A posteriori error estimates robust with respect to the strength of nonlinearities

André Harnist, Koondanibha Mitra, Ari Rappaport, and **Martin Vohralík**

Inria Paris & Ecole des Ponts

CMAP, 21 February 2023

Outline

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Numerical approximation of partial differential equations

Numerical methods

- mathematically-based algorithms evaluated by **computers**
- deliver **approximate solutions**
- conception: more effort $\Rightarrow$ closer to the unknown solution
- example: elastic string

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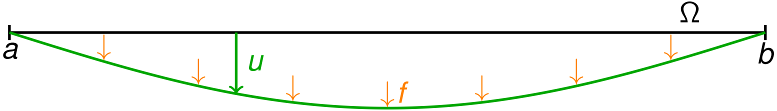

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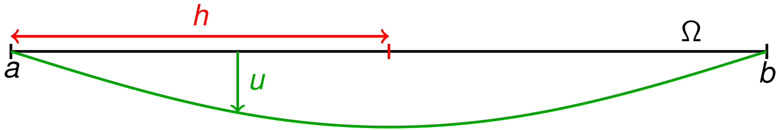

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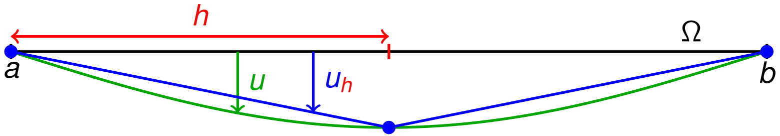

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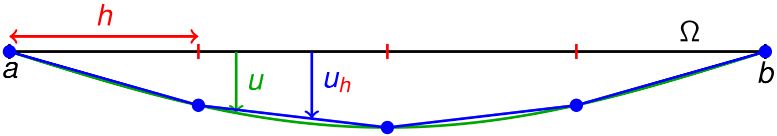

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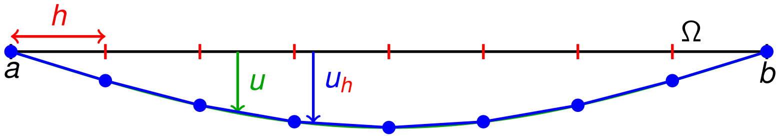

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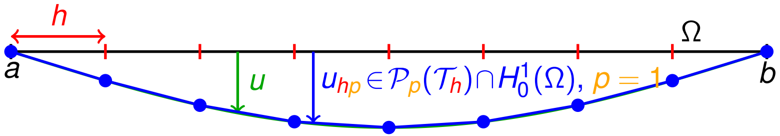

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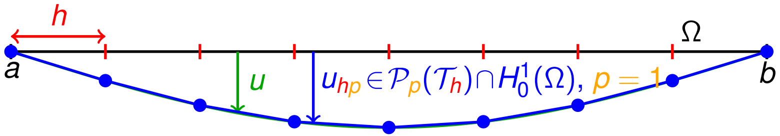

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Error

$$\|\nabla(u - u_{hp})\| = \left\{\int_a^b |(u - u_{hp})'|^2\right\}^{\frac{1}{2}}$$

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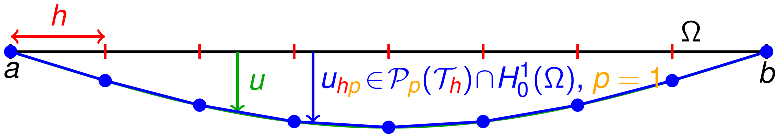

Numerical approximation u_{hp} and its convergence to u

Error

$$\|\nabla(u - u_{hp})\| = \left\{ \int_a^b |(u - u_{hp})'|^2 \right\}^{\frac{1}{2}}$$

Need to solve a linear system

$$\mathbb{A}_{hp} \mathsf{U}_{hp} = \mathsf{F}_{hp}$$

3 crucial questions

Crucial questions

1. How **large** is the overall **error**?
2. **Where** (model/space/time/linearization/algebra) is it **localized**?
3. Can we **decrease** it **efficiently**?

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Suggested answers

1. **A posteriori** error **estimates**.

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1. **A posteriori** error **estimates**.
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Suggested answers

1. **A posteriori** error **estimates**.
2. Identification of **error components**.
3. **Balancing** error components, **adaptivity** (working where needed).

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1. Introduction
   - Numerical approximation of partial differential equations
   - A posteriori error estimates
2. Setting and known results
   - Setting (gradient-dependent case)
   - Error measures
   - Known results
3. Iterative linearization
4. A posteriori error estimates for an augmented energy difference
5. Numerical experiments
6. Extensions
   - Setting (gradient-independent case)
   - A posteriori error estimates for an iteration-dependent norm
   - Numerical experiments
7. Conclusions

A posteriori error estimates: certify the error in a FE discretization

Laplacian: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\nabla u) &= f \quad \text{in} \quad \Omega,\\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Guaranteed error upper bound (reliability)

$$\underbrace{\|\nabla(u-u_\ell)\|}_{\text{unknown error}} \qquad \underbrace{\eta(u_\ell)}_{\text{estimator computable from } u_\ell}$$

Error lower bound (efficiency)

$$\eta(u_\ell) \le C_{\text{eff}} \|\nabla(u-u_\ell)\|$$

- C_{eff} a generic constant independent of Ω, u, u_ℓ and namely of the number of mesh elements $|\mathcal{T}_\ell|$ (h if $\mathcal{T}_\ell$ uniform) and of the polynomial degree p (for $d \le 3$)

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h_0	1	1.25

How **large** is the overall error? (model pb, known smooth solution)

$h \approx 1/\|\mathcal{T}_\ell\|^{\frac{1}{2}}$	p	$\eta(u_\ell)$	rel. error estimate $\frac{\eta(u_\ell)}{\|\nabla u_\ell\|}$	$\|\nabla(u-u_\ell)\|$	rel. error $\frac{\|\nabla(u-u_\ell)\|}{\|\nabla u_\ell\|}$	$I^{\mathrm{eff}} = \frac{\eta(u_\ell)}{\|\nabla(u-u_\ell)\|}$
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

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$h \approx 1/\lvert\mathcal{T}_\ell\rvert^{\frac{1}{2}}$	p	$\eta(u_\ell)$	rel. error estimate $\frac{\eta(u_\ell)}{\lVert\nabla u_\ell\rVert}$	$\lVert\nabla(u-u_\ell)\rVert$	rel. error $\frac{\lVert\nabla(u-u_\ell)\rVert}{\lVert\nabla u_\ell\rVert}$	$I^{\mathrm{eff}} = \frac{\eta(u_\ell)}{\lVert\nabla(u-u_\ell)\rVert}$
h_0	1	1.25	28%	1.07	24%	**1.17**
$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	**1.09**
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	**1.06**
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Linear problems

A posteriori error estimates **robust** with respect to the **discretization parameters** $|\mathcal{T}_\ell|$ (h if $\mathcal{T}_\ell$ uniform) and p ($d \leq 3$).

