef, July 1st, 2019



Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- analysis of mixed and nonconforming FEs:
 - ESI: interior
 - analysis of conforming FEs:
 - discontinuous pw pol. \approx_p continuous pw pols
- flux reconstruction
 - pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace
 - analysis of nonconforming FEs:

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ~ error
- *a posteriori* analysis of conforming FEs:
est. ~ error
- the local-equilibrated equilibration error estimator
approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ~ error
- *a posteriori* analysis of mixed (and nonconforming) FEs:
est. ~ error
- the local-equilibrated equilibration error estimator
approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error
- *a posteriori* analysis of conforming FEs:
est. ≈ error
- approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error
- *a posteriori* analysis of mixed (and nonconforming) FEs:
est. ≈ error
- approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:

est. ≈ error

- *a priori* analysis of conforming FEs:

approximation continuous pw pols \approx_p discontinuous pw pols

approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:

est. ≈ error

- *a priori* analysis of mixed (and nonconforming) FEs:

approximation continuous pw pols \approx_p discontinuous pw pols

approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. \approx error
- *a priori* analysis of conforming FEs:

balanced-equilibrium approximation

approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. \approx error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of (and nonconforming) FEs:

balanced-equilibrium approximation

approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. \approx error
- *a posteriori* analysis of conforming FEs:

Discontinuous functions → guaranteed equilibrated approximations

approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. \approx error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a posteriori* analysis of (and nonconforming) FEs:

Discontinuous functions → guaranteed equilibrated approximations

approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial \rightarrow continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. \approx error
- *a priori* analysis of conforming FEs:

• higher-order elements: equilibrated fluxes

approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium \rightarrow continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. \approx error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of conforming (and nonconforming) FEs:
• higher-order elements: equilibrated fluxes

approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:

• Discontinuous pw pol. with no equilibrium → no equilibrium

approximation continuous pw pols \approx_p discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)

• *a priori* analysis of mixed (and nonconforming) FEs:

• Discontinuous pw pol. with no equilibrium → no equilibrium

approximation continuous pw pols \approx_p discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:

global-best–local-best equivalence in H^1 Veeser (2016)

approximation continuous pw pols ≈ _{p} discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of mixed (and nonconforming) FEs:

global-best–local-best equivalence in L^2 Veeser (2016)

approximation continuous pw pols ≈ _{p} discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:

global-best–local-best equivalence in H^1 Veeser (2016)

approximation continuous pw pols ≈ _{p} discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of mixed (and nonconforming) FEs:
global-best–local-best equivalence in $H(\operatorname{div})$ Ern, Gudi, Smears, V. (2019)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:

global-best–local-best equivalence in H^1 Veeser (2016)

approximation continuous pw pols ≈ _{p} discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of mixed (and nonconforming) FEs:
global-best–local-best equivalence in $H(\text{div})$ Ern, Gudi, Smears, V. (2019)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:
global-best–local-best equivalence in H^1 Veeser (2016)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of mixed (and nonconforming) FEs:
global-best–local-best equivalence in $H(\text{div})$ Ern, Gudi, Smears, V. (2019)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw pol. ▶ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:
est. ≈ error: guaranteed & p -robust bounds Ern, V. (2015, 2016)
- *a priori* analysis of conforming FEs:
global-best–local-best equivalence in H^1 Veeser (2016)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▶ fl. rec.
- *a posteriori* analysis of conforming FEs:
est. ≈ error: guaranteed & p -robust bounds Braess, Pillwein, Schöberl (2009)
- *a priori* analysis of mixed (and nonconforming) FEs:
global-best–local-best equivalence in $H(\text{div})$ Ern, Gudi, Smears, V. (2019)
approximation continuous pw pols ≈ _{p} discontinuous pw pols

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $a \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^a := \arg \min_{v_h \in V_h^a} \|\nabla_h(\psi_a \xi_h - v_h)\|_{\omega_a}$$

$\psi_a \xi_h$ is the jump of ξ_h at a

Equivalent form: conforming FEs

Find $s_h^a \in V_h^a$ such that

$$(\nabla s_h^a, \nabla v_h)_{\omega_a} = (\nabla_h(\psi_a \xi_h), \nabla v_h)_{\omega_a} \quad \forall v_h \in V_h^a.$$

Key points

- localization to patches T_a
- cut-off by hat basis functions ψ_a
- projection of the discontinuous $\psi_a \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_a$: $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p+1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $a \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^a := \arg \min_{v_h \in V_h^a := \mathbb{P}_{p'}(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_h(v_h - \psi_a \xi_h) - v_h\|_{\omega_a}$$

and combine

$$s_h := \sum_{a \in \mathcal{V}} s_h^a.$$

Equivalent form: conforming FEs

Find $s_h^a \in V_h^a$ such that

$$(\nabla s_h^a, \nabla v_h)_{\omega_a} = (\nabla_h(v_h - \psi_a \xi_h), \nabla v_h)_{\omega_a} \quad \forall v_h \in V_h^a.$$

Key points

- localization to patches \mathcal{T}_a
- cut-off by hat basis functions ψ_a
- projection of the discontinuous $\psi_a \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_a$: $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p+1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

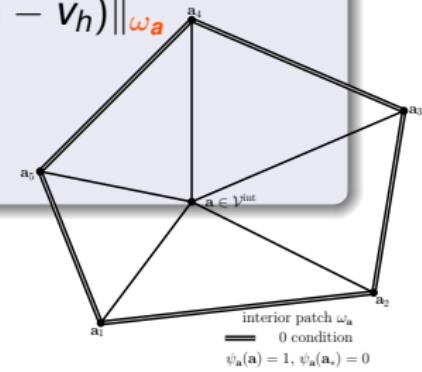
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(\psi_{\mathbf{a}} \xi_h - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



Equivalent form: conforming FEs

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h(\psi_{\mathbf{a}} \xi_h), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\mathbf{a}}$
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- projection of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p+1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

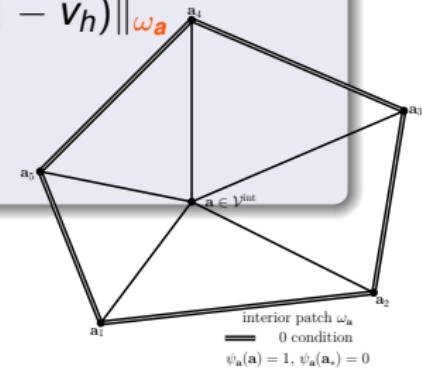
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $a \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^a := \arg \min_{v_h \in V_h^a := \mathbb{P}_{p'}(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_h(v_h - \psi_a \xi_h) - v_h\|_{\omega_a}$$

and combine

$$s_h := \sum_{a \in \mathcal{V}} s_h^a.$$



Equivalent form: **conforming FEs**

Find $s_h^a \in V_h^a$ such that

$$(\nabla s_h^a, \nabla v_h)_{\omega_a} = (\nabla_h(\psi_a \xi_h), \nabla v_h)_{\omega_a} \quad \forall v_h \in V_h^a.$$

Key points

- localization to patches \mathcal{T}_a
- cut-off by hat basis functions ψ_a
- projection of the discontinuous $\psi_a \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_a$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

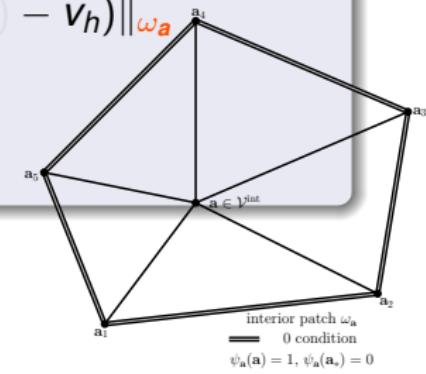
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(\|_p(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



Equivalent form: **conforming FEs**

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h(\|_p(\psi_{\mathbf{a}} \xi_h)), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\mathbf{a}}$
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- projection of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

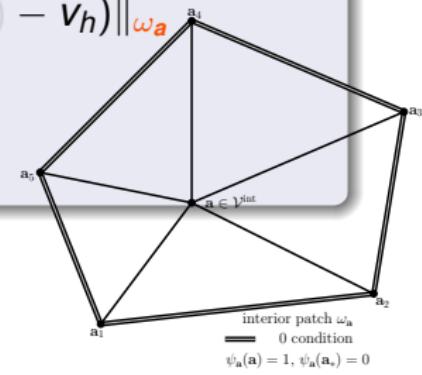
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



Equivalent form: conforming FEs

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h I_{p'}(\psi_{\mathbf{a}} \xi_h), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

Key points

- **localization** to patches $\mathcal{T}_{\mathbf{a}}$
- **cut-off** by hat basis functions $\psi_{\mathbf{a}}$
- **projection** of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$ or $p' = p$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

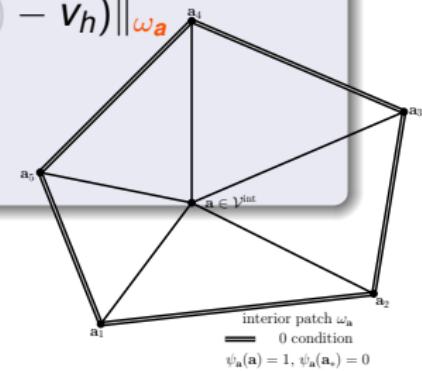
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



Equivalent form: conforming FEs

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h I_{p'}(\psi_{\mathbf{a}} \xi_h), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

Key points

- **localization** to patches $\mathcal{T}_{\mathbf{a}}$
- **cut-off** by hat basis functions $\psi_{\mathbf{a}}$
- **projection** of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conforming space
- homogeneous **Dirichlet BC** on $\partial \omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$ or $p' = p$

Potential reconstruction: datum $\xi_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$

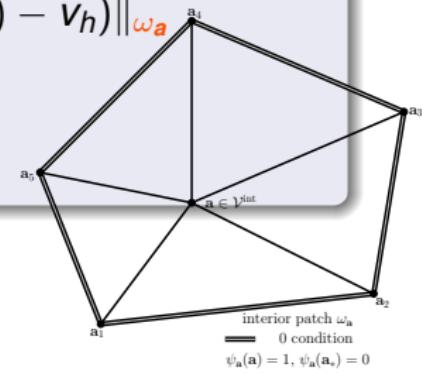
Definition (Construction of s_h EV (2015), \approx Carstensen and Merdon (2013))

For each vertex $\mathbf{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h(I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}}$$

and combine

$$s_h := \sum_{\mathbf{a} \in \mathcal{V}} s_h^{\mathbf{a}}.$$



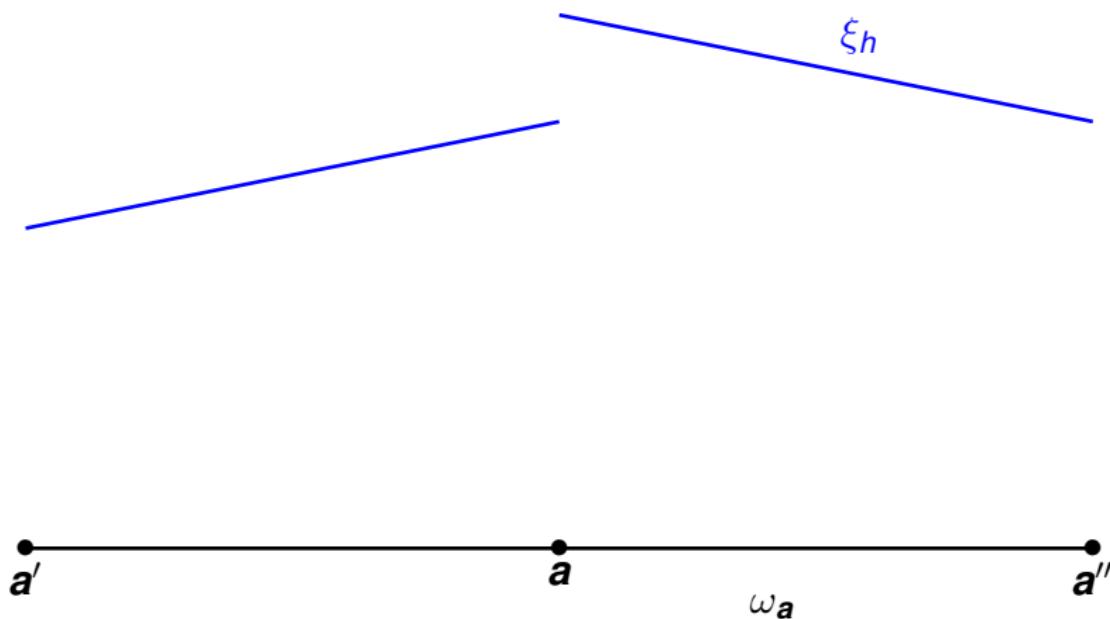
Equivalent form: conforming FEs

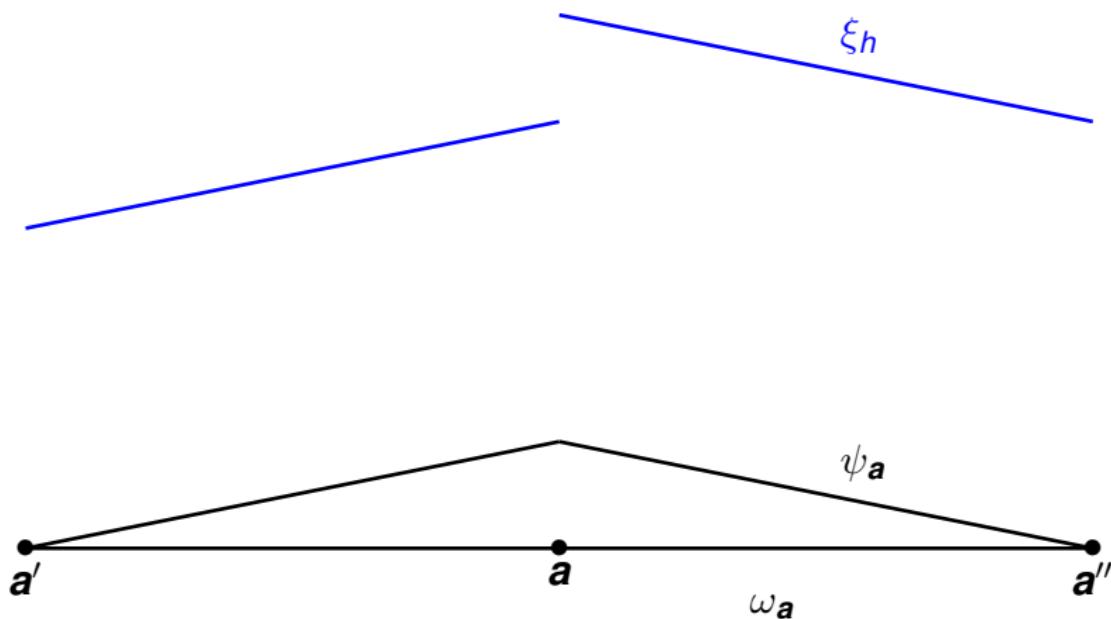
Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

