Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling

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joint work with **Gergina Pencheva, Mary Wheeler, Tim Wildey**
(CSM, ICES, Austin)

Linz, October 3, 2011

Outline

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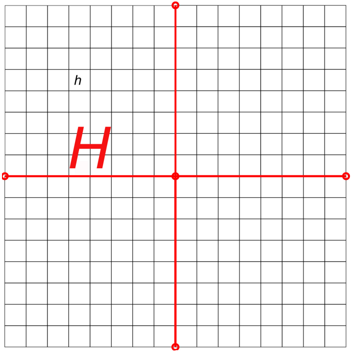
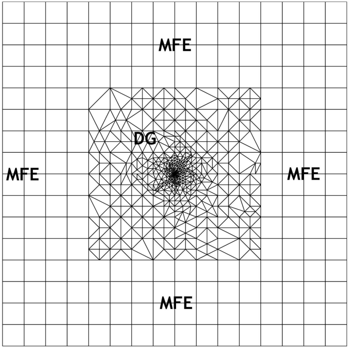
Multiscale



Multiscale

- subdomain meshes of size h (low order polynomials)
- interface meshes of size H (high order polynomials)

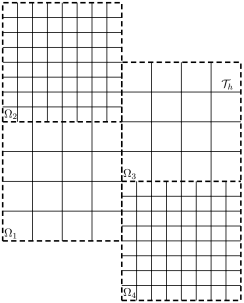
Multinumerics



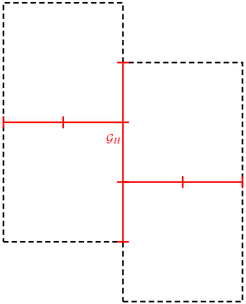
Multinumerics

- different numerical methods in different subdomains

Mortar coupling



Nonmatching subd. grids



Interface grid

Mortar coupling

- mortars used to enforce weakly mass conservation over the interface grid
- effective parallel implementation: independent local subd. problems, only the mortar unknowns globally coupled

Aims of this work

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- derive **guaranteed** a posteriori error estimates

$$\|p - p_h\| \leq \eta(p_h)$$

- ensure their **local efficiency**

$$\eta_T \leq C\|p - p_h\|_{\text{neighbors of } T}$$

- look for **robustness** with respect to the ratio H/h (the constant C is independent of the ratio H/h)
- bound separately the **subdomain** and **interface errors**
- propose an **adaptive strategy** which balances the subdomain and interface errors
- develop a **unified setting** encompassing different numerical methods

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Previous works

Multiscale/multinumerics/mortars

- Arbogast, Pencheva, Wheeler, Yotov (2007) (multiscale mortar mixed finite element method)
- Girault, Sun, Wheeler, Yotov (2008) (coupling DG and MFE by mortars)

A posteriori error estimates

- Prager and Synge (1947) (error equality)
- Ladevèze and Leguillon (1983) and Repin (1997) (application to a posteriori error estimation)
- Wohlmuth (1999) / Bernardi and Hecht (2002) (mortars)
- Wheeler and Yotov (2005) (mortar MFE)
- Aarnes and Efendiev (2006) / Larson and Målqvist (2007) (multiscale)
- Ainsworth / Kim / Ern, Nicaise, Vohralík (2007) (DG)
- Vohralík (2007, 2010) (MFE)
- Creusé and Nicaise (2008) (multinumerics)

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Setting

Model problem

$$-\nabla\cdot(\mathbf{K}\nabla p) = f \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, polygonal
- $\mathbf{K}$ is symmetric, bounded, and uniformly positive definite
- $f \in L^2(\Omega)$

Potential and flux

- p: **potential** (pressure head); $p \in H_0^1(\Omega)$
- $\mathbf{u} := -\mathbf{K}\nabla p$: **flux** (Darcy velocity); $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$, $\nabla\cdot\mathbf{u} = f$

Energy (semi-)norms

- $|||\varphi|||^2 := \|\mathbf{K}^{\frac{1}{2}}\nabla\varphi\|^2$, $\varphi \in H^1(\mathcal{T}_h)$
- $|||\mathbf{v}|||_*^2 := \|\mathbf{K}^{-\frac{1}{2}}\mathbf{v}\|^2$, $\mathbf{v} \in \mathbf{L}^2(\Omega)$

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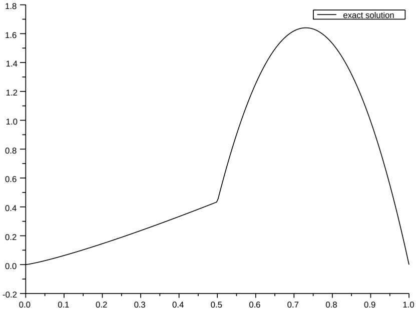
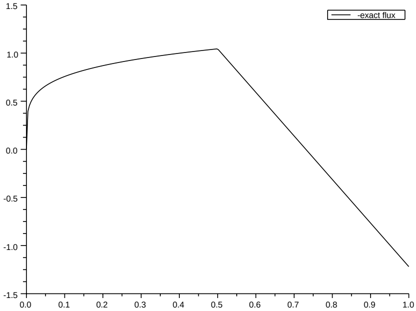
Outline

Exact potential and exact flux

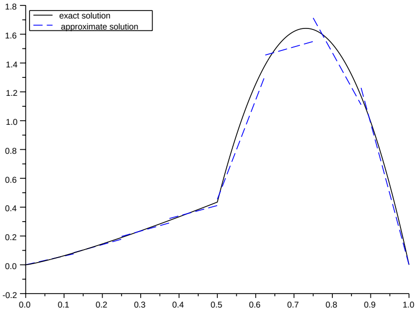


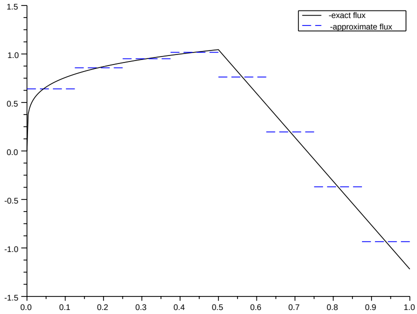


Potential p is in $H_0^1(\Omega)$

Flux $\mathbf{u}$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate potential and approximate flux