Nonlinear problems

A posteriori error estimates **robust** with respect to the **strength of nonlinearities**?

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Nonlinear problems

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Outline

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A model nonlinear problem

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(a(|\nabla u|)\nabla u) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $d \geq 1$, open polytope with Lipschitz boundary $\partial\Omega$
- $f \in L^2(\Omega)$

Assumption (Nonlinear function a)

Function $a : [0, \infty) \to (0, \infty)$, *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\begin{aligned} |a(|\mathbf{x}|)\mathbf{x} - a(|\mathbf{y}|)\mathbf{y}| &\leq a_c|\mathbf{x} - \mathbf{y}| && \textit{(Lipschitz continuity)}, \\ (a(|\mathbf{x}|)\mathbf{x} - a(|\mathbf{y}|)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) &\geq a_m|\mathbf{x} - \mathbf{y}|^2 && \textit{(strong monotonicity)}. \end{aligned}$$

- $a_m \leq a(r) \leq a_c$, $a_m \leq (a(r)r)'(r) \leq a_c$

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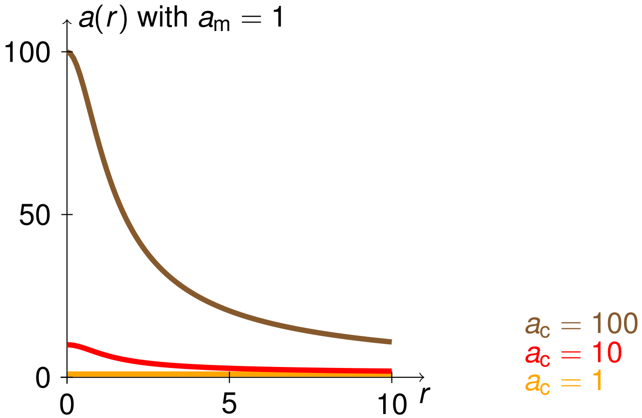
Strength of the nonlinearity, $0 < a_\mathrm{m} \leq a_\mathrm{c} < \infty$ real parameters

Example (Mean curvature nonlinearity)

$$a(r) := a_\mathrm{m} + \frac{a_\mathrm{c} - a_\mathrm{m}}{\sqrt{1 + r^2}}.$$



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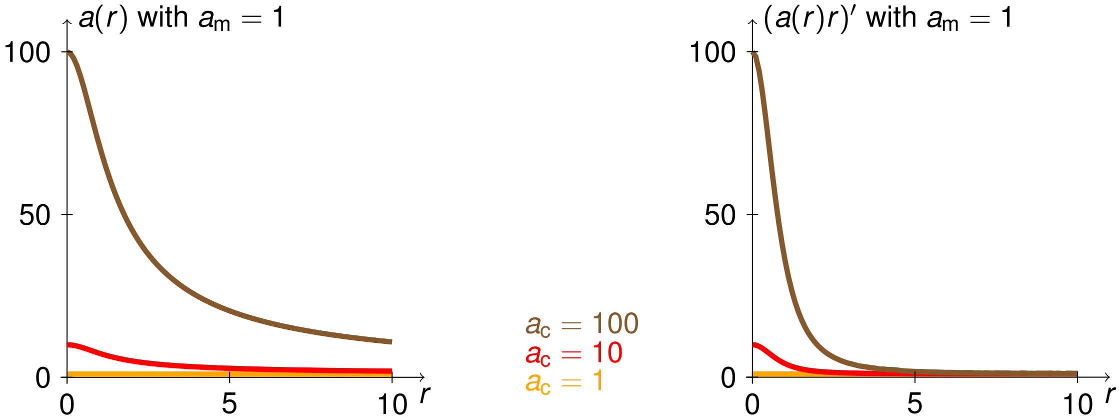

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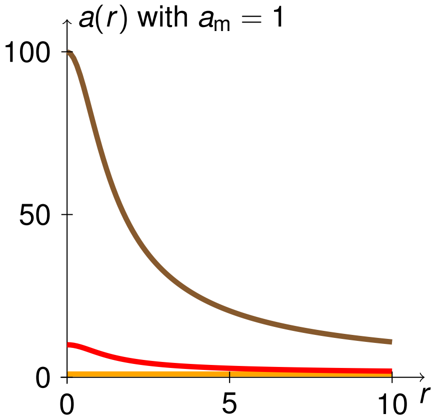

$a_{\mathrm{c}} = 100$
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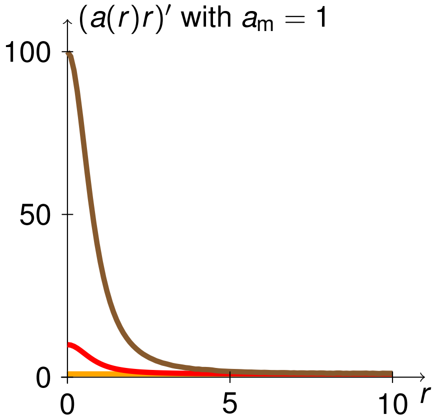

Weak solution and its finite element approximation

Definition (Weak solution)

$u \in H_0^1(\Omega)$ such that

$$(a(|\nabla u|)\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega).$$

Definition (Finite element approximation)

$u_\ell \in V_\ell^p$ such that

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- $\mathcal{T}_\ell$ simplicial mesh of Ω
- $p \geq 1$ polynomial degree
- conforming finite elements
- $V_\ell^p := \mathcal{P}_p(\mathcal{T}_\ell) \cap H_0^1(\Omega)$