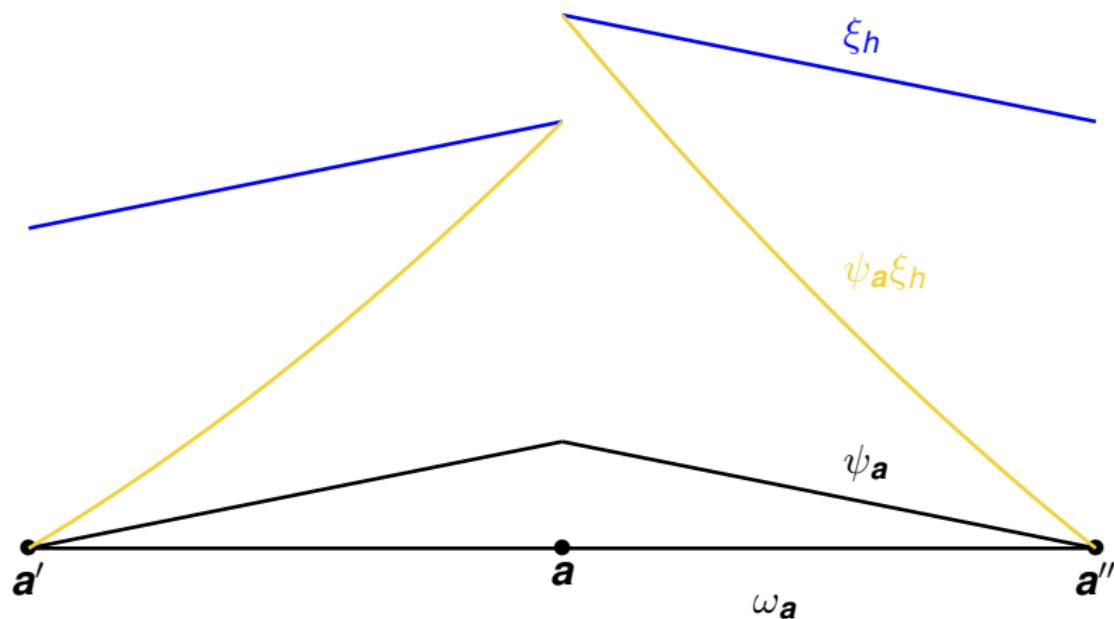
$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla_h I_{p'}(\psi_{\mathbf{a}} \xi_h), \nabla v_h)_{\omega_{\mathbf{a}}} \quad \forall v_h \in V_h^{\mathbf{a}}.$$

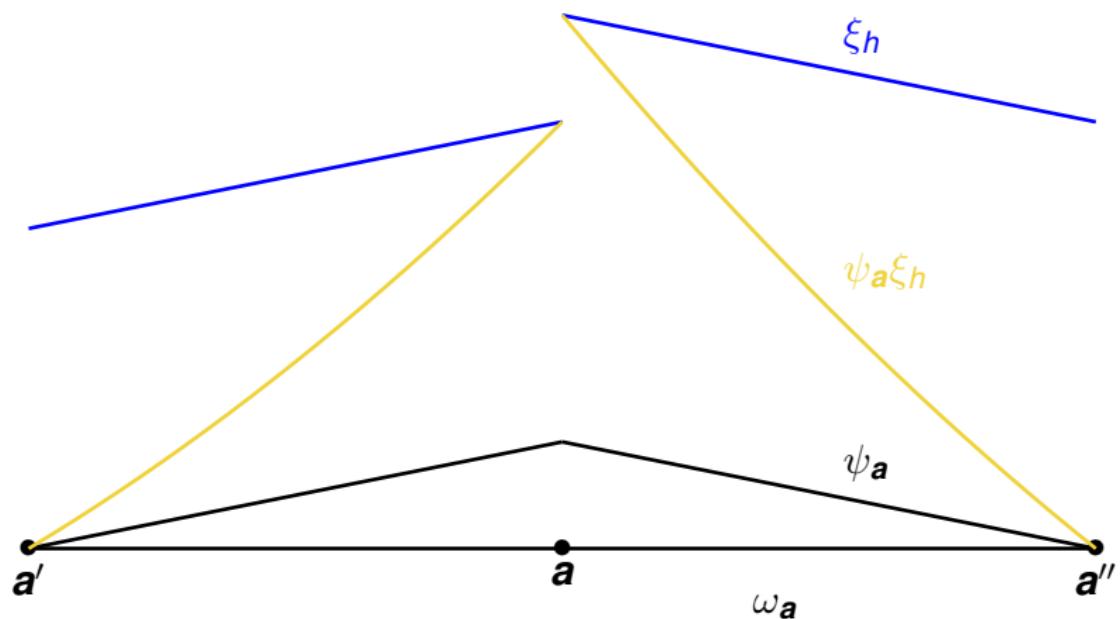
Key points

- **localization** to patches $\mathcal{T}_{\mathbf{a}}$
- **cut-off** by hat basis functions $\psi_{\mathbf{a}}$
- **projection** of the discontinuous $\psi_{\mathbf{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial \omega_{\mathbf{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
- $p' = p + 1$ or $p' = p$

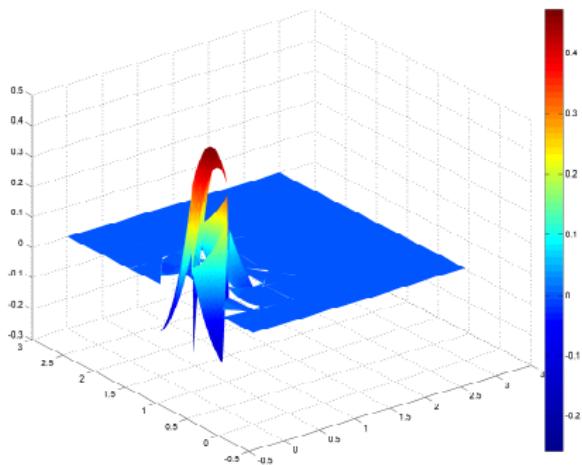
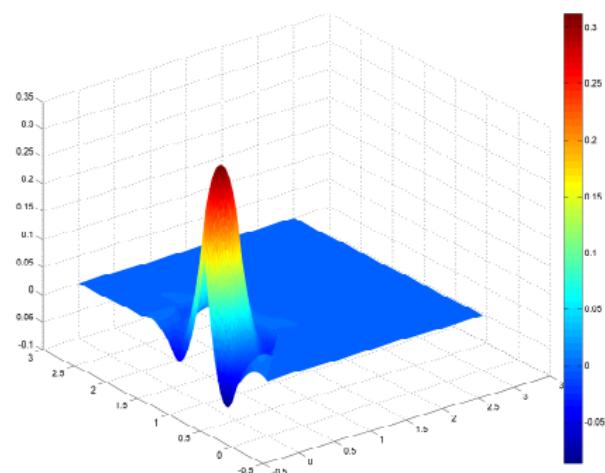
Potential reconstruction in 1D, $p = 1$, $p' = 2$ 

Potential reconstruction in 1D, $p = 1$, $p' = 2$ 

Potential reconstruction in 1D, $p = 1$, $p' = 2$ 

Potential reconstruction in 1D, $p = 1$, $p' = 2$ 

Potential reconstruction

Potential ξ_h Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \rightarrow s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Stability of the potential reconstruction

Theorem (Local stability EV (2015, 2016), using ▶ Tools)

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v_h)\|_{\omega_a} \lesssim \min_{v \in H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v)\|_{\omega_a}.$$

Stability of the potential reconstruction

Theorem (Local stability $\text{EV}(2015, 2016)$, using [Tools](#))

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(T_a) \cap H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v_h)\|_{\omega_a} \lesssim \min_{v \in H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v)\|_{\omega_a}.$$

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\xi_h]\|_F^2 \right\}^{1/2}.$$

Corollary (Global stability; $p' = p$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\xi_h]\|_F^2 \right\}^{1/2}.$$

Stability of the potential reconstruction

Theorem (Local stability $\text{EV}(2015, 2016)$, using [Tools](#))

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_a) \cap H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v_h)\|_{\omega_a} \lesssim \min_{v \in H_0^1(\omega_a)} \|\nabla_h(I_{p'}(\psi_a \xi_h) - v)\|_{\omega_a}.$$

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\xi_h]\|_F^2 \right\}^{1/2}.$$

Corollary (Global stability; $p' = p$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\xi_h]\|_F^2 \right\}^{1/2}.$$



Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the local constrained minimization pb

$$\sigma_h^a := \arg \min_{\substack{v_h \in V_h^a \\ \nabla \cdot v_h = 0}} \| \psi_a \xi_h - v_h \|_{\omega_a}$$

• hom. Dirichlet BC

• hom. Neumann BC

• jump BC

Key points

- hom. Neumann BC on $\partial\omega_\delta$: $\sigma_h \in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega)$

- equilibrium $\nabla \cdot \sigma_h = \sum_{\delta \in \mathcal{V}} \nabla \cdot \sigma_h^\delta = \sum_{\delta \in \mathcal{V}} \Pi_p(f \psi_a + \xi_h \nabla \psi_a) = \Pi_p f$

- $p' = p+1$

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the local constrained minimization pb

$$\sigma_h^a := \arg \min_{\begin{array}{c} v_h \in V_h^a = RTN_p(\mathcal{T}_h) \cap H(\text{div}, \omega_a) \\ \nabla \cdot v_h = f \text{ on } \partial \omega_a \cap \mathcal{E}_h \end{array}} \| \psi_a \xi_h - v_h \|_{\omega_a}$$

• hom. Dirichlet BC on $\partial \omega_a \setminus \mathcal{E}_h$

• hom. Neumann BC on $\partial \omega_a \setminus \mathcal{E}_h$

Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega)$

- equilibrium $\nabla \cdot \sigma_h = \sum_{\partial \mathcal{E} \in \mathcal{E}_h} \nabla \cdot \sigma_h^a = \sum_{\partial \mathcal{E} \in \mathcal{E}_h} \Pi_p(f \psi_a + \xi_h \nabla \psi_a) = \Pi_p f$

- $p' = p+1$

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

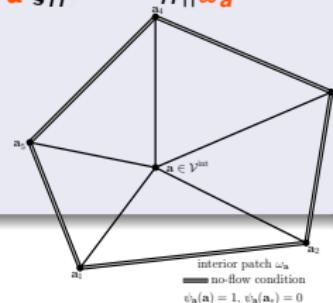
Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\begin{array}{l} \mathbf{v}_h \in \mathbf{V}_h^a := RTN_{p'}(\mathcal{T}_a) \cap H_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \end{array}} \| \mathbf{v}_h - \psi_a \xi_h \|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in RTN_p(\mathcal{T}) \cap H_0(\text{div}, \omega_a)$
- equilibrium $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a - \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = 0$
- $p = p+1$

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

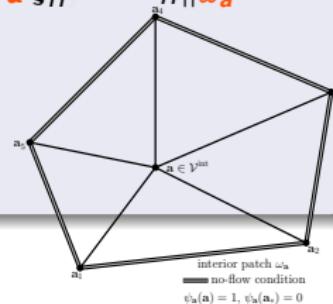
For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^a := RTN_{p'}(\mathcal{T}_a) \cap H_0(\text{div}, \omega_a)} \| \psi_a \xi_h - \mathbf{v}_h \|_{\omega_a}$$

$$\text{subject to } \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a)$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in RTN_{p'}(\mathcal{T}) \cap H(\text{div}, \omega_a)$
- equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = \Pi_{p'} f$
- $p' = p+1$

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

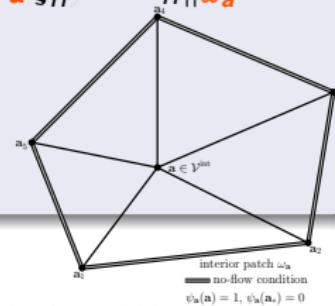
Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\begin{array}{l} \mathbf{v}_h \in \mathbf{V}_h^a := RTN_{p'}(\mathcal{T}_a) \cap H_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \end{array}} \| \mathcal{I}_{p'}(\psi_a \xi_h) - \mathbf{v}_h \|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in RTN_{p'}(\mathcal{T}) \cap H(\text{div}, \Omega)$
- equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = \Pi_{p'} f$
- $p' = p + 1$

Flux reconstruction: $\xi_h \in RTN_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

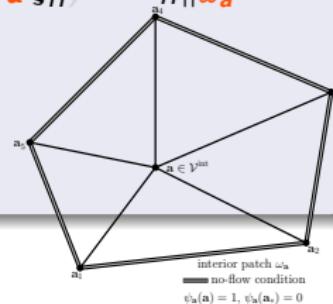
Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\begin{array}{l} \mathbf{v}_h \in \mathbf{V}_h^a := RTN_{p'}(\mathcal{T}_a) \cap H_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \end{array}} \| I_{p'}(\psi_a \xi_h) - \mathbf{v}_h \|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in RTN_{p'}(\mathcal{T}) \cap H(\text{div}, \Omega)$
- equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = \Pi_{p'} f$
- $p' = p + 1$ or $p' = p$

Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

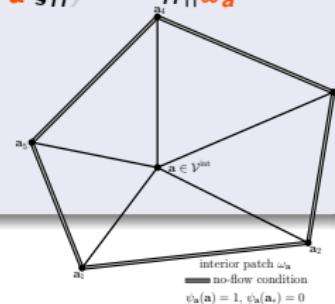
Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\begin{array}{l} \mathbf{v}_h \in \mathbf{V}_h^a := \mathbf{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \end{array}} \| I_{p'}(\psi_a \xi_h) - \mathbf{v}_h \|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in \mathbf{RTN}_{p'}(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$
- **equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = \Pi_{p'} f$
- $p' = p + 1$ or $p' = p$

Flux reconstruction: $\xi_h \in \mathbf{RTN}_p(\mathcal{T})$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}.$$

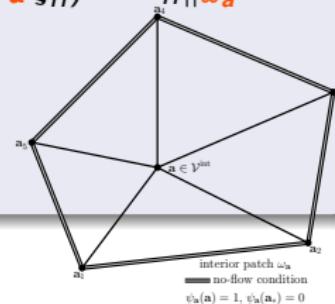
Definition (Constr. of σ_h , Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $a \in \mathcal{V}$, solve the **local constrained minimization pb**

$$\sigma_h^a := \arg \min_{\begin{array}{l} \mathbf{v}_h \in \mathbf{V}_h^a := \mathbf{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \end{array}} \| \mathbf{I}_{p'}(\psi_a \xi_h) - \mathbf{v}_h \|_{\omega_a}$$

and combine

$$\sigma_h := \sum_{a \in \mathcal{V}} \sigma_h^a.$$



Key points

- hom. Neumann BC on $\partial \omega_a$: $\sigma_h \in \mathbf{RTN}_{p'}(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$
- **equilibrium** $\nabla \cdot \sigma_h = \sum_{a \in \mathcal{V}} \nabla \cdot \sigma_h^a = \sum_{a \in \mathcal{V}} \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) = \Pi_{p'} f$
- $p' = p + 1$ or $p' = p$

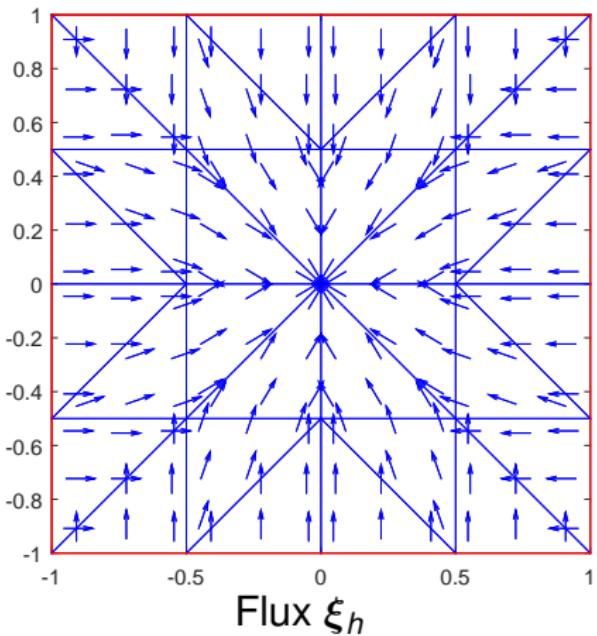
Equilibrated flux reconstruction

Equivalent form: mixed FEs

Find $(\sigma_h^{\mathbf{a}}, \gamma_h^{\mathbf{a}}) \in \mathbf{V}_h^{\mathbf{a}} \times \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}})$ such that

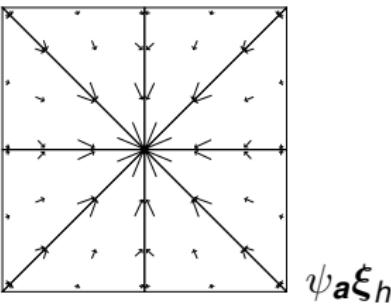
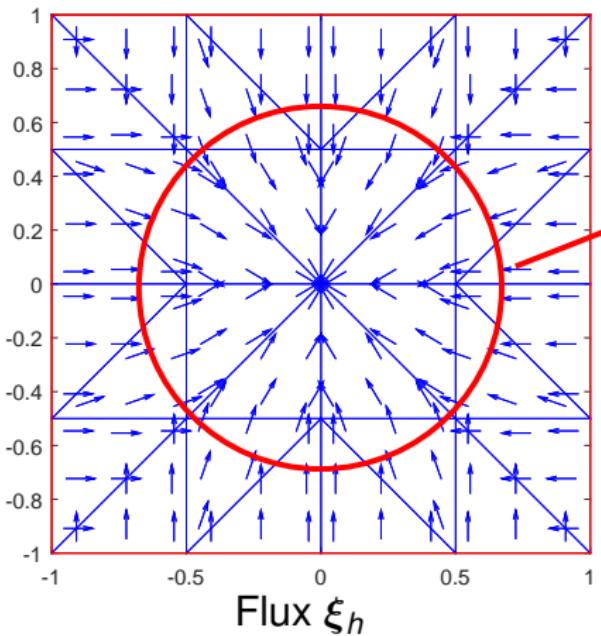
$$\begin{aligned} (\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\gamma_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h)_{\omega_{\mathbf{a}}} &= (\mathbf{I}_{p'}(\psi_{\mathbf{a}} \boldsymbol{\xi}_h), \mathbf{v}_h)_{\omega_{\mathbf{a}}} & \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla \cdot \sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (f \psi_{\mathbf{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} & \forall q_h \in \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \end{aligned}$$

Equilibrated flux reconstruction



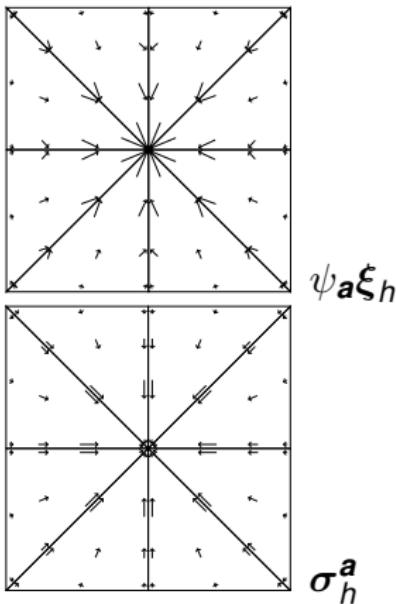
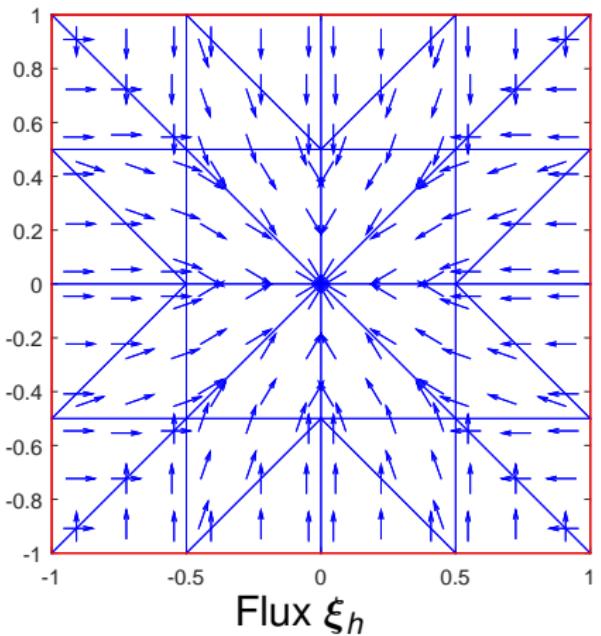
$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



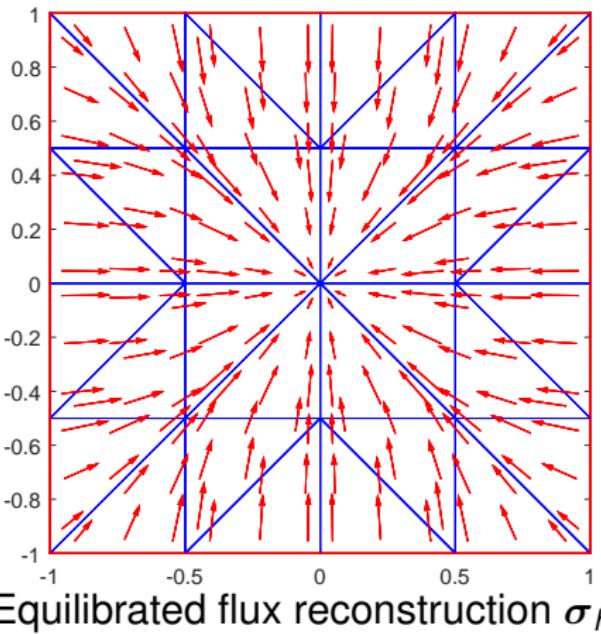
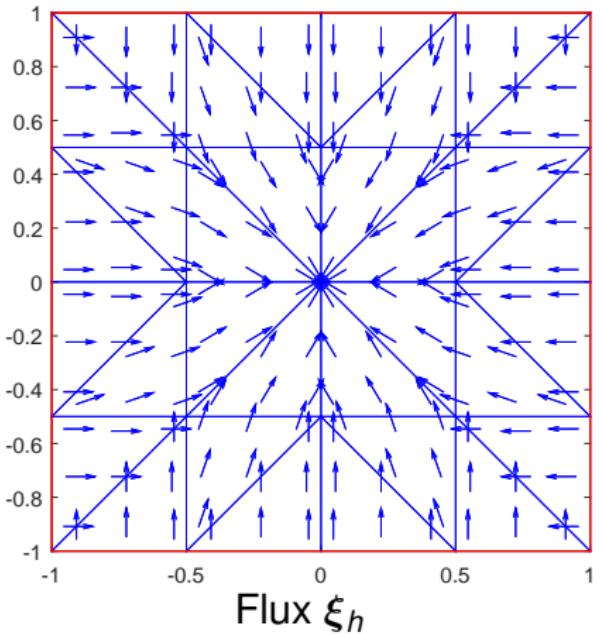
$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



$$\underbrace{\xi_h \in RTN_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}}$$

Equilibrated flux reconstruction



$$\underbrace{\xi_h \in \mathbf{RTN}_p(\mathcal{T}), f \in L^2(\Omega)}_{(f, \psi_a)_{\omega_a} + (\xi_h, \nabla \psi_a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}^{\text{int}}} \rightarrow \sigma_h \in \underbrace{\mathbf{RTN}_{p'}(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)}_{p' = p \text{ or } p' = p+1}, \nabla \cdot \sigma_h = \Pi_{p'} f$$

Stability of the flux reconstruction

Theorem (Local stability) Braess, Pillwein, Schöberl (2009; 2D), EV (2016; 3D), using ▶ Tools)

There holds

$$\min_{\mathbf{v}_h \in \mathbf{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)} \| I_{p'}(\psi_a \xi_h) - \mathbf{v}_h \|_{\omega_a} \lesssim \min_{\mathbf{v} \in \mathbf{H}_0(\text{div}, \omega_a)} \| I_{p'}(\psi_a \xi_h) - \mathbf{v} \|_{\omega_a}.$$

$$\nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a)$$

$$\nabla \cdot \mathbf{v} = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a)$$

Stability of the flux reconstruction

Theorem (Local stability) Braess, Pillwein, Schöberl (2009; 2D), EV (2016; 3D), using ▶ Tools)

There holds

$$\min_{\mathbf{v}_h \in \textcolor{red}{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)} \|I_{p'}(\psi_a \xi_h) - \mathbf{v}_h\|_{\omega_a} \lesssim \min_{\mathbf{v} \in \textcolor{red}{H}_0(\text{div}, \omega_a)} \|I_{p'}(\psi_a \xi_h) - \mathbf{v}\|_{\omega_a}.$$

$$\nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \quad \nabla \cdot \mathbf{v} = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a)$$

Corollary (Global stability; $p' = p + 1$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

Corollary (Global stability; $p' = p$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|f - \nabla \cdot \xi_h\|_K^2 \right\}^{1/2}.$$

Stability of the flux reconstruction

Theorem (Local stability) Braess, Pillwein, Schöberl (2009; 2D), EV (2016; 3D), using ▶ Tools)

There holds

$$\min_{\mathbf{v}_h \in \textcolor{red}{RTN}_{p'}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)} \|\mathbf{I}_{p'}(\psi_a \xi_h) - \mathbf{v}_h\|_{\omega_a} \lesssim \min_{\mathbf{v} \in \textcolor{red}{H}_0(\text{div}, \omega_a)} \|\mathbf{I}_{p'}(\psi_a \xi_h) - \mathbf{v}\|_{\omega_a}.$$

$$\nabla \cdot \mathbf{v}_h = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a) \quad \nabla \cdot \mathbf{v} = \Pi_{p'}(f \psi_a + \xi_h \cdot \nabla \psi_a)$$

Corollary (Global stability; $p' = p + 1$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

Corollary (Global stability; $p' = p$)

σ_h is closer to ξ_h than any $\sigma \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \sigma = f$:

$$\|\xi_h - \sigma_h\| \lesssim_p \|\xi_h - \sigma\| + \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|f - \nabla \cdot \xi_h\|_K^2 \right\}^{1/2}.$$

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $H(\text{div})$
- Stable commuting local projector in $H(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $H(\text{div})$
- Stable commuting local projector in $H(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Aurada, Feischl, Kemetmüller, Page, Praetorius (2013),
Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)} \end{array}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\begin{array}{l} \text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)} \end{array}}.$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p
- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying with $p' = p$, and using its

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Aurada, Feischl, Kemetmüller, Page, Praetorius (2013),
Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)} \end{array}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\begin{array}{l} \text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)} \end{array}}.$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p
- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying with $p' = p$, and using its