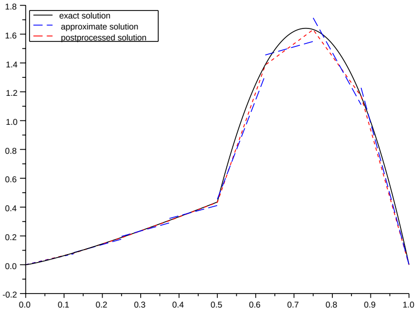




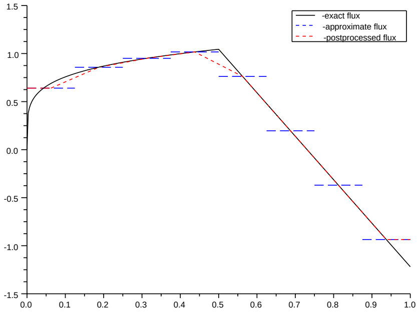
Approximate potential p_h is not in $H_0^1(\Omega)$

Approximate flux $\mathbf{u}_h$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Potential and flux reconstructions



A postprocessed potential s_h is in $H^1_0(\Omega)$



A postprocessed flux $\mathbf{t}_h$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Estimates for the flux

Theorem (Estimate for the flux)

Let $\mathbf{u}$ *be the exact flux and let* $\mathbf{u}_h \in \mathbf{L}^2(\Omega)$ *be arbitrary. Let* $s_h \in H_0^1(\Omega)$ *be arbitrary and let* $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be arbitrary s.t.*

$$(\nabla\cdot\mathbf{t}_h, 1)_T = (f, 1)_T \qquad \forall T \in \mathcal{T}_\hbar.$$

Then

$$|||\mathbf{u} - \mathbf{u}_h|||_* \le \eta_{\mathrm{P}} + \eta_{\mathrm{R},\hbar} + \eta_{\mathrm{M}},$$

with the potential, residual, and mortar estimators given by

$$\eta_{\mathrm{P}} := |||\mathbf{u}_h + \mathbf{K}\nabla s_h|||_*,$$

$$\eta_{\mathrm{R},\hbar} := \left\{ \sum_{T\in\mathcal{T}_\hbar} C_{\mathrm{P},T}^2 \hbar_T^2 c_{\mathbf{K},T}^{-1} \|f - \nabla\cdot\mathbf{t}_h\|_T^2 \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{M}} := |||\mathbf{u}_h - \mathbf{t}_h|||_*.$$

Properties

- s_h: potential reconstruction; $\mathbf{t}_h$: flux reconstruction
- $\eta_{\mathrm{R},\hbar}$: typically a higher-order data oscillation term
- $\mathcal{T}_\hbar$ to be specified, $\mathcal{T}_h$, $\widehat{\mathcal{T}}_h$, or $\mathcal{T}_H$

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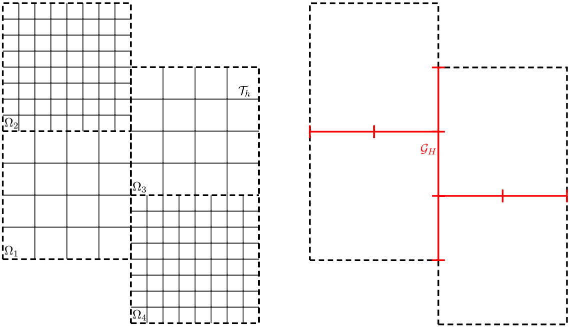
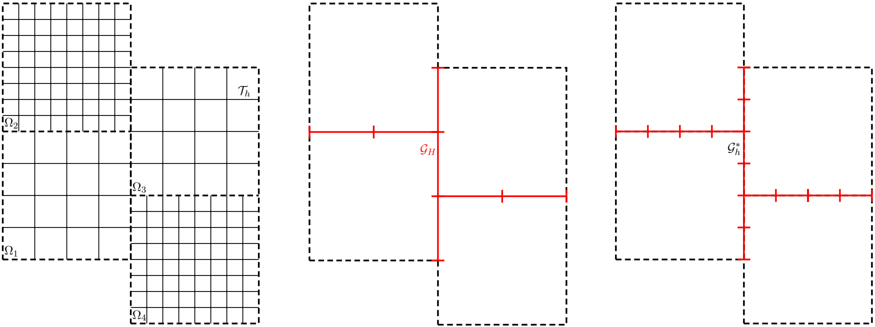
Interface meshes



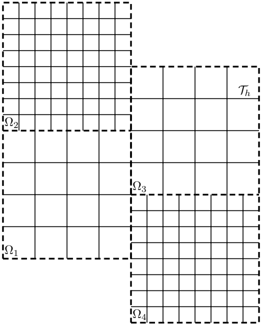
Nonmatching mesh $\mathcal{T}_h$ and given interface mesh $\mathcal{G}_H$

Interface meshes



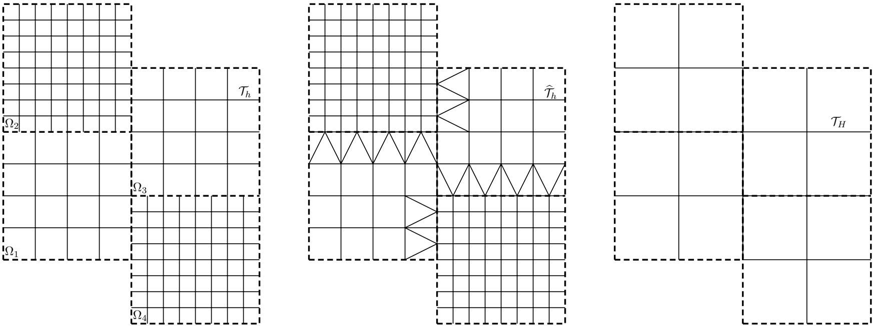
Nonmatching mesh $\mathcal{T}_h$, given interface mesh $\mathcal{G}_H$, and intersection interface mesh $\mathcal{G}_h^*$