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Outline

Sobolev space and error

Sobolev space

$$H_0^1(\Omega)$$

Sobolev norm error

$$\boxed{\|\nabla(u_\ell - u)\|}$$

Energy and energy differences

Definition (Energy functional)

$\mathcal{J} : H_0^1(\Omega) \to \mathbb{R}$

$$\mathcal{J}(v) := \int_\Omega \phi(|\nabla v|) - (f, v), \quad v \in H_0^1(\Omega),$$

with function $\phi : [0, \infty) \to [0, \infty)$ such that, for all $r \in [0, \infty)$,

$$\phi(r) := \int_0^r a(s) s \, \mathrm{d}s.$$

Equivalently

$$u = \arg \min_{v \in H_0^1(\Omega)} \mathcal{J}(v), \qquad u_\ell = \arg \min_{v_\ell \in V_\ell^p} \mathcal{J}(v_\ell).$$

Energy difference

$$\boxed{\mathcal{J}(u_\ell) - \mathcal{J}(u)}$$

- $\mathcal{J}(u_\ell) - \mathcal{J}(u) \geq 0$, $\mathcal{J}(u_\ell) - \mathcal{J}(u) = 0$ if and only if $u_\ell = u$
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Residual and its dual norm

Definition (Residual)

$\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$; for $w \in H_0^1(\Omega)$, $\mathcal{R}(w) \in H^{-1}(\Omega)$ is given by

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Definition (Dual norm of the finite element residual)

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- Zeidler (1992), Han (1994), Repin (1997), Ladevèze & Moës (1997), Diening & Kreuzer (2008), Bartels & Milicevic (2020), ...

Strength of the nonlinearity

Not robust with respect to $\frac{a_{\mathrm{c}}}{a_{\mathrm{m}}}$.

Known results

Sobolev norm

$$a_{\mathrm{m}} \|\nabla(u_\ell - u)\| \leq \eta(u_\ell) \leq C_{\mathrm{eff}} a_{\mathrm{c}} \|\nabla(u_\ell - u)\|$$

- Pousin & Rappaz (1994), Verfürth (1994), Kim (2007), Houston, Süli, & Wihler (2008), Garau, Morin, & Zuppa (2011), Gantner, Haberl, Praetorius, & Stiftner (2018), Heid & Wihler (2020), Botti & Riedlbeck (2020), ...

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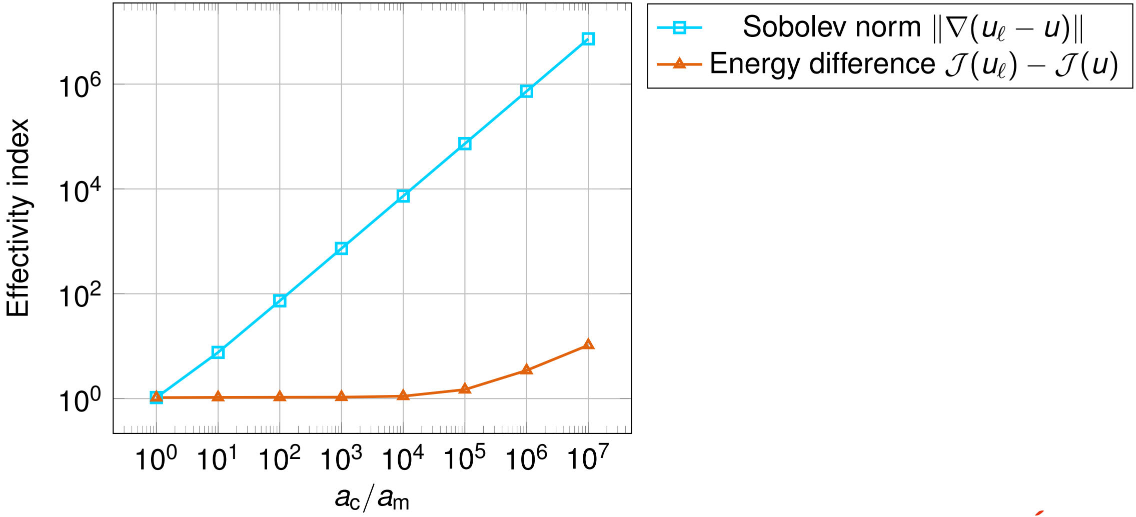

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$$|||\mathcal{R}(u_\ell)|||_{-1} \le \eta(u_\ell) \le C_{\text{eff}} |||\mathcal{R}(u_\ell)|||_{-1}$$

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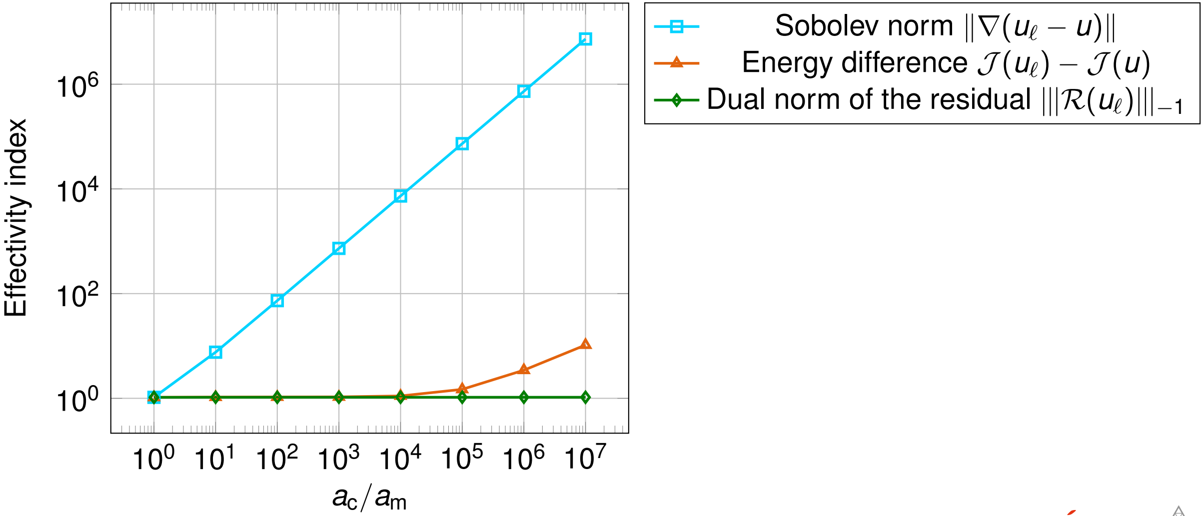

Nonlinear problems

A posteriori error estimates **robust** with respect to the **strength of nonlinearities** in more **physically-based error measures**?