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)} \end{array}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\begin{array}{l} \text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)} \end{array}}.$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p
- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying potential reconstruction with $p' = p$, and using its H^1 stability

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)} \end{array}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\begin{array}{l} \text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)} \end{array}}.$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p
- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying potential reconstruction with $p' = p$, and using its H^1 stability

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)} \end{array}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\begin{array}{l} \text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)} \end{array}}.$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T} , and polynomial degree p
- proof taking $\xi_h|_K := \arg \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying potential reconstruction with $p' = p$, and using its H^1 stability

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $\textcolor{red}{u} \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $\textcolor{red}{u}_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized *a priori* error estimate)

$$\underbrace{\|\nabla(u - u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2}$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $\mathbf{u} \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $\mathbf{u}_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized *a priori* error estimate)

From [Brenner et al., 2008], there holds

$$\underbrace{\|\nabla(u - u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2} \leq \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in V_h(K)} \|\nabla(u - v_h)\|_K^2}_{\text{local-best approximation of } u \text{ on each } K \atop \text{no interface constraints}} \leq \gamma^2$$

regularly only in K counts



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $\textcolor{red}{u} \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $\textcolor{red}{u}_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized *a priori* error estimate)

From $\hookrightarrow H_0^1(\Omega)$ global-local, there holds

$$\underbrace{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2}_{\text{local-best approximation of } u \text{ on each } K \text{ no interface constraints regularity only in } K \text{ counts}} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\text{no interface constraints regularity only in } K \text{ counts}} \lesssim h^p.$$



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $\mathbf{u} \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $\mathbf{u}_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized *a priori* error estimate)

From $\hookrightarrow H_0^1(\Omega)$ global – local, there holds

$$\underbrace{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2}_{\text{local-best approximation of } u \text{ on each } K \text{ no interface constraints regularity only in } K \text{ counts}} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\text{no interface constraints regularity only in } K \text{ counts}} \lesssim h^p.$$



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $\mathbf{u} \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $\mathbf{u}_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h$$

Corollary (Localized *a priori* error estimate)

From $\hookrightarrow H_0^1(\Omega)$ global – local, there holds

$$\underbrace{\min_{v_h \in V_h} \|\nabla(u - v_h)\|^2}_{\text{local-best approximation of } u \text{ on each } K \text{ no interface constraints regularity only in } K \text{ counts}} \lesssim_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\text{no interface constraints regularity only in } K \text{ counts}} \lesssim h^p.$$



Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- **Constrained global-best – local-best equivalence in $H(\text{div})$**
- Stable commuting local projector in $H(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

hp interpolation/stable local commuting projectors

hp interpolation estimates

- Demkowicz and Buffa (2005): $\log(p)$ factors
- Bespalov and Heuer (2011): low regularity but still not $H(\text{div})$
- Ern and Guermond (2017): $H(\text{div})$ regularity but not commuting and only optimal in h
- Melenk and Rojik (2019): optimal hp approximation estimates (no $\log(p)$ factors) but higher regularity requested

Stable local commuting projectors defined on $H(\text{div})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Falk and Winther (2014): local but not L^2 -stable
- Ern and Guermond (2016): not local
- Licht (2019): essential boundary conditions on part of $\partial\Omega$

hp interpolation/stable local commuting projectors

hp interpolation estimates

- Demkowicz and Buffa (2005): $\log(p)$ factors
- Bespalov and Heuer (2011): low regularity but still not $H(\text{div})$
- Ern and Guermond (2017): $H(\text{div})$ regularity but not commuting and only optimal in h
- Melenk and Rojik (2019): optimal *hp* approximation estimates (no $\log(p)$ factors) but higher regularity requested

Stable local commuting projectors defined on $H(\text{div})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Falk and Winther (2014): local but not L^2 -stable
- Ern and Guermond (2016): not local
- Licht (2019): essential boundary conditions on part of $\partial\Omega$

Global-best approx. \approx local-best approx. in $\mathbf{H}(\text{div})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\min_{\substack{\mathbf{v}_h \in RTN_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2$$

global-best on Ω

normal trace-continuity constraint

divergence constraint

MFE space (much smaller)

$$\approx p \sum_{K \in \mathcal{T}} \left[\min_{\mathbf{v}_h \in RTN_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right].$$

local-best on each K

no normal trace-continuity constraint

no divergence constraint

broken MFE space (much bigger)

- the right number (a priori) much smaller than the left one
- proof using ~~commuting~~ with $p' = p$ & ~~commuting~~

Global-best approx. \approx local-best approx. in $\mathbf{H}(\text{div})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx p \sum_{K \in \mathcal{T}} \left[\underbrace{\min_{\mathbf{v}_h \in \mathbf{RTN}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2}_{\begin{array}{l} \text{i} \text{ local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)} \end{array}} \right].$$

- the right number (a priori) much smaller than the left one
- proof using flux reconstruction with $p' = p$ & $H(\text{div})$ stability

Global-best approx. \approx local-best approx. in $\mathbf{H}(\text{div})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2}_{\begin{array}{l} \text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)} \end{array}}$$

$$\approx p \sum_{K \in \mathcal{T}} \underbrace{\left[\min_{\mathbf{v}_h \in \mathbf{RTN}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right]}_{\begin{array}{l} \text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)} \end{array}}.$$

- the right number (a priori) much smaller than the left one
- proof using flux reconstruction with $p' = p$ & $H(\text{div})$ stability

Global-best approx. \approx local-best approx. in $H(\text{div})$

Theorem (Constrained equivalence in $H(\text{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in H(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\min_{\substack{\mathbf{v}_h \in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx p \sum_{K \in \mathcal{T}} \left[\underbrace{\min_{\mathbf{v}_h \in RTN_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2}_{\begin{array}{l} \text{i} \text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)} \end{array}} \right].$$

- the right number (a priori) much smaller than the left one
- proof using \rightarrow flux reconstruction with $p' = p$ & $\rightarrow H(\text{div})$ stability

Optimal hp approximation estimate

Theorem (Localized hp approximation, Ern, Gudi, Smears, & V. (2019))

For any $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ s.t., locally on all $K \in \mathcal{T}$,

$$\mathbf{v}|_K \in \mathbf{H}^s(K), \quad s > 0,$$

there holds

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v}\|_K^2 & \text{if } s < 1, \\ \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 & \text{if } s \geq 1. \end{cases}$$

- \lesssim : only depends on d , shape-regularity of \mathcal{T} , and s
- $\mathbf{H}(\text{div})$ stability of flux reconstruction with $p' = p$ & $p' = p + 1$
- contours known (quasi-)interpolates
- fully optimal hp approximation estimate (minimal elementwise regularity, no logarithmic factor in p)

Optimal hp approximation estimate

Theorem (Localized hp approximation, Ern, Gudi, Smears, & V. (2019))

For any $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ s.t., locally on all $K \in \mathcal{T}$,

$$\mathbf{v}|_K \in \mathbf{H}^s(K), \quad s > 0,$$

there holds

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})}} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v}\|_K^2 & \text{if } s < 1, \\ \sum_{K \in \mathcal{T}} \frac{h_K^{2 \min(s, p+1)}}{(p+1)^{2s}} \|\mathbf{v}\|_{\mathbf{H}^s(K)}^2 & \text{if } s \geq 1. \end{cases}$$

- \lesssim : only depends on d , shape-regularity of \mathcal{T} , and s
- $\rightarrow \mathbf{H}(\text{div})$ stability of \rightarrow flux reconstruction with $p' = p$ & $p' = p + 1$
- contours known (quasi-)interpolates
- **fully optimal hp** approximation estimate (minimal elementwise regularity, no logarithmic factor in p)

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) &= 0 & \forall \mathbf{v} \in H(\text{div}, \Omega), \\ (\nabla \cdot \sigma, q) &= (f, q) & \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\sigma_h, u_h) \in V_h := RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\sigma_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in V_h, \\ (\nabla \cdot \sigma_h, q_h) &= (f, q_h) & \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal hp a priori error estimate, Ern, Gudi, Smears, & V. (2019))

$$\|\sigma - \sigma_h\| = \min_{\substack{\mathbf{v}_h \in V_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\sigma - \mathbf{v}_h\|$$



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) &= 0 & \forall \mathbf{v} \in H(\text{div}, \Omega), \\ (\nabla \cdot \sigma, q) &= (f, q) & \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\sigma_h, u_h) \in \mathbf{V}_h := RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\sigma_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \sigma_h, q_h) &= (f, q_h) & \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal hp a priori error estimate, Ern, Gudi, Smears, & V. (2019))

From the above, there holds

$$\|\sigma - \sigma_h\| = \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\sigma - \mathbf{v}_h\| \lesssim \frac{h^{\min(s,p+1)}}{(p+1)^s}.$$



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) &= 0 & \forall \mathbf{v} \in H(\text{div}, \Omega), \\ (\nabla \cdot \sigma, q) &= (f, q) & \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\sigma_h, u_h) \in \mathbf{V}_h := RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\sigma_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \sigma_h, q_h) &= (f, q_h) & \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal hp a priori error estimate, Ern, Gudi, Smears, & V. (2019))

From $\hookrightarrow H(\text{div}, \Omega)$ hp approx., there holds

$$\|\sigma - \sigma_h\| = \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\sigma - \mathbf{v}_h\| \lesssim_s \frac{h^{\min(s, p+1)}}{(p+1)^s}.$$



Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\sigma, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\sigma, \mathbf{v}) - (u, \nabla \cdot \mathbf{v}) &= 0 & \forall \mathbf{v} \in H(\text{div}, \Omega), \\ (\nabla \cdot \sigma, q) &= (f, q) & \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\sigma_h, u_h) \in \mathbf{V}_h := RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\sigma_h, \mathbf{v}_h) - (u_h, \nabla \cdot \mathbf{v}_h) &= 0 & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla \cdot \sigma_h, q_h) &= (f, q_h) & \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal hp a priori error estimate, Ern, Gudi, Smears, & V. (2019))

From $\hookrightarrow H(\text{div}, \Omega)$ hp approx., there holds

$$\|\sigma - \sigma_h\| = \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h \\ \nabla \cdot \mathbf{v}_h = \Pi_p f}} \|\sigma - \mathbf{v}_h\| \lesssim_s \frac{h^{\min(s, p+1)}}{(p+1)^s}.$$



Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Stable local commuting projector in $H(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in H(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \sigma_h$
 $\in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega) =$ flux reconstruction of

$\xi_h|_K := \arg \min_{\mathbf{v}_h \in RTN_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$
with $p' = p$ is locally defined,

$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v})$ commuting,

$P_p \mathbf{v} = \mathbf{v}$ if $\mathbf{v} \in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega)$ projector,

Proof idea: $\mathbf{v} = \mathbf{v}_h + \mathbf{v}_e$ with $\mathbf{v}_h \in RTN_p(K)$, $\nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})$, $\mathbf{v}_e \perp \mathbf{v}_h$ on ∂K

Stable local commuting projector in $\mathbf{H}(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \sigma_h$
 $\in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) =$ flux reconstruction of

$\xi_h|_K := \arg \min_{\mathbf{v}_h \in \mathbf{RTN}_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$
with $p' = p$ is locally defined,

$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v})$ commuting,

$P_p \mathbf{v} = \mathbf{v}$ if $\mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ projector,

$$\|P_p \mathbf{v}\| \lesssim_p \|\mathbf{v}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right\}^{1/2} \text{stable up to osc.}$$

Stable local commuting projector in $\mathbf{H}(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \sigma_h$
 $\in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) =$ flux reconstruction of

$\xi_h|_K := \arg \min_{\mathbf{v}_h \in \mathbf{RTN}_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$
with $p' = p$ is locally defined,

$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v})$ commuting,

$P_p \mathbf{v} = \mathbf{v}$ if $\mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ projector,

$$\|P_p \mathbf{v}\| \lesssim_p \|\mathbf{v}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right\}^{1/2} \text{stable up to osc.}$$

Stable local commuting projector in $\mathbf{H}(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \sigma_h$
 $\in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) =$ flux reconstruction of

$\xi_h|_K := \arg \min_{\mathbf{v}_h \in \mathbf{RTN}_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$
with $p' = p$ is locally defined,

$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v})$ commuting,

$P_p \mathbf{v} = \mathbf{v}$ if $\mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ projector,

$$\|P_p \mathbf{v}\| \lesssim_p \|\mathbf{v}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right\}^{1/2} \text{stable up to osc.}$$

Stable local commuting projector in $\mathbf{H}(\text{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \mathbf{v} := \sigma_h$
 $\in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega) =$ flux reconstruction of

$\xi_h|_K := \arg \min_{\mathbf{v}_h \in \mathbf{RTN}_p(K), \nabla \cdot \mathbf{v}_h = \Pi_p(\nabla \cdot \mathbf{v})} \|\mathbf{v} - \mathbf{v}_h\|_K^2$ for all $K \in \mathcal{T}$
with $p' = p$ is locally defined,

$\nabla \cdot (P_p \mathbf{v}) = \Pi_p(\nabla \cdot \mathbf{v})$ commuting,

$P_p \mathbf{v} = \mathbf{v}$ if $\mathbf{v} \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ projector,

$$\|P_p \mathbf{v}\| \lesssim_p \|\mathbf{v}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K^2 \right\}^{1/2} \text{stable up to osc.}$$

Comments

- P_p defined on the entire $\mathbf{H}(\text{div}, \Omega)$ (no regularity)
- \lesssim_p : only depends on d , shape-regularity of \mathcal{T} , and p
- $h_K \|\nabla \cdot \mathbf{v} - \Pi_p(\nabla \cdot \mathbf{v})\|_K / (p+1)$: data oscillation term,
disappears when $\nabla \cdot \mathbf{v}$ is a pw p -degree polynomial

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate) Prager and Syngel

(1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;

- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_a)_{\omega_a} = (f, \psi_a)_{\omega_a} \quad \forall a \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ potential reconstruction;

- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in RTN_p(\mathcal{T}) \cap H(\text{div}, \Omega)$ stress reconstruction.