Subdomain meshes



Nonmatching mesh $\mathcal{T}_h$

Subdomain meshes

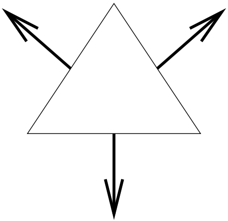


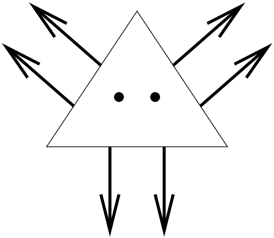
Nonmatching mesh $\mathcal{T}_h$, matching refinement $\widehat{\mathcal{T}}_h$ of $\mathcal{T}_h$, and a mesh $\mathcal{T}_H$

Function spaces

Function spaces

- $W_h := \mathbb{R}_k(\mathcal{T}_h)$: potential space, piecewise polynomials of order k
- $\mathbf{V}_h := \bigoplus_{i=1}^{n} \mathbf{V}_{h,i}$, $\mathbf{V}_{h,i} := \mathbf{RTN}^k(\mathcal{T}_{h,i})$: flux space, Raviart–Thomas–Nédélec spaces of order k inside each subdomain
- M_H: mortar space, discontinuous piecewise polynomials of order m on the interface mesh $\mathcal{G}_H$, $m > k$ when $h \ll H$





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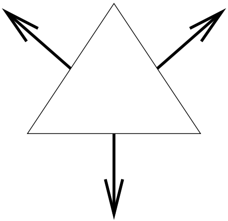
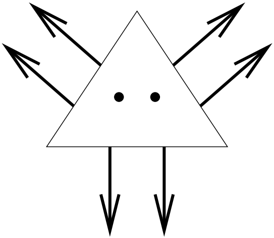




Function spaces

Function spaces

- $W_h := \mathbb{R}_k(\mathcal{T}_h)$: potential space, piecewise polynomials of order k
- $\mathbf{V}_h := \bigoplus_{i=1}^n \mathbf{V}_{h,i}$, $\mathbf{V}_{h,i} := \mathbf{RTN}^k(\mathcal{T}_{h,i})$: flux space, Raviart–Thomas–Nédélec spaces of order k inside each subdomain
- M_H: mortar space, discontinuous piecewise polynomials of order m on the interface mesh $\mathcal{G}_H$, $m > k$ when $h \ll H$

Outline

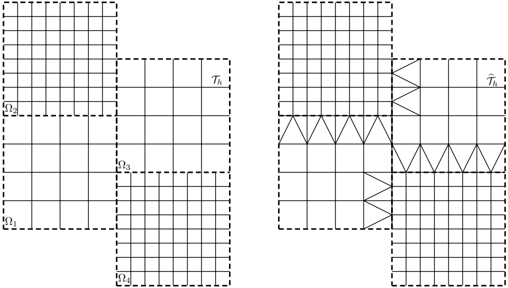
Potential reconstruction

Potential reconstruction

- every piecewise polynomial on $\mathcal{T}_h$ is also a piecewise polynomial on $\widehat{\mathcal{T}}_h$
- averaging interpolate $\mathcal{I}_{\mathrm{av}} : \mathbb{R}_{k'}(\widehat{\mathcal{T}}_h) \to \mathbb{R}_{k'}(\widehat{\mathcal{T}}_h) \cap H_0^1(\Omega)$:

$$\mathcal{I}_{\mathrm{av}}(\varphi_h)(V) = \frac{1}{|\widehat{\mathcal{T}}_V|} \sum_{T \in \widehat{\mathcal{T}}_V} \varphi_h|_T(V)$$

- $s_h := \mathcal{I}_{\mathrm{av}}(p_h)$



General assumption on the approximate flux

Assumption (Properties of $\mathbf{u}_h$)

We suppose that

1. $\mathbf{u}_h \in \mathbf{V}_h$ *$\mathbf{u}_h$ is from the $\mathbf{RTN}^k(\mathcal{T}_{h,i})$ space, $i \in \{1, \ldots, n\}$;*
2. $(\nabla\cdot\mathbf{u}_h, 1)_T = (f, 1)_T \quad \forall T \in \mathcal{T}_h$ *local conservation inside each subdomain Ω_i on the elements of $\mathcal{T}_{h,i}$;*
3. $\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot \mathbf{n}_{\Omega_i}, \mu_H \rangle_{\Gamma_i} = 0 \quad \forall \mu_H \in M_H$ *normal trace of $\mathbf{u}_h$ weakly continuous (in the sense of the mortar space) across the interface sides.*

Consequences

- $$F|_{\Gamma_{i,j}} := P_{M_H}((\mathbf{u}_h|_{\Omega_i}\cdot\mathbf{n}_\Gamma)|_{\Gamma_{i,j}}) = P_{M_H}((\mathbf{u}_h|_{\Omega_j}\cdot\mathbf{n}_\Gamma)|_{\Gamma_{i,j}}).$$
- $$\langle \mathbf{u}_h|_{\Omega_i}\cdot\mathbf{n}_g, 1\rangle_g = \langle \mathbf{u}_h|_{\Omega_j}\cdot\mathbf{n}_g, 1\rangle_g = \langle \{\!\{\mathbf{u}_h\cdot\mathbf{n}_g\}\!\}, 1\rangle_g = \langle F, 1\rangle_g,\ g \in \mathcal{G}_{H,i,j}$$