Iterative linearization

Addressing iterative linearization

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Observation

None of the above approaches employ **in the analysis**, to define norms, the **iterative linearization**, i.e., **how** do we solve the nonlinear system $\mathcal{A}_\ell(\mathsf{U}_\ell) = \mathsf{F}_\ell$.

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Iterative linearization

Definition (Linearized finite element approximation)

$u_\ell^k \in V_\ell^p$ such that

$$(\mathbf{A}_\ell^{k-1}\nabla u_\ell^k, \nabla v_\ell) = (f, v_\ell) + (\mathbf{b}_\ell^{k-1}, \nabla v_\ell) \qquad \forall v_\ell \in V_\ell^p.$$

- $u_\ell^0 \in V_\ell^p$ a given initial guess
- iterative linearization index $k \geq 1$
- **linearization**: $\mathbf{A}_\ell^{k-1}: \Omega \to \mathbb{R}^{d\times d}$ diffusion matrix, $\mathbf{b}_\ell^{k-1}: \Omega \to \mathbb{R}^d$ RHS vector

Definition (Linearized energy functional)

$\mathcal{J}_\ell^{k-1}: H_0^1(\Omega) \to \mathbb{R}$

$$\mathcal{J}_\ell^{k-1}(v) := \frac{1}{2}\left\|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}}\nabla v\right\|^2 - (f, v) - (\mathbf{b}_\ell^{k-1}, \nabla v), \quad v \in H_0^1(\Omega).$$

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Examples

Example (Picard (fixed-point))

$$\boldsymbol{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\boldsymbol{I}_d, \quad \boldsymbol{b}_\ell^{k-1} = \boldsymbol{0}.$$

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$$\boldsymbol{A}_\ell^{k-1} = \gamma \boldsymbol{I}_d, \quad \boldsymbol{b}_\ell^{k-1} = (\gamma - a(|\nabla u_\ell^{k-1}|))\nabla u_\ell^{k-1},$$

with $\gamma \geq \frac{a_c^2}{a_m}$ *a constant parameter.*

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A posteriori error estimates for an augmented energy difference

Theorem (A posteriori estimate of augmented energy)

Let $f \in \mathcal{P}_{p-1}(\mathcal{T}_\ell)$ for simplicity. For all linearization steps $k \geq 1$,

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$$C_\ell^k := \max_{\boldsymbol{a} \in \mathcal{V}_\ell} \left(\frac{\sup_{\omega_\ell^{\boldsymbol{a}}} A_{\mathrm{c},\ell}^{k-1}}{\inf_{\omega_\ell^{\boldsymbol{a}}} A_{\mathrm{m},\ell}^{k-1}} \right) \begin{cases} = 1 & \text{Zarantonello} \\ \leq \frac{A_\mathrm{c}}{A_\mathrm{m}} \leq \frac{a_\mathrm{c}}{a_\mathrm{m}} & \text{in general.} \end{cases}$$

- λ_ℓ^k computable weight to make the two components comparable
- $C_\ell^k = 1$ for Zarantonello $\Longrightarrow$ **robustness** wrt the **strength of nonlinearities**
- C_ℓ^k given by **local** (patch) **properties**: typically **much better** than $a_\mathrm{c}/a_\mathrm{m}$
- C_ℓ^k **computable**: we can affirm **robustness *a posteriori***, for the given case

Outline

Smooth solution

Setting

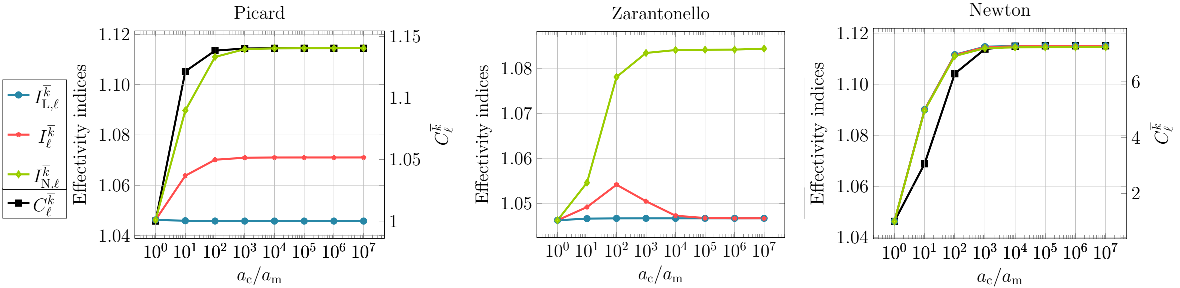
- unit square $\Omega = (0,1)^2$
- known smooth solution $u(x,y) := 10\,x(x-1)y(y-1)$
- mean curvature or exponential nonlinearity

$$a(r) = a_{\mathrm{m}} + \frac{a_{\mathrm{c}} - a_{\mathrm{m}}}{\sqrt{1+r^2}} \qquad \text{or} \qquad a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}}) \frac{1 - e^{-\frac{3}{2}r^2}}{1 + 2e^{-\frac{3}{2}}}$$

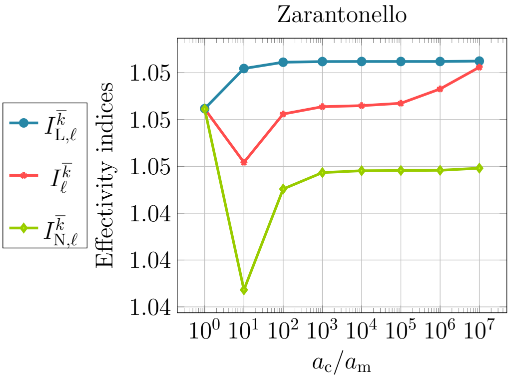
- $p = 1$, 3969 DOFs
- stopping iteration $\overline{k}$ such that $\|\nabla(u_\ell^{\overline{k}-1} - u_\ell^{\overline{k}})\| < 10^{-6}$
- effectivity indices