Then

$$\|\nabla_h(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2$$

$$+ \sum_{K \in \mathcal{T}} \|\nabla_h(u_h - s_h)\|_K^2.$$

panel constraint

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate) Prager and Syngel

(1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ (potential reconstruction);
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in RTN_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ (flux reconstruction).

Then

$$\begin{aligned} \|\nabla_h(u - u_h)\|^2 &\leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 \\ &\quad + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}. \end{aligned}$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate) Prager and Syngel

(1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_a)_{\omega_a} = (f, \psi_a)_{\omega_a} \quad \forall a \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ ▶ potential reconstruction ;
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in RTN_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ ▶ flux reconstruction .

Then

$$\begin{aligned} \|\nabla_h(u - u_h)\|^2 &\leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 \\ &\quad + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}. \end{aligned}$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate) Prager and Syngel

(1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ ▶ potential reconstruction ;
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ ▶ flux reconstruction .

Then

$$\begin{aligned} \|\nabla_h(u - u_h)\|^2 &\leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 \\ &\quad + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}. \end{aligned}$$

Laplace model problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate) Prager and Syngel

(1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...)

- Let $u \in H_0^1(\Omega)$ be the weak solution;
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, be arbitrary subject to

$$(\nabla_h u_h, \nabla \psi_a)_{\omega_a} = (f, \psi_a)_{\omega_a} \quad \forall a \in \mathcal{V}^{\text{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H_0^1(\Omega)$ ▶ potential reconstruction ;
- $\xi_h := -\nabla_h u_h$, f : $\sigma_h \in RTN_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$ ▶ flux reconstruction .

Then

$$\begin{aligned} \|\nabla_h(u - u_h)\|^2 &\leq \sum_{K \in \mathcal{T}} \left(\underbrace{\|\nabla_h u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \right)^2 \\ &\quad + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}. \end{aligned}$$

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- **Polynomial-degree-robust local efficiency**
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity) Braess, Pillwein, and Schöberl (2009), EV (2015, 2016)

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F[u_h]\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \sigma_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of H^1 stability and $H(\text{div})$ stability with $p' = p + 1$
- p -robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity) Braess, Pillwein, and Schöberl (2009), EV (2015, 2016)

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F[u_h]\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \sigma_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of $\rightarrow H^1$ stability and $\rightarrow H(\text{div})$ stability with $p' = p + 1$
- p -robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Applications

Discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ mixed finite elements

Numerics: smooth case

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(x, y) = \sin(2\pi x) \sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method:
 $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
- uniform h and p refinement

Numerics: smooth case

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(x, y) = \sin(2\pi x) \sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method:
 $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
- uniform h and p refinement

Numerics: smooth case

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(x, y) = \sin(2\pi x) \sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method:
 $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
- uniform h and p refinement

How large is the overall error?

h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$
h_0	1	1.3	$2.8 \times 10^{11}\%$	1.1	$2.4 \times 10^{11}\%$

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$\frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.1
$h_0/2$	2	8.1×10^{-2}	$2.4 \times 10^{-1}\%$	5.5×10^{-2}	$1.2 \times 10^{-1}\%$	1.5×10^0
$h_0/4$	3	3.1×10^{-3}	$2.4 \times 10^{-3}\%$	2.7×10^{-3}	$6.5 \times 10^{-4}\%$	1.1×10^0
$h_0/8$	4	1.5×10^{-4}	$2.4 \times 10^{-4}\%$	1.3×10^{-4}	$3.2 \times 10^{-5}\%$	4.6×10^{-1}
$h_0/16$	5	6.1×10^{-5}	$2.4 \times 10^{-5}\%$	6.5×10^{-5}	$1.6 \times 10^{-6}\%$	3.7×10^{-1}
$h_0/32$	6	2.4×10^{-6}	$2.4 \times 10^{-6}\%$	3.2×10^{-6}	$8.0 \times 10^{-7}\%$	3.0×10^{-1}
$h_0/64$	7	9.6×10^{-7}	$2.4 \times 10^{-7}\%$	1.6×10^{-7}	$4.0 \times 10^{-8}\%$	2.4×10^{-1}
$h_0/128$	8	3.8×10^{-8}	$2.4 \times 10^{-8}\%$	8.0×10^{-8}	$2.0 \times 10^{-9}\%$	1.9×10^{-1}
$h_0/256$	9	1.5×10^{-9}	$2.4 \times 10^{-9}\%$	4.0×10^{-9}	$1.0 \times 10^{-10}\%$	1.5×10^{-1}
$h_0/512$	10	5.8×10^{-10}	$2.4 \times 10^{-10}\%$	2.0×10^{-10}	$5.0 \times 10^{-11}\%$	1.1×10^{-1}
$h_0/1024$	11	2.3×10^{-11}	$2.4 \times 10^{-11}\%$	1.0×10^{-11}	$2.5 \times 10^{-12}\%$	9.2×10^{-2}
$h_0/2048$	12	9.2×10^{-12}	$2.4 \times 10^{-12}\%$	5.0×10^{-12}	$1.2 \times 10^{-13}\%$	7.7×10^{-2}
$h_0/4096$	13	3.6×10^{-13}	$2.4 \times 10^{-13}\%$	2.5×10^{-13}	$6.0 \times 10^{-14}\%$	6.0×10^{-2}
$h_0/8192$	14	1.4×10^{-14}	$2.4 \times 10^{-14}\%$	1.3×10^{-14}	$3.0 \times 10^{-15}\%$	4.7×10^{-2}
$h_0/16384$	15	5.6×10^{-15}	$2.4 \times 10^{-15}\%$	6.5×10^{-15}	$1.5 \times 10^{-16}\%$	3.6×10^{-2}
$h_0/32768$	16	2.2×10^{-16}	$2.4 \times 10^{-16}\%$	3.2×10^{-16}	$7.0 \times 10^{-18}\%$	3.1×10^{-2}
$h_0/65536$	17	8.8×10^{-17}	$2.4 \times 10^{-17}\%$	1.6×10^{-17}	$3.5 \times 10^{-19}\%$	2.5×10^{-2}
$h_0/131072$	18	3.5×10^{-18}	$2.4 \times 10^{-18}\%$	8.0×10^{-18}	$1.7 \times 10^{-20}\%$	2.1×10^{-2}
$h_0/262144$	19	1.4×10^{-19}	$2.4 \times 10^{-19}\%$	4.0×10^{-19}	$8.0 \times 10^{-21}\%$	1.7×10^{-2}
$h_0/524288$	20	5.6×10^{-20}	$2.4 \times 10^{-20}\%$	2.0×10^{-20}	$4.0 \times 10^{-22}\%$	1.4×10^{-2}
$h_0/1048576$	21	2.2×10^{-21}	$2.4 \times 10^{-21}\%$	1.0×10^{-21}	$2.0 \times 10^{-23}\%$	1.1×10^{-2}
$h_0/2097152$	22	8.8×10^{-22}	$2.4 \times 10^{-22}\%$	5.0×10^{-22}	$1.0 \times 10^{-24}\%$	9.0×10^{-3}
$h_0/4194304$	23	3.5×10^{-23}	$2.4 \times 10^{-23}\%$	2.5×10^{-23}	$5.0 \times 10^{-26}\%$	7.0×10^{-3}
$h_0/8388608$	24	1.4×10^{-24}	$2.4 \times 10^{-24}\%$	1.3×10^{-24}	$2.5 \times 10^{-27}\%$	5.6×10^{-3}
$h_0/16777216$	25	5.6×10^{-25}	$2.4 \times 10^{-25}\%$	6.5×10^{-25}	$1.2 \times 10^{-28}\%$	4.5×10^{-3}
$h_0/33554432$	26	2.2×10^{-26}	$2.4 \times 10^{-26}\%$	3.2×10^{-26}	$6.0 \times 10^{-29}\%$	3.6×10^{-3}
$h_0/67108864$	27	8.8×10^{-27}	$2.4 \times 10^{-27}\%$	1.6×10^{-27}	$3.0 \times 10^{-30}\%$	2.8×10^{-3}
$h_0/134217728$	28	3.5×10^{-28}	$2.4 \times 10^{-28}\%$	8.0×10^{-28}	$1.5 \times 10^{-31}\%$	2.2×10^{-3}
$h_0/268435456$	29	1.4×10^{-29}	$2.4 \times 10^{-29}\%$	4.0×10^{-29}	$7.0 \times 10^{-32}\%$	1.7×10^{-3}
$h_0/536870912$	30	5.6×10^{-30}	$2.4 \times 10^{-30}\%$	2.0×10^{-30}	$3.5 \times 10^{-33}\%$	1.3×10^{-3}
$h_0/1073741824$	31	2.2×10^{-31}	$2.4 \times 10^{-31}\%$	1.0×10^{-31}	$1.7 \times 10^{-34}\%$	1.0×10^{-3}
$h_0/2147483648$	32	8.8×10^{-32}	$2.4 \times 10^{-32}\%$	5.0×10^{-32}	$8.0 \times 10^{-35}\%$	7.0×10^{-4}
$h_0/4294967296$	33	3.5×10^{-33}	$2.4 \times 10^{-33}\%$	2.5×10^{-33}	$4.0 \times 10^{-36}\%$	5.6×10^{-4}
$h_0/8589934592$	34	1.4×10^{-34}	$2.4 \times 10^{-34}\%$	1.3×10^{-34}	$2.0 \times 10^{-37}\%$	4.5×10^{-4}
$h_0/17179869184$	35	5.6×10^{-35}	$2.4 \times 10^{-35}\%$	6.5×10^{-35}	$1.0 \times 10^{-38}\%$	3.6×10^{-4}
$h_0/34359738368$	36	2.2×10^{-36}	$2.4 \times 10^{-36}\%$	3.2×10^{-36}	$5.0 \times 10^{-39}\%$	2.8×10^{-4}
$h_0/68719476736$	37	8.8×10^{-37}	$2.4 \times 10^{-37}\%$	1.6×10^{-37}	$2.5 \times 10^{-40}\%$	2.2×10^{-4}
$h_0/137438953472$	38	3.5×10^{-38}	$2.4 \times 10^{-38}\%$	8.0×10^{-38}	$1.2 \times 10^{-41}\%$	1.7×10^{-4}
$h_0/274877906944$	39	1.4×10^{-39}	$2.4 \times 10^{-39}\%$	4.0×10^{-39}	$6.0 \times 10^{-42}\%$	1.3×10^{-4}
$h_0/549755813888$	40	5.6×10^{-40}	$2.4 \times 10^{-40}\%$	2.0×10^{-40}	$3.0 \times 10^{-43}\%$	9.6×10^{-5}
$h_0/1099511627776$	41	2.2×10^{-41}	$2.4 \times 10^{-41}\%$	1.0×10^{-41}	$1.5 \times 10^{-44}\%$	7.2×10^{-5}
$h_0/2199023255552$	42	8.8×10^{-42}	$2.4 \times 10^{-42}\%$	5.0×10^{-42}	$7.0 \times 10^{-45}\%$	5.6×10^{-5}
$h_0/4398046511104$	43	3.5×10^{-43}	$2.4 \times 10^{-43}\%$	2.0×10^{-43}	$3.5 \times 10^{-46}\%$	4.2×10^{-5}
$h_0/8796093022208$	44	1.4×10^{-44}	$2.4 \times 10^{-44}\%$	1.0×10^{-44}	$1.7 \times 10^{-47}\%$	3.0×10^{-5}
$h_0/17592186044416$	45	5.6×10^{-45}	$2.4 \times 10^{-45}\%$	5.0×10^{-45}	$8.0 \times 10^{-48}\%$	2.2×10^{-5}
$h_0/35184372088832$	46	2.2×10^{-46}	$2.4 \times 10^{-46}\%$	2.0×10^{-46}	$4.0 \times 10^{-49}\%$	1.5×10^{-5}
$h_0/70368744177664$	47	8.8×10^{-47}	$2.4 \times 10^{-47}\%$	1.0×10^{-47}	$2.0 \times 10^{-50}\%$	1.0×10^{-5}
$h_0/140737488355328$	48	3.5×10^{-48}	$2.4 \times 10^{-48}\%$	5.0×10^{-48}	$1.0 \times 10^{-51}\%$	6.7×10^{-6}
$h_0/281474976710656$	49	1.4×10^{-49}	$2.4 \times 10^{-49}\%$	2.0×10^{-49}	$5.0 \times 10^{-52}\%$	4.5×10^{-6}
$h_0/562949953421312$	50	5.6×10^{-50}	$2.4 \times 10^{-50}\%$	1.0×10^{-50}	$2.0 \times 10^{-53}\%$	3.0×10^{-6}
$h_0/1125899906842624$	51	2.2×10^{-51}	$2.4 \times 10^{-51}\%$	5.0×10^{-51}	$1.0 \times 10^{-54}\%$	1.7×10^{-6}
$h_0/2251799813685248$	52	8.8×10^{-52}	$2.4 \times 10^{-52}\%$	2.0×10^{-52}	$5.0 \times 10^{-55}\%$	1.1×10^{-6}
$h_0/4503599627370496$	53	3.5×10^{-53}	$2.4 \times 10^{-53}\%$	1.0×10^{-53}	$2.0 \times 10^{-56}\%$	6.7×10^{-7}
$h_0/9007199254740992$	54	1.4×10^{-54}	$2.4 \times 10^{-54}\%$	5.0×10^{-54}	$1.0 \times 10^{-57}\%$	4.5×10^{-7}
$h_0/18014398509481984$	55	5.6×10^{-55}	$2.4 \times 10^{-55}\%$	2.0×10^{-55}	$5.0 \times 10^{-58}\%$	3.0×10^{-7}
$h_0/36028797018963968$	56	2.2×10^{-56}	$2.4 \times 10^{-56}\%$	1.0×10^{-56}	$2.0 \times 10^{-59}\%$	1.7×10^{-7}
$h_0/72057594037927936$	57	8.8×10^{-57}	$2.4 \times 10^{-57}\%$	5.0×10^{-57}	$1.0 \times 10^{-60}\%$	1.1×10^{-7}
$h_0/144115188075855872$	58	3.5×10^{-58}	$2.4 \times 10^{-58}\%$	2.0×10^{-58}	$5.0 \times 10^{-61}\%$	7.0×10^{-8}
$h_0/288230376151711744$	59	1.4×10^{-59}	$2.4 \times 10^{-59}\%$	1.0×10^{-59}	$2.0 \times 10^{-62}\%$	4.5×10^{-8}
$h_0/576460752303423488$	60	5.6×10^{-60}	$2.4 \times 10^{-60}\%$	5.0×10^{-60}	$1.0 \times 10^{-63}\%$	3.0×10^{-8}
$h_0/1152921504606846976$	61	2.2×10^{-61}	$2.4 \times 10^{-61}\%$	2.0×10^{-61}	$5.0 \times 10^{-64}\%$	1.7×10^{-8}
$h_0/2305843009213693952$	62	8.8×10^{-62}	$2.4 \times 10^{-62}\%$	1.0×10^{-62}	$2.0 \times 10^{-65}\%$	1.1×10^{-8}
$h_0/4611686018427387904$	63	3.5×10^{-63}	$2.4 \times 10^{-63}\%$	5.0×10^{-63}	$1.0 \times 10^{-66}\%$	7.0×10^{-9}
$h_0/9223372036854775808$	64	1.4×10^{-64}	$2.4 \times 10^{-64}\%$	2.0×10^{-64}	$5.0 \times 10^{-67}\%$	4.5×10^{-9}
$h_0/18446744073709551616$	65	5.6×10^{-65}	$2.4 \times 10^{-65}\%$	1.0×10^{-65}	$2.0 \times 10^{-68}\%$	3.0×10^{-9}
$h_0/36893488147419103232$	66	2.2×10^{-66}	$2.4 \times 10^{-66}\%$	5.0×10^{-66}	$1.0 \times 10^{-69}\%$	1.7×10^{-9}
$h_0/73786976294838206464$	67	8.8×10^{-67}	$2.4 \times 10^{-67}\%$	2.0×10^{-67}	$5.0 \times 10^{-70}\%$	1.1×10^{-9}
$h_0/147573952589676412928$	68	3.5×10^{-68}	$2.4 \times 10^{-68}\%$	1.0×10^{-68}	$2.0 \times 10^{-71}\%$	7.0×10^{-10}
$h_0/295147905179352825856$	69	1.4×10^{-69}	$2.4 \times 10^{-69}\%$	5.0×10^{-69}	$1.0 \times 10^{-72}\%$	4.5×10^{-10}
$h_0/590295810358705651712$	70	5.6×10^{-70}	$2.4 \times 10^{-70}\%$	2.0×10^{-70}	$5.0 \times 10^{-73}\%$	3.0×10^{-10}
$h_0/118059162071741130344$	71	2.2×10^{-71}	$2.4 \times 10^{-71}\%$	1.0×10^{-71}	$2.0 \times 10^{-74}\%$	1.7×10^{-10}
$h_0/236118324143482260688$	72	8.8×10^{-72}	$2.4 \times 10^{-72}\%$	5.0×10^{-72}	$1.0 \times 10^{-75}\%$	1.1×10^{-10}
$h_0/472236648286964521376$	73	3.5×10^{-73}	$2.4 \times 10^{-73}\%$	2.0×10^{-73}	$5.0 \times 10^{-76}\%$	7.0×10^{-11}
$h_0/944473296573929042752$	74	1.4×10^{-74}	$2.4 \times 10^{-74}\%$	1.0×10^{-74}	$2.0 \times 10^{-77}\%$	4.5×10^{-11}
$h_0/1888946593147858085504$	75	5.6×10^{-75}	$2.4 \times 10^{-75}\%$	5.0×10^{-75}	$1.0 \times 10^{-78}\%$	3.0×10^{-11}
$h_0/3777893186295716171008$	76	2.2×10^{-76}	$2.4 \times 10^{-76}\%$	2.0×10^{-76}	$5.0 \times 10^{-79}\%$	1.7×10^{-11}
$h_0/7555786372591432342016$	77	8.8×10^{-77}	$2.4 \times 10^{-77}\%$	1.0×10^{-77}	$2.0 \times 10^{-80}\%$	1.1×10^{-11}
$h_0/15111572745182664684032$	78	3.5×10^{-78}	$2.4 \times 10^{-78}\%$	5.0×10^{-78}	$1.0 \times 10^{-81}\%$	7.0×10^{-12}
$h_0/30223145490365329368064$	79	1.4×10^{-79}	$2.4 \times 10^{-79}\%$	2.0×10^{-79}	$5.0 \times 10^{-82}\%$	4.5×10^{-12}
$h_0/60446290980730658736128$	80	5.6×10^{-80}	$2.4 \times 10^{-80}\%$	1.0×10^{-80}	$2.0 \times 10^{-83}\%$	3.0×10^{-12}
$h_0/120892581961461317472256$	81	2.2×10^{-81}	$2.4 \times 10^{-81}\%$	5.0×10^{-81}	$1.0 \times 10^{-84}\%$	1.7×10^{-12}
$h_0/241785163922922634944512$	82	8.8×10^{-82}	$2.4 \times 10^{-82}\%$	2.0×10^{-82}	$5.0 \times 10^{-85}\%$	1.1×10^{-12}
$h_0/483570327845845269889024$	83	3.5×10^{-83}	$2.4 \times 10^{-83}\%$	1.0×10^{-83}	$2.0 \times 10^{-86}\%$	7.0×10^{-13}
$h_0/967140655691690539778048$	84	1.4×10^{-84}	$2.4 \times 10^{-84}\%$	5.0×10^{-84}	$1.0 \times 10^{-87}\%$	4.5×10^{-13}
$h_0/1934281311383381079556096$	85	5.6×10^{-85}	$2.4 \times 10^{-85}\%$	2.0×10^{-85}	$5.0 \times 10^{-88}\%$	3.0×10^{-13}
$h_0/3868562622766762159112192$	86	2.2×10^{-86}	2.4			