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Flux reconstruction 1. MFE low order h-grid-size local Neumann problems

Flux reconstruction by MFE solution of local Neumann problems (Ern and Vohralík (2009))

- $\mathbf{t}_h \in \mathbf{V}_{\widehat{h}}$, Neumann BCs given by $\langle \{\{\mathbf{u}_h \cdot \mathbf{n}_g\}\}, 1\rangle_g$
-

$$(\mathbf{K}^{-1}(\mathbf{t}_h - \mathbf{u}_h), \mathbf{v}_h)_T - (q_h, \nabla\cdot\mathbf{v}_h)_T = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_{\widehat{h},0,T},$$

$$(\nabla\cdot\mathbf{t}_h, w_h)_T = (f, w_h)_T \qquad \forall w_h \in W_{\widehat{h}}(T) \text{ such that } (w_h, 1)_T = 0.$$

Properties

- $\nabla\cdot\mathbf{t}_h = P_{W_{\widehat{h}}}(f)$
- low order (k-th order RTN) polynomial $\mathbf{t}_h$
- local linear system to be solved (H-sized macroelements T with h-sized grids)
- overestimation in the multiscale setting when $h \ll H$

Flux reconstruction 1. MFE low order *h*-grid-size local Neumann problems

Flux reconstruction by MFE solution of local Neumann problems (Ern and Vohralík (2009))

- $\mathbf{t}_h \in \mathbf{V}_{\widehat{h}}$, Neumann BCs given by $\langle \{\!\{\mathbf{u}_h \cdot \mathbf{n}_g\}\!\}, 1\rangle_g$
-

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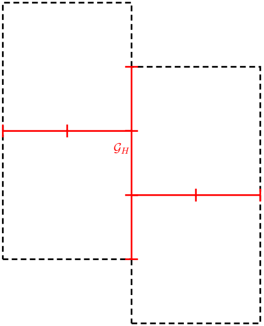

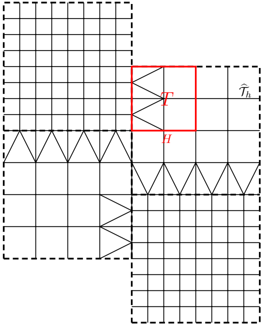

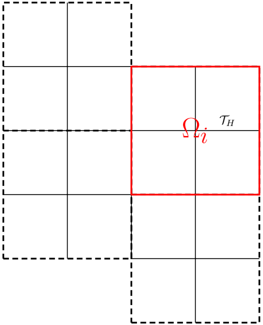

Interface mesh $\mathcal{G}_H$ and: flux reconstruction 1 flux reconstruction 2

Flux reconstruction 2. MFE high order *H*-grid-size local Neumann problems

Flux reconstruction by MFE solution of local Neumann problems (Ern and Vohralík (2009))

- $\mathbf{t}_h \in \mathbf{V}_H$, Neumann BCs given by F
-

$$(\mathbf{K}^{-1}(\mathbf{t}_h - \mathbf{u}_h), \mathbf{v}_H)_{\Omega_i} - (q_H, \nabla\cdot\mathbf{v}_H)_{\Omega_i} = 0 \qquad \forall \mathbf{v}_H \in \mathbf{V}_{H,0,\Omega_i},$$
$$(\nabla\cdot\mathbf{t}_h, w_H)_{\Omega_i} = (f, w_H)_{\Omega_i} \qquad \forall w_H \in W_H(\Omega_i) \text{ such that } (w_H, 1)_{\Omega_i} = 0.$$

Properties

- $\nabla\cdot\mathbf{t}_h = P_{W_H}(f)$
- high order (m-th order RTN) polynomial $\mathbf{t}_h$
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- local linear system to be solved (subdomains Ω_i with H-sized grids)
- optimal estimation in the multiscale setting when $h \ll H$

Outline

General assumption on the approximate potential

Assumption (Properties of $\tilde{p}_h$)

Let

1. $\tilde{p}_h \in \mathbb{R}_r(\mathcal{T}_h)$ *for some* $r \geq 1$ $\tilde{p}_h$ *is a piecewise polynomial,*
2. $\langle [\![\tilde{p}_h]\!], 1\rangle_e = 0 \quad \forall e \in \mathcal{E}_h^{\mathrm{int}} \cup \mathcal{E}_h^{\mathrm{ext}}$ *means of traces of* $\tilde{p}_h$ *on interior sides in each subdomain are continuous, zero on the boundary,*
3. $\langle [\![\tilde{p}_h]\!], 1\rangle_g = 0 \quad \forall g \in \mathcal{G}_h^*$ *means of traces on collections of sides inside the interface* Γ *are continuous.*

Local efficiency

Theorem (Local efficiency, part I)

Let $\tilde{p}_h \in H^1(\mathcal{T}_h)$, $\mathbf{u}_h \in \mathbf{L}^2(\Omega)$, $s_h \in H_0^1(\Omega)$, and $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ be arbitrary. Then, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} \eta_{\mathrm{DF},T} &\leq |||\mathbf{u} - \mathbf{u}_h|||_{*,T} + |||p - \tilde{p}_h|||_T, \\ \eta_{\mathrm{P},T} &\leq \eta_{\mathrm{DF},T} + \eta_{\mathrm{NC},T}, \\ \eta_{\mathrm{DFM},T} &\leq \eta_{\mathrm{DF},T} + \eta_{\mathrm{M},T}. \end{aligned}$$

Let the Assumption on $\tilde{p}_h$ hold and let $s_h \in \mathbb{R}_{r'}(\widehat{\mathcal{T}}_h)$ be given by $s_h := \mathcal{I}_{\mathrm{av}}(\tilde{p}_h)$. Then, for all $T \in \mathcal{T}_h$,

$$\begin{aligned} \eta_{\mathrm{NC},T} &\lesssim |||p - \tilde{p}_h|||_{\mathfrak{T}_T} && \text{if } T \cap \Gamma = \emptyset, \\ \eta_{\mathrm{NC},T} &\lesssim |||p - \tilde{p}_h|||_{\mathfrak{T}_{T,\Gamma}} && \text{if } T \cap \Gamma \neq \emptyset. \end{aligned}$$