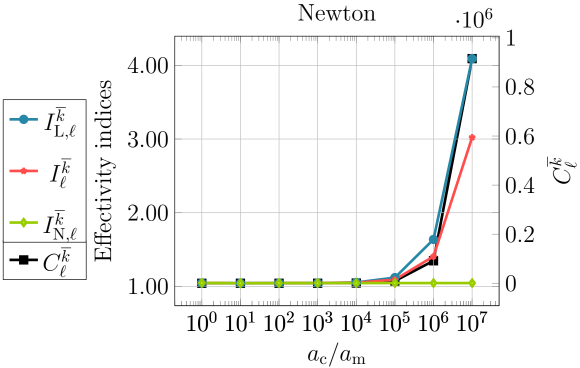
$$\underbrace{I_\ell^k := \left(\frac{\eta_\ell^k}{\mathcal{E}_\ell^k}\right)^{\frac{1}{2}}}_{\text{total}}, \quad \underbrace{I_{\mathrm{N},\ell}^k := \left(\frac{\mathcal{J}(u_\ell^k) - \mathcal{J}^*(\boldsymbol{\sigma}_\ell^k)}{\mathcal{J}(u_\ell^k) - \mathcal{J}(u)}\right)^{\frac{1}{2}}}_{\text{energy difference}}, \quad \underbrace{I_{\mathrm{L},\ell}^k := \left(\frac{\mathcal{J}_\ell^{k-1}(u_\ell^k) - \mathcal{J}_\ell^{*,k-1}(\boldsymbol{\sigma}_\ell^k)}{\mathcal{J}_\ell^{k-1}(u_\ell^k) - \mathcal{J}_\ell^{k-1}(u_{\langle\ell\rangle}^k)}\right)^{\frac{1}{2}}}_{\text{energy difference linearization}}$$

Error certification robust wrt the nonlinearities (mean curvature)

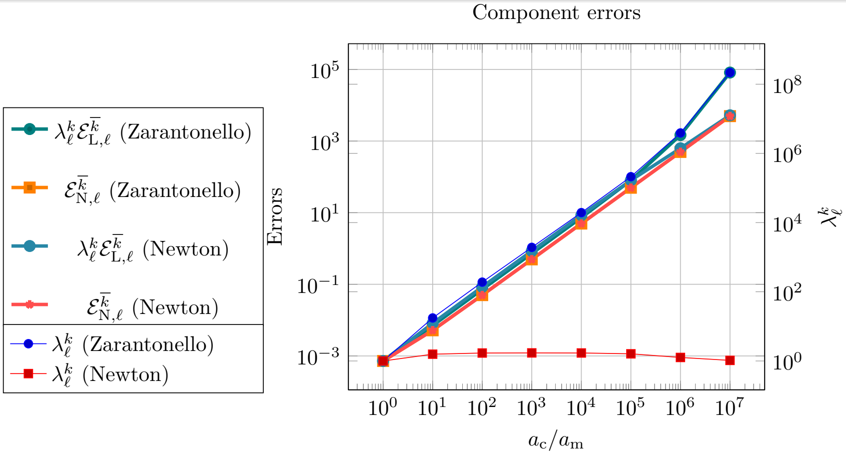


Error certification robust wrt the nonlinearities (exponential, robustness only for Zarantonello)





Error certification robust wrt the nonlinearities (exponential)



Components of $\mathcal{E}_\ell^{\overline{k}}$: the energy difference $\mathcal{E}_{\mathrm{N},\ell}^{\overline{k}} = \mathcal{J}(u_\ell^{\overline{k}}) - \mathcal{J}(u)$, the energy difference of the linearized problem $\mathcal{E}_{\mathrm{L},\ell}^{\overline{k}} = \mathcal{J}_\ell^{\overline{k}-1}(u_\ell^{\overline{k}}) - \mathcal{J}_\ell^{\overline{k}-1}(u_{(\ell)}^{\overline{k}})$, & the weight λ_ℓ^k

Singular solution

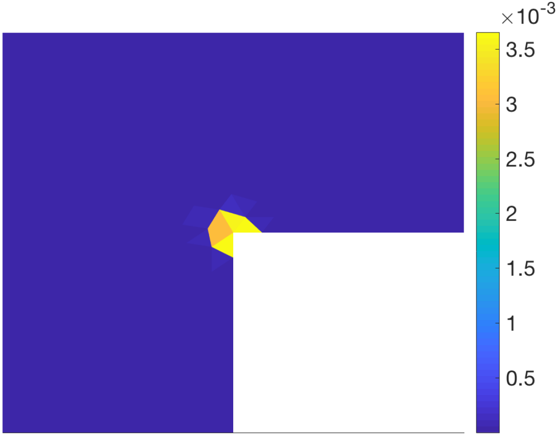
Setting

- L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$
- known singular solution $u(\rho,\theta) = \rho^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$
- exponential nonlinearity

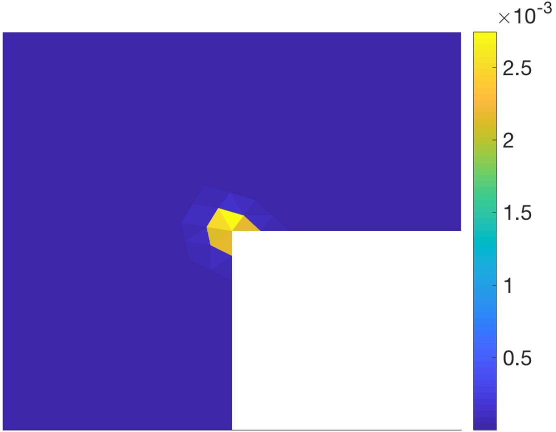
$$a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}}) \frac{1 - e^{-\frac{3}{2}r^2}}{1 + 2e^{-\frac{3}{2}}}$$

- $p = 1$, uniform or adaptive mesh refinement

Where is the error **localized**?