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$\text{ref} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.1
$h_1/2$	2	8.1×10^{-2}	$1.9 \times 10^{-1}\%$			
$h_1/4$	3	3.1×10^{-3}	$7.4 \times 10^{-4}\%$			
$h_1/8$	4	5×10^{-4}	$1.2 \times 10^{-4}\%$			
$h_1/16$	5	9.5×10^{-5}	$2.4 \times 10^{-5}\%$			
$h_1/32$	6	1.9×10^{-5}	$4.8 \times 10^{-6}\%$			
$h_1/64$	7	3.8×10^{-6}	$9.6 \times 10^{-7}\%$			
$h_1/128$	8	7.5×10^{-7}	$1.9 \times 10^{-7}\%$			
$h_1/256$	9	1.5×10^{-8}	$3.8 \times 10^{-8}\%$			
$h_1/512$	10	3×10^{-9}	$7.6 \times 10^{-9}\%$			
$h_1/1024$	11	6×10^{-10}	$1.5 \times 10^{-10}\%$			
$h_1/2048$	12	1.2×10^{-10}	$3.0 \times 10^{-11}\%$			
$h_1/4096$	13	2.4×10^{-11}	$6.0 \times 10^{-12}\%$			
$h_1/8192$	14	4.8×10^{-12}	$1.2 \times 10^{-12}\%$			
$h_1/16384$	15	9.6×10^{-13}	$2.4 \times 10^{-13}\%$			
$h_1/32768$	16	1.9×10^{-13}	$4.8 \times 10^{-14}\%$			
$h_1/65536$	17	3.8×10^{-14}	$9.6 \times 10^{-15}\%$			
$h_1/131072$	18	7.5×10^{-15}	$1.9 \times 10^{-15}\%$			
$h_1/262144$	19	1.5×10^{-15}	$3.8 \times 10^{-16}\%$			
$h_1/524288$	20	3×10^{-16}	$7.6 \times 10^{-17}\%$			
$h_1/1048576$	21	6×10^{-17}	$1.5 \times 10^{-17}\%$			
$h_1/2097152$	22	1.2×10^{-17}	$3.0 \times 10^{-18}\%$			
$h_1/4194304$	23	2.4×10^{-18}	$6.0 \times 10^{-19}\%$			
$h_1/8388608$	24	4.8×10^{-19}	$1.2 \times 10^{-19}\%$			
$h_1/16777216$	25	9.6×10^{-20}	$2.4 \times 10^{-20}\%$			
$h_1/33554432$	26	1.9×10^{-20}	$4.8 \times 10^{-21}\%$			
$h_1/67108864$	27	3.8×10^{-21}	$9.6 \times 10^{-22}\%$			
$h_1/134217728$	28	7.5×10^{-22}	$1.9 \times 10^{-22}\%$			
$h_1/268435456$	29	1.5×10^{-22}	$3.8 \times 10^{-23}\%$			
$h_1/536870912$	30	3×10^{-23}	$7.6 \times 10^{-24}\%$			
$h_1/1073741824$	31	6×10^{-24}	$1.5 \times 10^{-24}\%$			
$h_1/2147483648$	32	1.2×10^{-24}	$3.0 \times 10^{-25}\%$			
$h_1/4294967296$	33	2.4×10^{-25}	$6.0 \times 10^{-26}\%$			
$h_1/8589934592$	34	4.8×10^{-26}	$1.2 \times 10^{-26}\%$			
$h_1/17179869184$	35	9.6×10^{-27}	$2.4 \times 10^{-27}\%$			
$h_1/34359738368$	36	1.9×10^{-27}	$4.8 \times 10^{-28}\%$			
$h_1/68719476736$	37	3.8×10^{-28}	$9.6 \times 10^{-29}\%$			
$h_1/137438953472$	38	7.5×10^{-29}	$1.9 \times 10^{-29}\%$			
$h_1/274877906944$	39	1.5×10^{-29}	$3.8 \times 10^{-30}\%$			
$h_1/549755813888$	40	3×10^{-30}	$7.6 \times 10^{-31}\%$			
$h_1/1099511627776$	41	6×10^{-31}	$1.5 \times 10^{-31}\%$			
$h_1/2199023255552$	42	1.2×10^{-31}	$3.0 \times 10^{-32}\%$			
$h_1/4398046511104$	43	2.4×10^{-32}	$6.0 \times 10^{-33}\%$			
$h_1/8796093022208$	44	4.8×10^{-33}	$1.2 \times 10^{-33}\%$			
$h_1/17592186044416$	45	9.6×10^{-34}	$2.4 \times 10^{-34}\%$			
$h_1/35184372088832$	46	1.9×10^{-34}	$4.8 \times 10^{-35}\%$			
$h_1/70368744177664$	47	3.8×10^{-35}	$9.6 \times 10^{-36}\%$			
$h_1/140737488355328$	48	7.5×10^{-36}	$1.9 \times 10^{-36}\%$			
$h_1/281474976710656$	49	1.5×10^{-36}	$3.8 \times 10^{-37}\%$			
$h_1/562949953421312$	50	3×10^{-37}	$7.6 \times 10^{-38}\%$			
$h_1/1125899906842624$	51	6×10^{-38}	$1.5 \times 10^{-38}\%$			
$h_1/2251799813685248$	52	1.2×10^{-38}	$3.0 \times 10^{-39}\%$			
$h_1/4503599627370496$	53	2.4×10^{-39}	$6.0 \times 10^{-40}\%$			
$h_1/9007199254740992$	54	4.8×10^{-39}	$1.2 \times 10^{-40}\%$			
$h_1/18014398509481984$	55	9.6×10^{-40}	$2.4 \times 10^{-41}\%$			
$h_1/36028797018963968$	56	1.9×10^{-40}	$4.8 \times 10^{-42}\%$			
$h_1/72057594037927936$	57	3.8×10^{-41}	$9.6 \times 10^{-43}\%$			
$h_1/144115188075855872$	58	7.5×10^{-41}	$1.9 \times 10^{-43}\%$			
$h_1/288230376151711744$	59	1.5×10^{-41}	$3.8 \times 10^{-44}\%$			
$h_1/576460752303423488$	60	3×10^{-42}	$7.6 \times 10^{-45}\%$			
$h_1/1152921504606846976$	61	6×10^{-42}	$1.5 \times 10^{-45}\%$			
$h_1/2305843009213693952$	62	1.2×10^{-42}	$3.0 \times 10^{-46}\%$			
$h_1/4611686018427387904$	63	2.4×10^{-43}	$6.0 \times 10^{-47}\%$			
$h_1/9223372036854775808$	64	4.8×10^{-43}	$1.2 \times 10^{-47}\%$			
$h_1/18446744073709551616$	65	9.6×10^{-44}	$2.4 \times 10^{-48}\%$			
$h_1/36893488147419103232$	66	1.9×10^{-44}	$4.8 \times 10^{-49}\%$			
$h_1/73786976294838206464$	67	3.8×10^{-45}	$9.6 \times 10^{-50}\%$			
$h_1/147573952589676412928$	68	7.5×10^{-45}	$1.9 \times 10^{-50}\%$			
$h_1/295147905179352825856$	69	1.5×10^{-45}	$3.8 \times 10^{-51}\%$			
$h_1/590295810358705651712$	70	3×10^{-46}	$7.6 \times 10^{-52}\%$			
$h_1/1180591620717411303424$	71	6×10^{-46}	$1.5 \times 10^{-52}\%$			
$h_1/2361183241434822606848$	72	1.2×10^{-46}	$3.0 \times 10^{-53}\%$			
$h_1/4722366482869645213696$	73	2.4×10^{-47}	$6.0 \times 10^{-54}\%$			
$h_1/9444732965739290427392$	74	4.8×10^{-47}	$1.2 \times 10^{-54}\%$			
$h_1/18889465931478580854784$	75	9.6×10^{-48}	$2.4 \times 10^{-55}\%$			
$h_1/37778931862957161709568$	76	1.9×10^{-48}	$4.8 \times 10^{-56}\%$			
$h_1/75557863725914323419136$	77	3.8×10^{-49}	$9.6 \times 10^{-57}\%$			
$h_1/151115727451826646838272$	78	7.5×10^{-49}	$1.9 \times 10^{-57}\%$			
$h_1/302231454903653293676544$	79	1.5×10^{-49}	$3.8 \times 10^{-58}\%$			
$h_1/604462909807306587353088$	80	3×10^{-50}	$7.6 \times 10^{-59}\%$			
$h_1/1208925819614613174706176$	81	6×10^{-50}	$1.5 \times 10^{-59}\%$			
$h_1/2417851639229226349412352$	82	1.2×10^{-50}	$3.0 \times 10^{-60}\%$			
$h_1/4835703278458452698824704$	83	2.4×10^{-51}	$6.0 \times 10^{-61}\%$			
$h_1/9671406556916905397649408$	84	4.8×10^{-51}	$1.2 \times 10^{-61}\%$			
$h_1/1934281311383381079529816$	85	9.6×10^{-52}	$2.4 \times 10^{-62}\%$			
$h_1/3868562622766762159059632$	86	1.9×10^{-52}	$4.8 \times 10^{-63}\%$			
$h_1/7737125245533524318119264$	87	3.8×10^{-53}	$9.6 \times 10^{-64}\%$			
$h_1/15474250491067048636238528$	88	7.5×10^{-53}	$1.9 \times 10^{-64}\%$			
$h_1/30948500982134097272477056$	89	1.5×10^{-53}	$3.8 \times 10^{-65}\%$			
$h_1/61897001964268194544954112$	90	3×10^{-54}	$7.6 \times 10^{-66}\%$			
$h_1/12379400392853638908988824$	91	6×10^{-54}	$1.5 \times 10^{-66}\%$			
$h_1/24758800785707277817977648$	92	1.2×10^{-54}	$3.0 \times 10^{-67}\%$			
$h_1/49517601571414555635955296$	93	2.4×10^{-55}	$6.0 \times 10^{-68}\%$			
$h_1/99035203142829071271910592$	94	4.8×10^{-55}	$1.2 \times 10^{-68}\%$			
$h_1/19807040628565814254382184$	95	9.6×10^{-56}	$2.4 \times 10^{-69}\%$			
$h_1/39614081257131628508764368$	96	1.9×10^{-56}	$4.8 \times 10^{-69}\%$			
$h_1/79228162514263257017528736$	97	3.8×10^{-57}	$9.6 \times 10^{-70}\%$			
$h_1/15845632522852651403505752$	98	7.5×10^{-57}	$1.9 \times 10^{-70}\%$			
$h_1/31691265045705302807011504$	99	1.5×10^{-57}	$3.8 \times 10^{-71}\%$			
$h_1/63382530091410605614023008$	100	3×10^{-58}	$7.6 \times 10^{-71}\%$			