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Local efficiency

Theorem (Local efficiency, part II)

Let the Assumption on $\mathbf{u}_h$ hold. Let construction 1 of $\mathbf{t}_h$ be used. Then

$$\eta_{\mathrm{R},\widehat{h},T} \lesssim |||\mathbf{u} - \mathbf{u}_h|||_{*,T},$$

$$\eta_{\mathrm{M},T} \lesssim \sqrt{\frac{H_T}{h_{\mathfrak{T}_{T,\Gamma}}}}\,|||\mathbf{u} - \mathbf{u}_h|||_{*,\mathfrak{T}_{T,\Gamma}}.$$

Let construction 2 of $\mathbf{t}_h$ be used. Let the exact solution be smooth enough. Then

$$\eta_{\mathrm{R},H,T} \lesssim (\eta_{\mathrm{M},T} + |||\mathbf{u} - \mathbf{u}_h|||_T),$$

$$\eta_{\mathrm{M},\Omega_i} \leq |||\mathbf{u}_h - \mathbf{u}|||_{*,\Omega_i} + \eta_{\mathrm{R},h,\Omega_i} + CH^{m+1}.$$

Observation

- the term CH^{m+1} is superconvergent in the multiscale mortar mixed finite element method

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Outline

Outline

Multiscale mortar mixed finite element method

Multiscale mortar mixed finite element method (Arbogast, Pencheva, Wheeler, Yotov (2007))

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_H \in M_H$ such that,

$$
\begin{aligned}
&(\mathbf{K}^{-1}\mathbf{u}_h, \mathbf{v}_h)_{\Omega_i} - (p_h, \nabla\cdot\mathbf{v}_h)_{\Omega_i} + \langle \lambda_H, \mathbf{v}_h\cdot\mathbf{n}_{\Omega_i}\rangle_{\Gamma_i} = 0 && \forall\, \mathbf{v}_h \in \mathbf{V}_{h,i},\ \forall i,\\
&(\nabla\cdot\mathbf{u}_h, w_h)_{\Omega_i} = (f, w_h)_{\Omega_i} && \forall w_h \in W_{h,i},\ \forall i,\\
&\sum_{i=1}^{n} \langle \mathbf{u}_h\cdot\mathbf{n}_{\Omega_i}, \mu_H\rangle_{\Gamma_i} = 0 && \forall \mu_H \in M_H.
\end{aligned}
$$

Remarks

- p_h needs to be postprocessed to $\tilde{p}_h$
- direct application of the framework (both $\tilde{p}_h$ and $\mathbf{u}_h$ satisfy perfectly our Assumptions)

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$$\begin{aligned}
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&(\nabla\cdot\mathbf{u}_h, w_h)_{\Omega_i} = (f, w_h)_{\Omega_i} && \forall w_h \in W_{h,i},\ \forall i,\\
&\sum_{i=1}^{n} \langle \mathbf{u}_h\cdot\mathbf{n}_{\Omega_i}, \mu_H\rangle_{\Gamma_i} = 0 && \forall \mu_H \in M_H.
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- p_h needs to be postprocessed to $\tilde{p}_h$
- direct application of the framework (both $\tilde{p}_h$ and $\mathbf{u}_h$ satisfy perfectly our Assumptions)

Outline

Multiscale mortar discontinuous Galerkin method

Multiscale mortar discontinuous Galerkin method

Find $p_h \in W_h$ and $\lambda_H \in M_H$ such that

$$\mathcal{B}_{h,i}(p_h, \lambda_H; \varphi_h) = (f, \varphi_h)_{\Omega_i} \qquad \forall \varphi_h \in W_{h,i}, \forall i \in \{1, \ldots, n\},$$

$$\sum_{i=1}^{n} \sum_{g \in \mathcal{G}_{H,i}} \left\langle -\mathbf{K}\nabla p_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i} + \alpha_g \frac{\sigma_{\mathbf{K},g}}{H_g} (p_h|_{\Omega_i} - \pi_{k,\mathcal{E}^{\Gamma}_{h,i}}(\lambda_H)), \mu_H \right\rangle_g = 0 \qquad \forall \mu_H \in M_H,$$

where

$$\begin{aligned}
\mathcal{B}_{h,i}(p_h, \lambda_H; \varphi_h) := &- \sum_{e \in \mathcal{E}^{\mathrm{int}}_{h,i}} \{\langle \{\!\{\mathbf{K}\nabla p_h \cdot \mathbf{n}_e\}\!\}, [\![\varphi_h]\!] \rangle_e + \theta \langle \{\!\{\mathbf{K}\nabla \varphi_h \cdot \mathbf{n}_e\}\!\}, [\![p_h]\!] \rangle_e\} \\
&- \sum_{g \in \mathcal{G}_{H,i}} \left\{ \left\langle \mathbf{K}\nabla p_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i} - \alpha_g \frac{\sigma_{\mathbf{K},g}}{H_g} (p_h|_{\Omega_i} - \lambda_H), \varphi_h|_{\Omega_i} \right\rangle_g \right. \\
&\left. + \bar{\theta} \langle \mathbf{K}\nabla \varphi_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i}, p_h|_{\Omega_i} - \lambda_H \rangle_g \right\} \\
&+ (\mathbf{K}\nabla p_h, \nabla \varphi_h)_{\Omega_i} + \sum_{e \in \mathcal{E}^{\mathrm{int}}_{h,i}} \left\langle \alpha_e \frac{\sigma_{\mathbf{K},e}}{h_e} [\![p_h]\!], [\![\varphi_h]\!] \right\rangle_e.
\end{aligned}$$