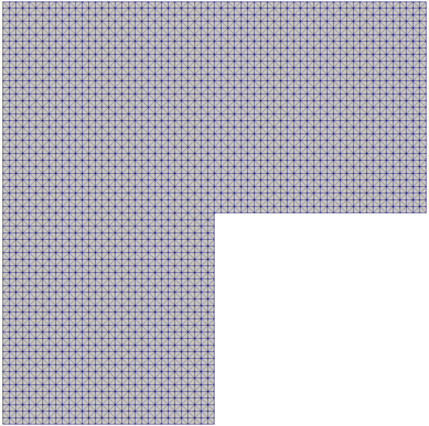
Estimated total errors $\eta_K(u_\ell^i)$

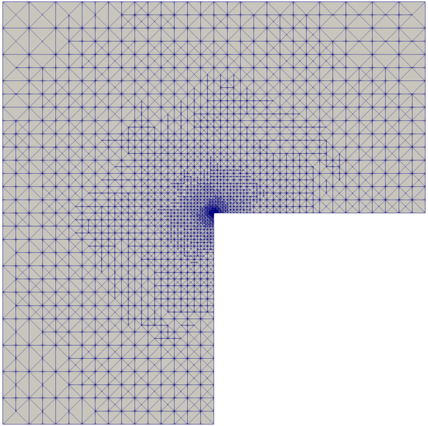


Exact total errors $\|\nabla(u - u_\ell^i)\|_K$

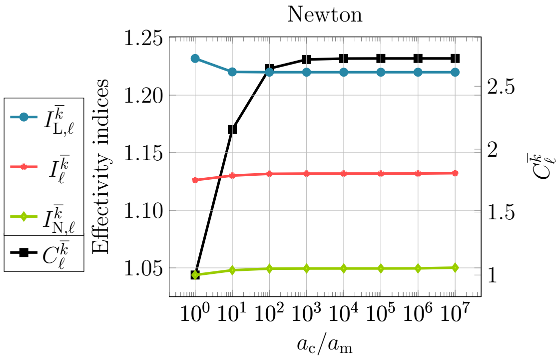
J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

Adaptive mesh refinement

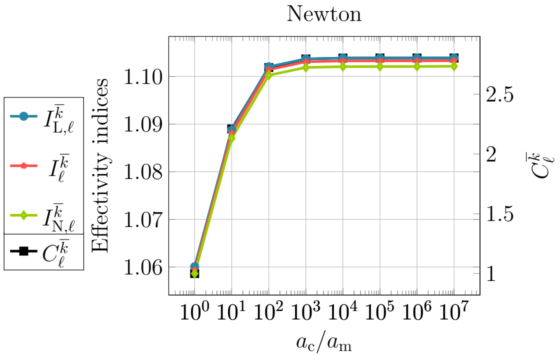




Error certification robust wrt the nonlinearities

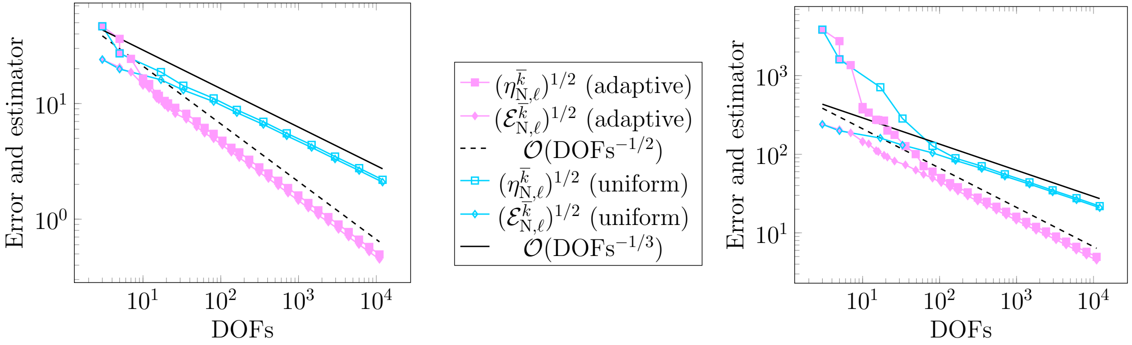


Uniform mesh refinement



Adaptive mesh refinement

Decreasing the error efficiently: optimal decay rate wrt DoFs



$\frac{a_c}{a_m} = 10^3$ $\frac{a_c}{a_m} = 10^6$

Outline

Outline

A model nonlinear problem

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot\Big(\tau \underline{\boldsymbol{K}}(\boldsymbol{x})\big(\underbrace{\mathcal{D}(\boldsymbol{x},u)}_{\text{diffusion}}\nabla u + \underbrace{\boldsymbol{q}(\boldsymbol{x},u)}_{\text{advection}}\big)\Big) + \underbrace{f(\boldsymbol{x},u)}_{\text{reaction}} = 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

- $\tau > 0$ a parameter (time step size in transient problems)

Assumption (Nonlinear functions $\mathcal{D}$, $\boldsymbol{q}$, and f)

$$|\mathcal{D}(\boldsymbol{x}_1,\xi_1) - \mathcal{D}(\boldsymbol{x}_2,\xi_2)| \leq \mathcal{D}_{\mathrm{M}}(|\boldsymbol{x}_1 - \boldsymbol{x}_2| + |\xi_1 - \xi_2|) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega \text{ and } \xi_1, \xi_2 \in \mathcal{R},$$

$$0 \leq f(\boldsymbol{x},\xi_2) - f(\boldsymbol{x},\xi_1) \leq f_{\mathrm{M}}(\xi_2 - \xi_1) \quad \forall \boldsymbol{x} \in \Omega \text{ and } \xi_2 \geq \xi_1,$$

$\boldsymbol{q}$ is “small” wrt $\underline{\boldsymbol{K}}\mathcal{D}$.

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Strength of the nonlinearity

ratio $a_{\rm c}/a_{\rm m}$

Iterative linearization

Definition (Linearized finite element approximation)

$u_\ell^k \in V_\ell^p$ such that

$$((u_\ell^k - u_\ell^{k-1}, v_\ell))_{u_\ell^{k-1}} = -\langle \underbrace{\mathcal{R}(u_\ell^{k-1})}_{\text{residual}}, v_\ell \rangle \qquad \forall v_\ell \in V_\ell^p.$$

- linearization: **reaction–diffusion scalar product**

$$((w, v))_{u_\ell^{k-1}} := (\underbrace{L(\boldsymbol{x}, u_\ell^{k-1})}_{\text{reaction coef.}} w, v) + (\underbrace{\mathfrak{a}(\boldsymbol{x}, u_\ell^{k-1})}_{\text{diffusion coef.}} \nabla w, \nabla v), \quad w, v \in H_0^1(\Omega)$$

- covers many linearization schemes: Picard (fixed-point), L & M-schemes, ...