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$I^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1\%}$	5.6×10^{-1}	$1.2 \times 10^{1\%}$	
$\approx h_0/4$		3.1×10^{-1}		2.9×10^{-1}		
$\approx h_0/8$		1.5×10^{-1}		1.4×10^{-1}		
$\approx h_0/16$		7.5×10^{-2}		7.5×10^{-2}		
$\approx h_0/32$		3.8×10^{-2}		3.8×10^{-2}		
$\approx h_0/64$		1.9×10^{-2}		1.9×10^{-2}		
$\approx h_0/128$		9.5×10^{-3}		9.5×10^{-3}		
$\approx h_0/256$		4.7×10^{-3}		4.7×10^{-3}		

© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France
© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$I^{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1\%}$	5.6×10^{-1}	$1.3 \times 10^{1\%}$	
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	
$\approx h_0/8$		1.5×10^{-1}	3.7%	1.4×10^{-1}	3.1%	
h_0	2	1.3×10^{-1}	$1.0 \times 10^{1\%}$	1.0×10^{-1}	$1.0 \times 10^{1\%}$	
$\approx h_0/2$		4.2×10^{-2}	$3.1 \times 10^{1\%}$	4.1×10^{-2}	$9.2 \times 10^{-2\%}$	
$\approx h_0/4$		1.4×10^{-2}	$1.0 \times 10^{1\%}$	1.4×10^{-2}	$3.1 \times 10^{-2\%}$	
$\approx h_0/8$		2.6×10^{-3}	$2.0 \times 10^{1\%}$	2.6×10^{-3}	$5.8 \times 10^{-3\%}$	
h_0	3	1.0×10^{-2}	$1.0 \times 10^{1\%}$	9.9×10^{-3}	$9.9 \times 10^{-3\%}$	
$\approx h_0/2$		2.6×10^{-3}	$2.0 \times 10^{1\%}$	2.6×10^{-3}	$5.8 \times 10^{-3\%}$	

© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France
© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1\%}$	5.6×10^{-1}	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.3×10^{-1}	$1.3 \times 10^{1\%}$	1.3×10^{-1}	3.0%	1.08
$\approx h_0/2$		4.2×10^{-2}	$3.1 \times 10^{1\%}$	4.1×10^{-2}	$9.2 \times 10^{-2\%}$	1.08
$\approx h_0/4$		1.4×10^{-2}	$1.3 \times 10^{1\%}$	1.4×10^{-2}	$3.1 \times 10^{-2\%}$	1.08
$\approx h_0/8$		4.6×10^{-3}	$4.6 \times 10^{1\%}$	4.6×10^{-3}	$1.1 \times 10^{-2\%}$	1.08
h_0	3	1.0×10^{-2}	$1.0 \times 10^{1\%}$	9.9×10^{-3}	$2.2 \times 10^{-2\%}$	1.08
$\approx h_0/2$		3.3×10^{-3}	$3.3 \times 10^{1\%}$	3.3×10^{-3}	$8.8 \times 10^{-3\%}$	1.08
$\approx h_0/4$		1.1×10^{-3}	$1.1 \times 10^{1\%}$	1.1×10^{-3}	$3.1 \times 10^{-3\%}$	1.08
$\approx h_0/8$		3.7×10^{-4}	$3.7 \times 10^{1\%}$	3.7×10^{-4}	$1.1 \times 10^{-4\%}$	1.08

© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France
© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1\%}$	5.6×10^{-1}	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1\%}$	4.1×10^{-2}	$9.2 \times 10^{-1\%}$	1.04
$\approx h_0/4$		1.4×10^{-2}	$3.4 \times 10^{-2\%}$	1.4×10^{-2}	$3.3 \times 10^{-2\%}$	1.03
$\approx h_0/8$		2.8×10^{-3}	$6.8 \times 10^{-3\%}$	2.8×10^{-3}	$5.8 \times 10^{-3\%}$	1.01
$\approx h_0/16$		1.0×10^{-3}	$2.1 \times 10^{-3\%}$	9.9×10^{-4}	$2.2 \times 10^{-3\%}$	1.01
$\approx h_0/32$		2.5×10^{-4}	$5.3 \times 10^{-4\%}$	2.5×10^{-4}	$5.8 \times 10^{-4\%}$	1.01

© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France
© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla \mathbf{u}_h\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^{1\%}$	5.6×10^{-1}	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1\%}$	4.1×10^{-2}	$9.2 \times 10^{-1\%}$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1\%}$	1.4×10^{-2}	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3\%}$	2.6×10^{-4}	$5.9 \times 10^{-3\%}$	1.01
h_0	4	1.0×10^{-3}	1.0%	9.9×10^{-4}	$2.2 \times 10^{-3\%}$	1.01
$\approx h_0/8$	4	2.5×10^{-5}	0.6%	2.5×10^{-5}	$5.8 \times 10^{-5\%}$	1.01

© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France
© Institut Élie Cartan de Lorraine, Université de Lorraine, Nancy, France

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla \mathbf{u}_h\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	1.02
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
 V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla \mathbf{u}_h\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	1.02
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla \mathbf{u}_h\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	1.02
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How large is the overall error? (model pb, known sol.)

h	p	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla \mathbf{u}_h\ }$	$J_{\text{eff}} = \frac{\eta(\mathbf{u}_h)}{\ \nabla(u - u_h)\ }$
h_0	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.17
$\approx h_0/2$		6.1×10^{-1}	$1.4 \times 10^1\%$	5.6×10^{-1}	$1.3 \times 10^1\%$	1.09
$\approx h_0/4$		3.1×10^{-1}	7.0%	2.9×10^{-1}	6.6%	1.06
$\approx h_0/8$		1.5×10^{-1}	3.3%	1.4×10^{-1}	3.1%	1.04
h_0	2	1.6×10^{-1}	3.7%	1.5×10^{-1}	3.5%	1.06
$\approx h_0/2$	2	4.2×10^{-2}	$9.5 \times 10^{-1}\%$	4.1×10^{-2}	$9.2 \times 10^{-1}\%$	1.04
h_0	3	1.4×10^{-2}	$3.2 \times 10^{-1}\%$	1.4×10^{-2}	$3.1 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	2.6×10^{-4}	$5.9 \times 10^{-3}\%$	1.01
h_0	4	1.0×10^{-3}	$2.3 \times 10^{-2}\%$	9.9×10^{-4}	$2.2 \times 10^{-2}\%$	1.02
$\approx h_0/8$	4	2.6×10^{-7}	$5.9 \times 10^{-6}\%$	2.6×10^{-7}	$5.8 \times 10^{-6}\%$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

Numerics: smooth case with localized features

Model problem

$$\begin{aligned}-\Delta u &= f \quad \text{in } \Omega := (-1, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

Exact solution

$$u(x, y) = (x^2 - 1)(y^2 - 1) \exp(-100(x^2 + y^2))$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- hp -adaptive refinement

Numerics: smooth case with localized features

Model problem

$$\begin{aligned}-\Delta u &= f \quad \text{in } \Omega := (-1, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

Exact solution

$$u(x, y) = (x^2 - 1)(y^2 - 1) \exp(-100(x^2 + y^2))$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- hp -adaptive refinement

Numerics: smooth case with localized features

Model problem

$$\begin{aligned}-\Delta u &= f \quad \text{in } \Omega := (-1, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega\end{aligned}$$

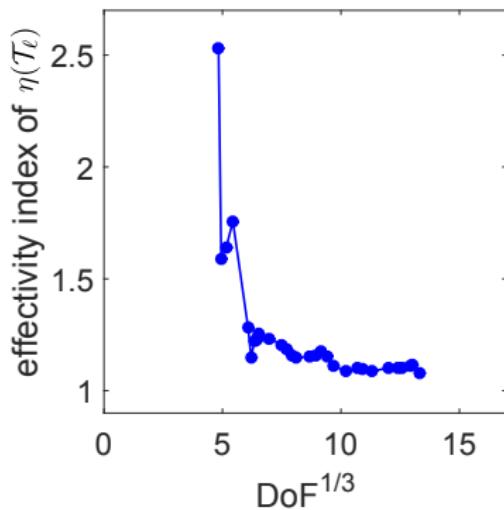
Exact solution

$$u(x, y) = (x^2 - 1)(y^2 - 1) \exp(-100(x^2 + y^2))$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- hp -adaptive refinement

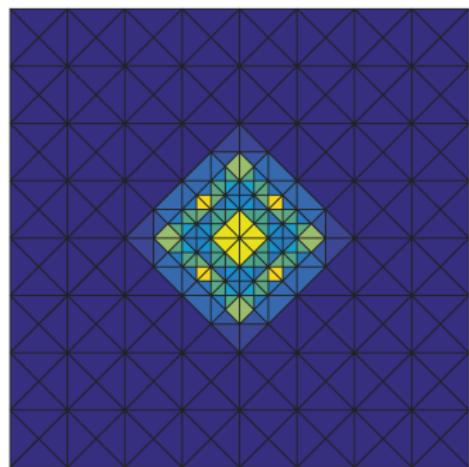
How precise are the estimates?



Effectivity indices on *hp* meshes

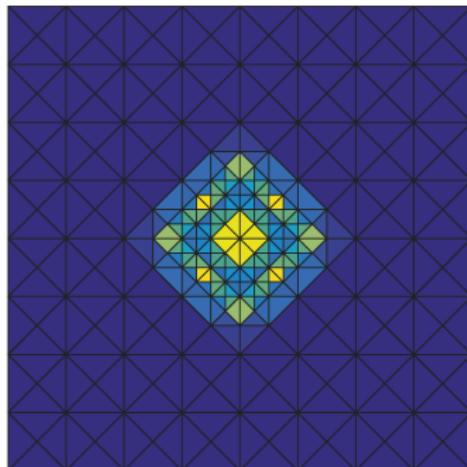
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Where (in space) is the error localized?



Estimated error distribution

$$\eta_K(u_h)$$

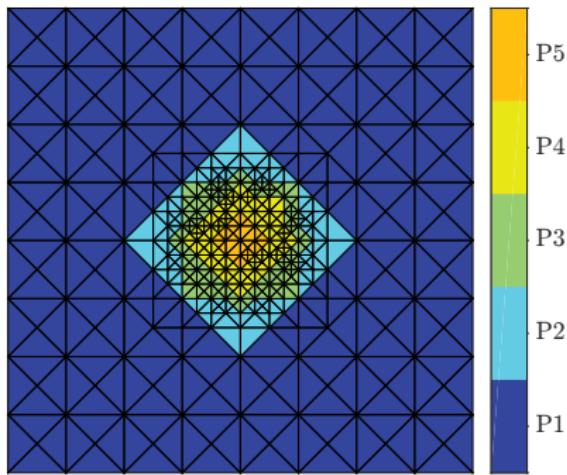


Exact error distribution

$$\|\nabla(u - u_h)\|_K$$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

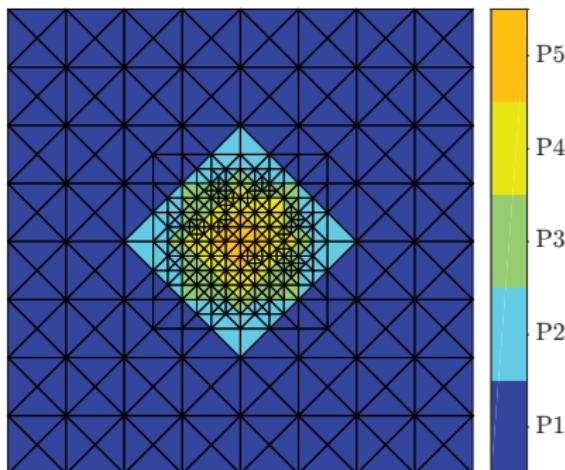
Can we decrease the error efficiently?