Remarks

- the flux $\mathbf{u}_h$ satisfying our Assumption needs to be recovered first

Multiscale mortar discontinuous Galerkin method

Multiscale mortar discontinuous Galerkin method

Find $p_h \in W_h$ and $\lambda_H \in M_H$ such that

$$\mathcal{B}_{h,i}(p_h, \lambda_H; \varphi_h) = (f, \varphi_h)_{\Omega_i} \qquad \forall \varphi_h \in W_{h,i},\ \forall i \in \{1, \ldots, n\},$$

$$\sum_{i=1}^{n} \sum_{g \in \mathcal{G}_{H,i}} \left\langle -\mathbf{K}\nabla p_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i} + \alpha_g \frac{\sigma_{\mathbf{K},g}}{H_g} (p_h|_{\Omega_i} - \pi_{k,\mathcal{E}^{\Gamma}_{h,i}}(\lambda_H)), \mu_H \right\rangle_g = 0 \qquad \forall \mu_H \in M_H,$$

where

$$\begin{aligned} \mathcal{B}_{h,i}(p_h, \lambda_H; \varphi_h) := & - \sum_{e \in \mathcal{E}^{\mathrm{int}}_{h,i}} \{ \langle \{\!\{ \mathbf{K}\nabla p_h \cdot \mathbf{n}_e \}\!\}, [\![\varphi_h]\!] \rangle_e + \theta \langle \{\!\{ \mathbf{K}\nabla \varphi_h \cdot \mathbf{n}_e \}\!\}, [\![p_h]\!] \rangle_e \} \\ & - \sum_{g \in \mathcal{G}_{H,i}} \left\{ \left\langle \mathbf{K}\nabla p_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i} - \alpha_g \frac{\sigma_{\mathbf{K},g}}{H_g} (p_h|_{\Omega_i} - \lambda_H), \varphi_h|_{\Omega_i} \right\rangle_g \right. \\ & \left. + \bar{\theta} \langle \mathbf{K}\nabla \varphi_h|_{\Omega_i} \cdot \mathbf{n}_{\Omega_i}, p_h|_{\Omega_i} - \lambda_H \rangle_g \right\} \\ & + (\mathbf{K}\nabla p_h, \nabla \varphi_h)_{\Omega_i} + \sum_{e \in \mathcal{E}^{\mathrm{int}}_{h,i}} \left\langle \alpha_e \frac{\sigma_{\mathbf{K},e}}{h_e} [\![p_h]\!], [\![\varphi_h]\!] \right\rangle_e . \end{aligned}$$

Remarks

- the flux $\mathbf{u}_h$ satisfying our Assumption needs to be recovered first

Flux recovery in MS MDG

Flux recovery (Ern, Nicaise, and Vohralík (2007))

Let $T \in \mathcal{T}_h$. The recovered flux $\mathbf{u}_h|_T \in \mathbf{V}_h(T)$ is given by

$$\langle \mathbf{u}_h{\cdot}\mathbf{n}_e, q_h\rangle_e = \left\langle -\{\!\{\mathbf{K}\nabla p_h{\cdot}\mathbf{n}_e\}\!\} + \alpha_e \frac{\sigma_{\mathbf{K},e}}{h_e}[\![p_h]\!], q_h\right\rangle_e \quad \forall q_h \in \mathbb{R}_k(e),\ \forall e \in \mathcal{E}_T,\ e \not\subset \Gamma,$$

$$\langle \mathbf{u}_h{\cdot}\mathbf{n}_e, q_h\rangle_e = \left\langle -\mathbf{K}\nabla p_h{\cdot}\mathbf{n}_e + \alpha_g \frac{\sigma_{\mathbf{K},g}}{H_g}(p_h - \lambda_H), q_h\right\rangle_e \quad \forall q_h \in \mathbb{R}_k(e),\ \forall e \in \mathcal{E}_T,\ e \subset g \in \mathcal{G}_H,$$

$$(\mathbf{u}_h, \mathbf{r}_h)_T = -(\mathbf{K}\nabla p_h, \mathbf{r}_h)_T + \theta \sum_{e \in \mathcal{E}_T,\, e \not\subset \Gamma} \omega_e \langle \mathbf{K}\mathbf{r}_h{\cdot}\mathbf{n}_e, [\![p_h]\!]\rangle_e + \bar{\theta} \sum_{e \in \mathcal{E}_T,\, e \subset \Gamma} \langle \mathbf{K}\mathbf{r}_h{\cdot}\mathbf{n}_e, (p_h - \lambda_H)\mathbf{n}_T{\cdot}\mathbf{n}_e\rangle_e \quad \forall \mathbf{r}_h \in \mathbb{R}_{k-1,*,d}(T).$$

Outline

Discontinuous Galerkin elements coupled with mixed finite elements

Principle of the application of our framework

- recover the flux in the DG method so that $\mathbf{u}_h \in \mathbf{H}(\mathrm{div}, \Omega_i)$ for all i, $\nabla\cdot\mathbf{u}_h = \pi_k(f)$, and $\sum_{i=1}^{n} \langle \mathbf{u}_h\cdot\mathbf{n}_{\Omega_i}, \mu_H\rangle_{\Gamma_i} = 0$ for all $\mu_H \in M_H$ (satisfied by the recovery above)
- rewrite the mortar coupling with the aid of the DG flux $\mathbf{u}_h$ and the MFE flux $\mathbf{u}_h$
- use the previous results on MS MMFE / MS MDG

Outline

A simplification without flux reconstruction

Theorem (Simplified estimate without flux reconstruction)