Iteration-dependent norm

$$|||v|||_{1,u_\ell^{k-1}} := ((v, v))_{u_\ell^{k-1}}^{\frac{1}{2}}, \quad v \in H_0^1(\Omega)$$

- induced by the linearization scalar product

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An orthogonal decomposition of the total residual/error

Theorem (Orthogonal decomposition of the total error into linearization and discretization components)

For all linearization steps $k \geq 1$, there holds

$$\underbrace{|||\mathcal{R}(u_\ell^{k-1})|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{total residual/error} \\ |||u_\ell^{k-1}-u_{\langle\ell\rangle}^k|||_{1,u_\ell^{k-1}}}} = \underbrace{|||u_\ell^{k-1} - u_\ell^k|||^2_{1,u_\ell^{k-1}}}_{\substack{\text{linearization} \\ \text{error}}} + \underbrace{|||\mathcal{R}_{\text{disc}}^{u_\ell^{k-1}}(u_\ell^k)|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{discretization residual/error} \\ |||u_\ell^k-u_{\langle\ell\rangle}^k|||_{1,u_\ell^{k-1}}}}.$$

- orthogonal decomposition
- error components

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- **orthogonal decomposition**
- **error components**

Outline

1. Introduction
   - Numerical approximation of partial differential equations
   - A posteriori error estimates
2. Setting and known results
   - Setting (gradient-dependent case)
   - Error measures
   - Known results
3. Iterative linearization
4. A posteriori error estimates for an augmented energy difference
5. Numerical experiments
6. Extensions
   - Setting (gradient-independent case)
   - A posteriori error estimates for an iteration-dependent norm
   - Numerical experiments
7. Conclusions

A posteriori error estimates for an iteration-dependent norm

Theorem (A posteriori estimate of iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta_\ell^k.$$

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Moreover, for all linearization steps $k \geq 1$, there holds

$$\eta_\ell^k \leq C_{\mathrm{eff}}(d, \kappa_{\mathcal{T}}, p) C_\ell^k |||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} + \textit{quadrature terms} + \textit{data oscillation terms},$$

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$$C_\ell^k \quad \begin{cases} = 1 & \text{Zarantonello} \\ & \end{cases}$$

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- $C_\ell^k = 1$ for Zarantonello $\Longrightarrow$ **robustness** wrt the **strength of nonlinearities**
- C_ℓ^k given by **local** (patch) **properties**: typically **much better** than $a_{\mathrm{c}}/a_{\mathrm{m}}$
- C_ℓ^k **computable**: we can affirm **robustness *a posteriori***, for the given case

A posteriori error estimates for an iteration-dependent norm

Theorem (A posteriori estimate of iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta_\ell^k.$$

Moreover, for all linearization steps $k \geq 1$, there holds

$$\eta_\ell^k \leq C_{\text{eff}}(d, \kappa_{\mathcal{T}}, p) C_\ell^k |||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} + \textit{quadrature terms} + \textit{data oscillation terms},$$

where

$$C_\ell^k := \max_{\boldsymbol{a} \in \mathcal{V}_\ell} \left(\frac{h_{\omega_{\boldsymbol{a}}}^2 \sup_{\omega_{\boldsymbol{a}}} L_{\mathrm{M}}^{k-1} + \pi^2 \sup_{\omega_{\boldsymbol{a}}} \mathfrak{a}_{\mathrm{M}}^{k-1}}{h_{\omega_{\boldsymbol{a}}}^2 \inf_{\omega_{\boldsymbol{a}}} L_{\mathrm{m}}^{k-1} + \pi^2 \inf_{\omega_{\boldsymbol{a}}} \mathfrak{a}_{\mathrm{m}}^{k-1}}, \frac{\sup_{\omega_{\boldsymbol{a}}} \mathfrak{a}_{\mathrm{M}}^{k-1}}{\inf_{\omega_{\boldsymbol{a}}} \mathfrak{a}_{\mathrm{m}}^{k-1}} \right) \begin{cases} = 1 & \textit{Zarantonello} \\ \leq \dfrac{a_{\mathrm{c}}}{a_{\mathrm{m}}} & \textit{in general.} \end{cases}$$

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- also **local efficiency**

Outline

The Richards equation

Setting

- one time step of the Richards equation
- unit square $\Omega = (0,1)^2$
- realistic data

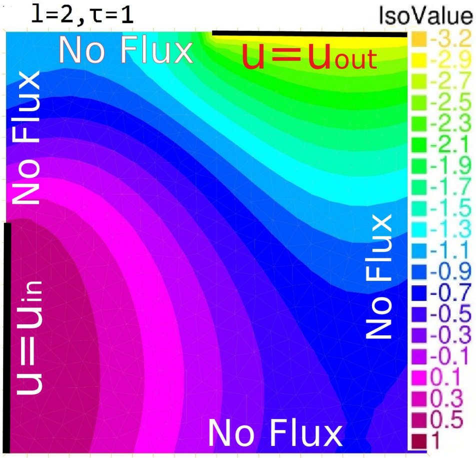
$$f(\boldsymbol{x},\xi) = S(\xi) - S(u_\ell^{n-1}(\boldsymbol{x})), \quad \mathcal{D}(\boldsymbol{x},\xi) = \kappa(S(\xi)), \quad \boldsymbol{q}(\boldsymbol{x},\xi) = -\kappa(S(\xi))\,\boldsymbol{g},$$

$$\underline{\boldsymbol{K}} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