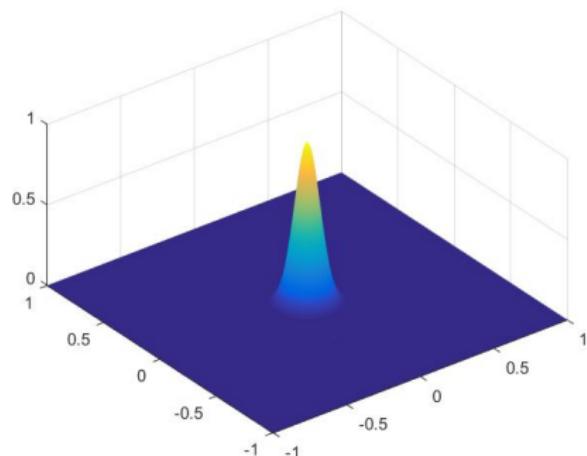
Mesh \mathcal{T} and pol. degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently?



Mesh \mathcal{T} and pol. degrees p_K



Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Numerics: singular case

Model problem

$$\begin{aligned}-\Delta u &= 0 \quad \text{in } \Omega := (-1, 1)^2 \setminus [0, 1]^2, \\ u &= u_D \quad \text{on } \partial\Omega\end{aligned}$$

Exact solution

$$u(r, \phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- hp -adaptive refinement

Numerics: singular case

Model problem

$$\begin{aligned}-\Delta u &= 0 \quad \text{in } \Omega := (-1, 1)^2 \setminus [0, 1]^2, \\ u &= u_D \quad \text{on } \partial\Omega\end{aligned}$$

Exact solution

$$u(r, \phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- hp -adaptive refinement

Numerics: singular case

Model problem

$$\begin{aligned}-\Delta u &= 0 \quad \text{in } \Omega := (-1, 1)^2 \setminus [0, 1]^2, \\ u &= u_D \quad \text{on } \partial\Omega\end{aligned}$$

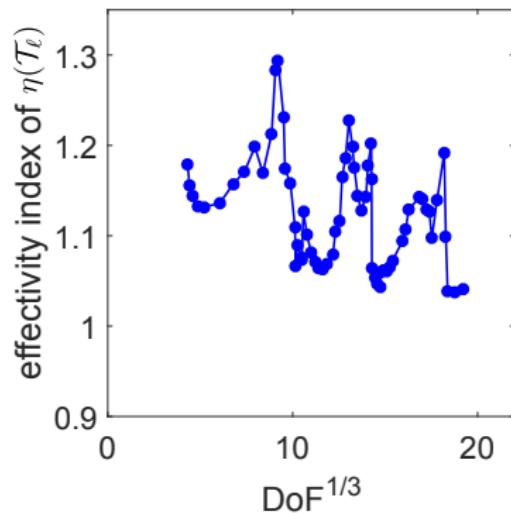
Exact solution

$$u(r, \phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

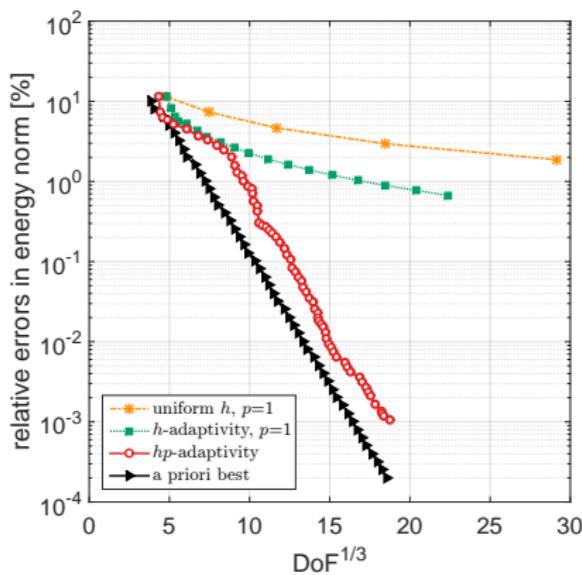
- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- *hp*-adaptive refinement

How precise are the estimates?



P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

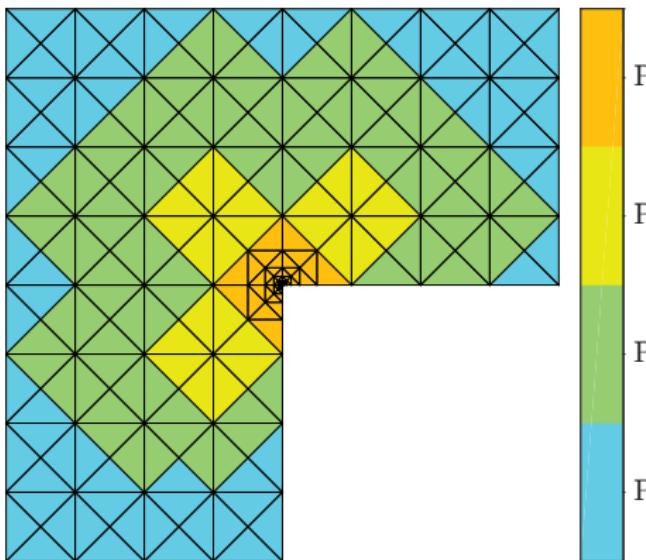
Can we decrease the error efficiently?



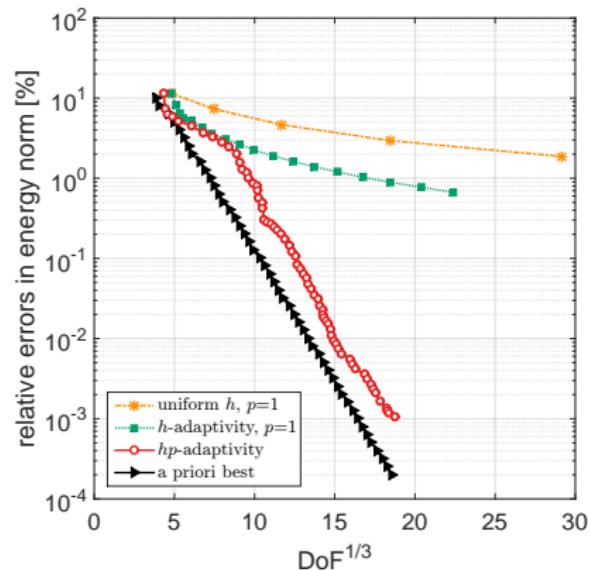
Relative error as a function of
no. of unknowns

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently?



Mesh \mathcal{T} and polynomial degrees p_K



Relative error as a function of no. of unknowns

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^D \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^D)$ be continuous on \mathcal{F}_K^D . Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^D \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^D)$ be continuous on \mathcal{F}_K^D . Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

Context

$$-\Delta \zeta_K = 0 \quad \text{in } K,$$

$$\zeta_K = r_F \quad \text{on all } F \in \mathcal{F}_K^D,$$

$$-\nabla \zeta_K \cdot \mathbf{n}_K = 0 \quad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^D.$$

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^D \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^D)$ be continuous on \mathcal{F}_K^D . Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

Context

$$-\Delta \zeta_K = 0 \quad \text{in } K,$$

$$\zeta_K = r_F \quad \text{on all } F \in \mathcal{F}_K^D,$$

$$-\nabla \zeta_K \cdot \mathbf{n}_K = 0 \quad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^D.$$

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^D \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^D)$ be continuous on \mathcal{F}_K^D . Then

$$\|\nabla \zeta_{h,K}\|_K \stackrel{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^D}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

Context

- $-\Delta \zeta_K = 0$ in K ,
- $\zeta_K = r_F$ on all $F \in \mathcal{F}_K^D$,
- $-\nabla \zeta_K \cdot \mathbf{n}_K = 0$ on all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^D$.

Potentials: patch

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2016))

For $p \geq 1$ and $\mathbf{a} \in \mathcal{V}^{\text{int}}$, let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_{\mathbf{a}}^{\text{int}})$. Suppose the compatibility

$$r_F|_{F \cap \partial\omega_{\mathbf{a}}} = 0 \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{int}},$$

$$\sum_{F \in \mathcal{F}_e} \iota_{F,e} r_F|_e = 0 \quad \forall e \in \mathcal{E}_{\mathbf{a}}.$$

Then

$$\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\mathbf{a}}) \\ v_h=0 \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{ext}} \\ [[v_h]] = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{int}}}} \|\nabla_h v_h\|_{\omega_{\mathbf{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\mathbf{a}}) \\ v=0 \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{ext}} \\ [[v]] = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{int}}}} \|\nabla_h v\|_{\omega_{\mathbf{a}}}.$$

Fluxes: one element

Lemma ($H(\text{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^N \subset \mathcal{F}_K$. Let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_K^N) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^N = \mathcal{F}_K$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \lesssim \min_{\substack{\mathbf{v} \in H(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K .$$

Fluxes: one element

Lemma ($H(\text{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^N \subset \mathcal{F}_K$. Let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_K^N) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K}(r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^N = \mathcal{F}_K$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \lesssim \min_{\substack{\mathbf{v} \in H(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K.$$

Context

- $-\Delta \zeta_K = r_K$ in K ,
- $-\nabla \zeta_K \cdot \mathbf{n}_K = r_F$ on all $F \in \mathcal{F}_K^N$,
- $\zeta_K = 0$ on all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^N$.

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: one element

Lemma ($H(\text{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^N \subset \mathcal{F}_K$. Let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_K^N) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K}(r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^N = \mathcal{F}_K$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \lesssim \min_{\substack{\mathbf{v} \in H(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K = \|\varphi_K\|_K.$$

Context

- $-\Delta \zeta_K = r_K$ in K ,
- $-\nabla \zeta_K \cdot \mathbf{n}_K = r_F$ on all $F \in \mathcal{F}_K^N$,
- $\zeta_K = 0$ on all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^N$.

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: one element

Lemma ($H(\text{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^N \subset \mathcal{F}_K$. Let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_K^N) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^N = \mathcal{F}_K$. Then

$$\|\varphi_{h,K}\|_K \stackrel{\text{MFEs}}{=} \min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \lesssim \min_{\substack{\mathbf{v} \in H(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K = \|\varphi_K\|_K.$$

Context

- $-\Delta \zeta_K = r_K$ in K ,
- $-\nabla \zeta_K \cdot \mathbf{n}_K = r_F$ on all $F \in \mathcal{F}_K^N$,
- $\zeta_K = 0$ on all $F \in \mathcal{F}_K \setminus \mathcal{F}_K^N$.

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: patch

Theorem (Broken $\mathbf{H}(\text{div})$ polynomial extension on a patch) Braess,

Pillwein, & Schöberl (2009; 2D), Ern & V. (2016; 3D)

For $p \geq 0$ and $\mathbf{a} \in \mathcal{V}^{\text{int}}$, let $\mathbf{r} \in \mathbb{P}_p(\mathcal{F}_{\mathbf{a}}) \times \mathbb{P}_p(\mathcal{T}_{\mathbf{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\mathbf{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\mathbf{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(\mathcal{T}_{\mathbf{a}}) \\ \mathbf{v}_h \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{ext}} \\ [\![\mathbf{v}_h \cdot \mathbf{n}_F]\!] = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{int}} \\ \nabla_h \cdot \mathbf{v}_h|_K = r_K \quad \forall K \in \mathcal{T}_{\mathbf{a}}}} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}} \lesssim \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, \mathcal{T}_{\mathbf{a}}) \\ \mathbf{v} \cdot \mathbf{n}_F = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{ext}} \\ [\![\mathbf{v} \cdot \mathbf{n}_F]\!] = r_F \quad \forall F \in \mathcal{F}_{\mathbf{a}}^{\text{int}} \\ \nabla_h \cdot \mathbf{v}|_K = r_K \quad \forall K \in \mathcal{T}_{\mathbf{a}}}} \|\mathbf{v}\|_{\omega_{\mathbf{a}}}.$$

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates

- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- Stable commuting local projector in $\mathbf{H}(\text{div})$

5 A posteriori estimates

- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Conclusions and outlook

Conclusions

- simple proof of global-best – local-best equivalence in H^1
- constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- incidentally leads to stable commuting local projectors
- optimal hp *a priori* error estimates
- p -robust *a posteriori* error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, and others carried out

Ongoing work

- extensions to other settings

Conclusions and outlook

Conclusions

- simple proof of global-best – local-best equivalence in H^1
- constrained global-best – local-best equivalence in $\mathbf{H}(\text{div})$
- incidentally leads to stable commuting local projectors
- optimal hp *a priori* error estimates
- p -robust *a posteriori* error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, and others carried out

Ongoing work

- extensions to other settings

References

-  ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, *SIAM J. Numer. Anal.* **53** (2015), 1058–1081.
-  ERN A., VOHRALÍK M., Stable broken H^1 and $\mathbf{H}(\text{div})$ polynomial extensions for polynomial-degree-robust potential and flux reconstructions in three space dimensions, HAL Preprint 01422204, submitted for publication, 2016.
-  ERN A., SMEARS, I., VOHRALÍK M., Discrete p -robust $\mathbf{H}(\text{div})$ -liftings and a posteriori estimates for elliptic problems with H^{-1} source terms, *Calcolo* **54** (2017), 1009–1025.
-  ERN A., GUDI T., SMEARS I., VOHRALÍK M., Equivalence of local- and global-best approximations, a simple stable local commuting projector, and optimal hp approximation estimates in $\mathbf{H}(\text{div})$, to be submitted, 2019.

Thank you for your attention!