Let $\mathbf{u}$ be the exact flux and let p be the exact potential. Let the Assumption on $\mathbf{u}_h$ be satisfied and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $s_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq \eta_{\mathrm{P}} + \eta_{\mathrm{R},h} + \tilde{\eta}_{\mathrm{M}},$$

$$|||p - \tilde{p}_h||| \leq \eta_{\mathrm{NC}} + \eta_{\mathrm{R},h} + \tilde{\eta}_{\mathrm{M}} + \eta_{\mathrm{DF}},$$

where

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Properties

- no flux reconstruction needed
- contains the (explicitly known) constants $C_{\mathrm{t},T_{i,g},g}$
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A simplification without flux reconstruction

Theorem (Simplified estimate without flux reconstruction)

Let $\mathbf{u}$ be the exact flux and let p be the exact potential. Let the Assumption on $\mathbf{u}_h$ be satisfied and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $s_h \in H_0^1(\Omega)$ be arbitrary. Then

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Outline

Outline

Mortar MFEs

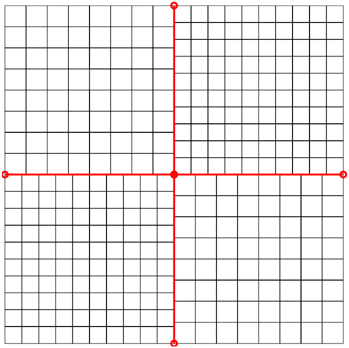
Setting

- $\Omega := (0,1) \times (0,1)$,
-

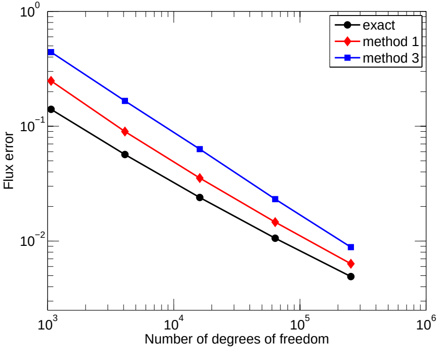
$$\mathbf{K} := \begin{cases} 15 - 10\sin(10\pi x)\sin(10\pi y), & x, y \in (0, 1/2) \\ & \text{or } x, y \in (1/2, 1), \\ 15 - \sin(2\pi x)\sin(2\pi y), & \text{otherwise}, \end{cases}$$

- $p(x, y) = x(1-x)y(1-y)$
- mortar MFEs, $k = 0$, $m = 1$
- H/h fixed

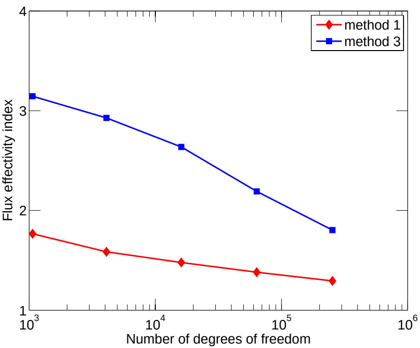
Initial mesh



Estimates, error, and effectivity indices

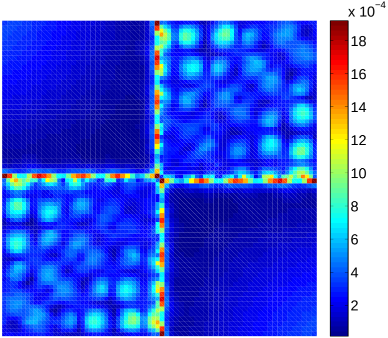


Estimated and exact flux error

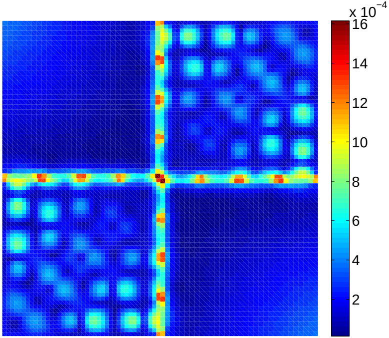


Effectivity indices

Error distribution



Estimated error distribution inside the subdomains and along the mortar interfaces



Exact error distribution inside the subdomains and along the mortar interfaces

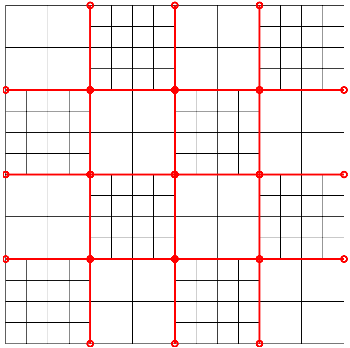
Outline

Multiscale mortar MFEs

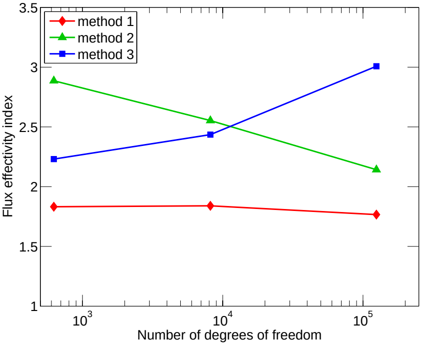
Setting

- $\Omega := (0,1) \times (0,1)$,
- $$\mathbf{K} := \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$
- $p(x,y) = \sin(2\pi x)\sin(2\pi y)$
- multiscale mortar MFEs, $k = 0$, $m = 2$ or even $m = 1$
- $H \approx \sqrt{h}$