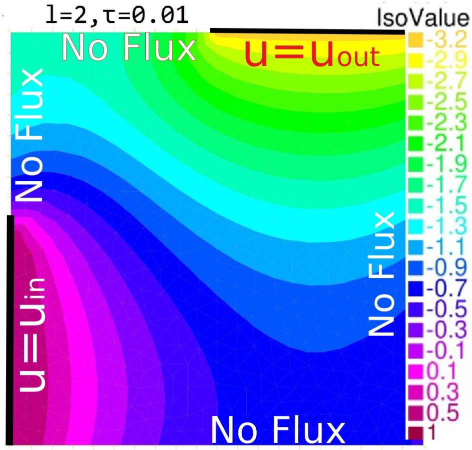
- time step length $\tau \in [10^{-3}, 1]$
- van Genuchten saturation and permeability laws

$$S(\xi) := \left(1 + (2-\xi)^{\frac{1}{1-\lambda}}\right)^{-\lambda}, \quad \kappa(s) := \sqrt{s}\left(1 - (1 - s^{\frac{1}{\lambda}})^{\lambda}\right)^2, \quad \lambda = 0.5$$

One time step of the Richards equation with realistic data

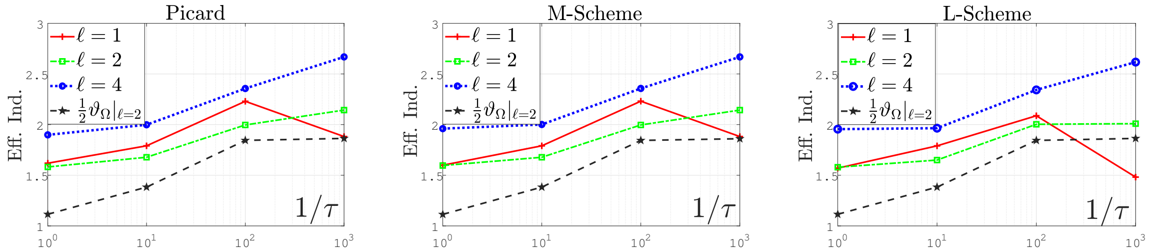


Time step length $\tau = 1$

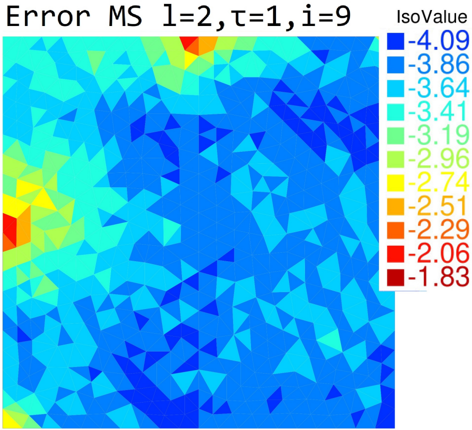


Time step length $\tau = 0.01$

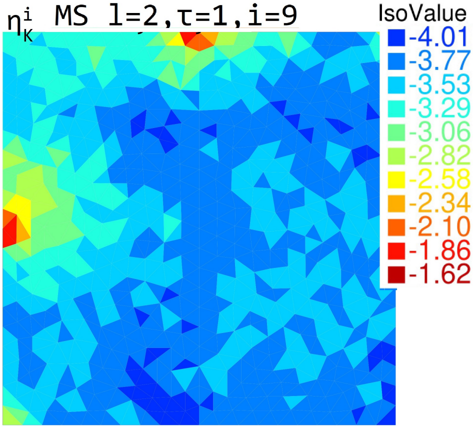
Error certification robust wrt the nonlinearities



Where is the error **localized**?

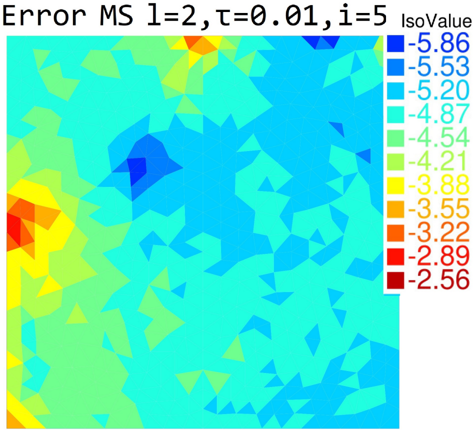


Error, $\tau = 1$

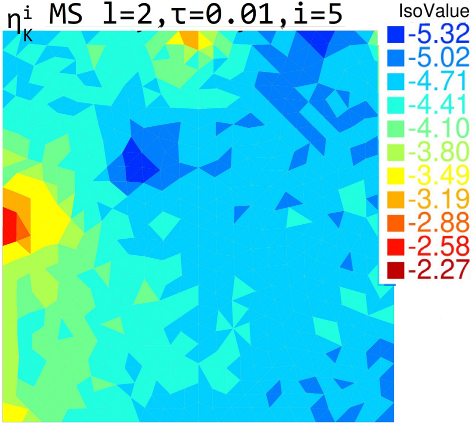


Estimate, $\tau = 1$

Where is the error **localized**?

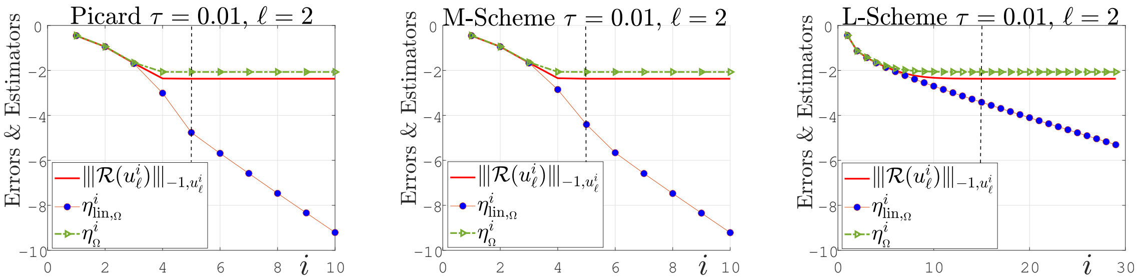


Error, $\tau = 0.01$



Estimate, $\tau = 0.01$

Error components and adaptivity via **stopping criteria**



Time step length $\tau = 0.01$

Outline

Conclusions

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- a posteriori **certification** of the **error** for nonlinear problems
- **robustness** with respect to the **strength of nonlinearities**
- augmenting the energy difference by the (discretization) error on the given linearization step
- employing iteration-dependent norms

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Thank you for your attention!