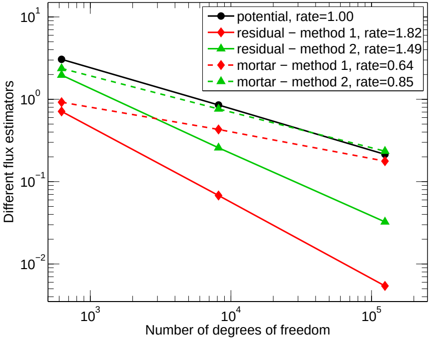
Initial mesh



Estimates, error, and effectivity indices

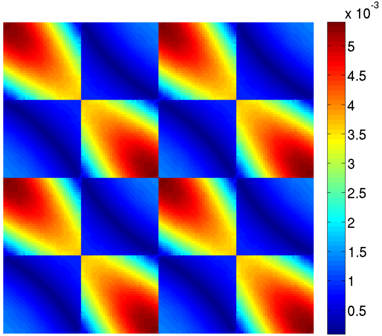


Effectivity indices

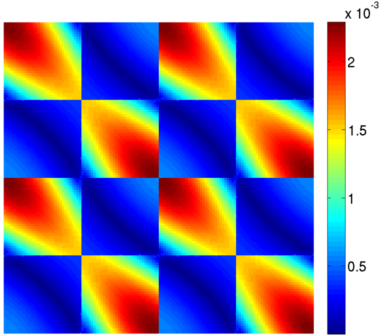


Different estimators

Error distribution



Estimated error distribution



Exact error distribution

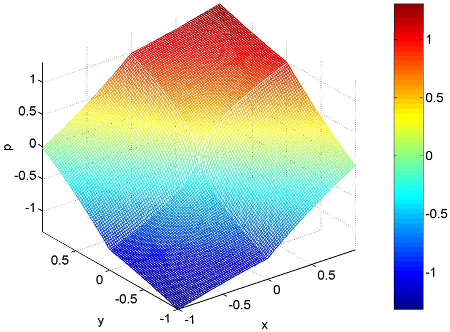
Outline

Coupled DG–MFE

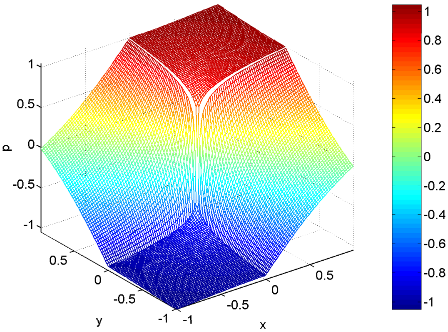
Setting

- $\Omega := (-1,1) \times (-1,1)$,
- $$\mathbf{K} := \begin{cases} 5 & (x,y) \in (-1,0) \times (-1,0) \\ & \text{or } (x,y) \in (0,1) \times (0,1), \\ 1 & \text{otherwise}, \end{cases}$$
- $p(r,\theta)|_i = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$,
- the exact solution has a singularity at the origin
- coupled DG–MFE

Exact solution

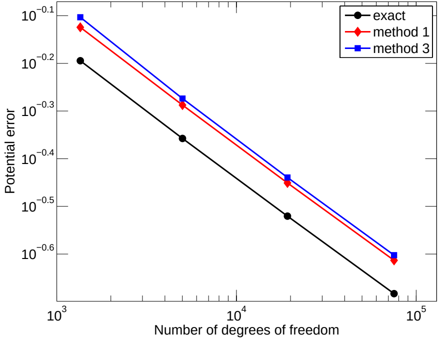


$\alpha = 0.53$

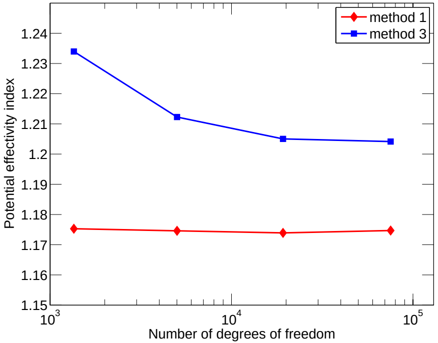


$\alpha = 0.12$

Estimates, error, and effectivity indices for uniform refinement

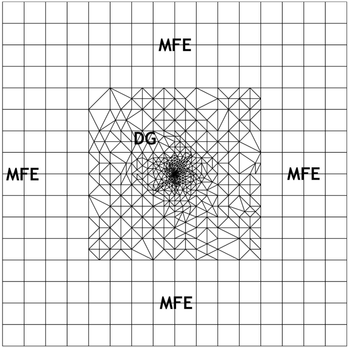


Estimated and exact potential error

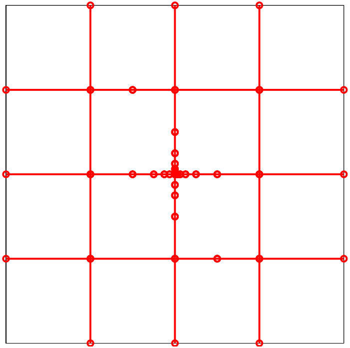


Effectivity indices

Adaptive meshes

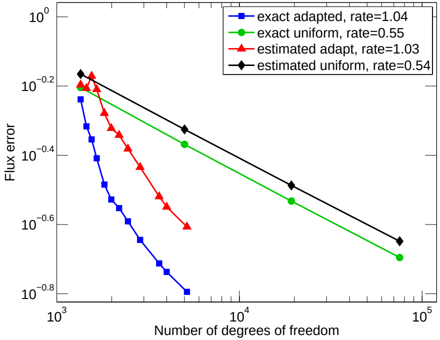


Adapted mesh in multinumerics DG–MFE discretization

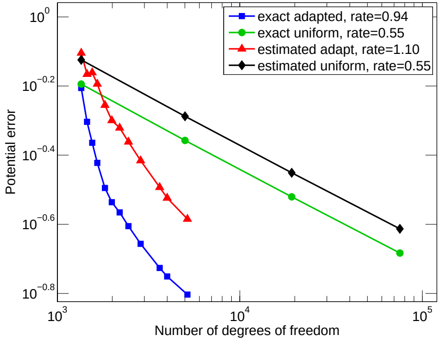


Corresponding adapted mortar mesh

Estimates and errors for adaptive refinement



Estimated and actual flux error



Estimated and actual potential error

Outline

Conclusions and future work

Conclusions

- guaranteed, locally efficient, and possibly robust estimates
- unified setting (two conditions need to be verified in order to apply the framework)

Future work

- robustness without subdomain solves and sufficient regularity?
- upscaling?

Thank you for your attention!

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