A posteriori error estimates in numerical simulations

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INRIA Paris-Rocquencourt

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Outline

1. Introduction
2. Steady linear problems: space mesh adaptation
   - Potential and flux reconstructions
   - A guaranteed a posteriori error estimate
   - Polynomial-degree-robust local efficiency
   - Application and numerical results
3. Nonlinear problems: stopping algebraic & linearization solvers
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Application and numerical results
4. Unsteady problems: time step adaptation
   - A guaranteed a posteriori error estimate
   - Application and numerical results
5. Conclusions and future directions

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What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE.
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|u - u_h\|_{\cdot,\Omega} \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|u - u_h\|_{\cdot,\Omega} \lesssim f(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u-u_h)\| \le \left\{ \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \le C\|\nabla(u-u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{ \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u-u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- **error components identification**

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Model diffusion problem

Model diffusion problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where

- $\underline{\mathbf{K}} : \Omega \to \mathbb{R}^{d\times d}$ is a diffusion tensor ($\underline{\mathbf{K}} \in [L^\infty(\Omega)]^{d\times d}$, uniformly positive definite),
- $f : \Omega \to \mathbb{R}$ is a source term ($f \in L^2(\Omega)$).

Form in 1D

Let $\Omega =]a, b[$, $a < b$. Let $k :]a, b[\to \mathbb{R}$ and $f :]a, b[\to \mathbb{R}$ be two given functions. Find $u :]a, b[\to \mathbb{R}$ such that

$$-(ku')' = f,$$
$$u(a) = u(b) = 0.$$

Weak formulation

Find $u \in V := H_0^1(\Omega)$ such that

$$(\underline{\mathbf{K}}\nabla u, \nabla v) = (f, v) \qquad \forall v \in V.$$

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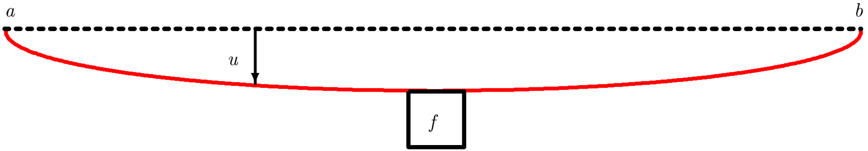
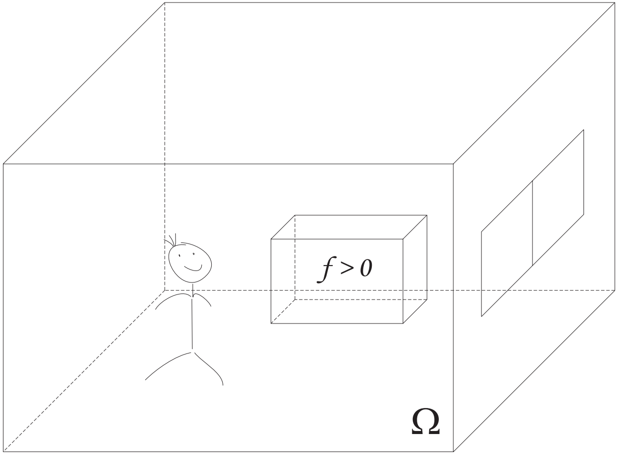
Example: elastic string



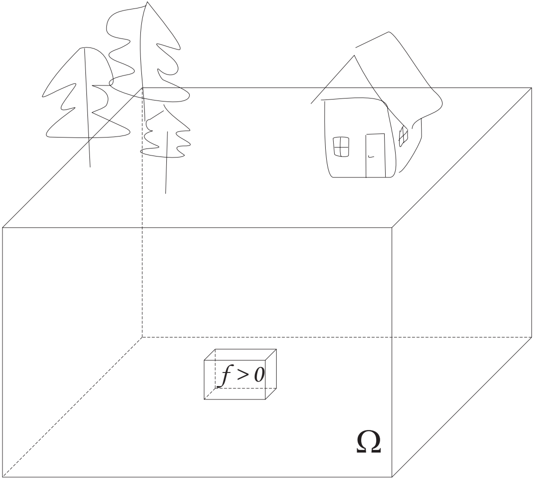
Elastic string with displacement u and weight f

Example: heat flow



A room with a heater of $f > 0$ and temperature u

Example: underground water flow



Underground with a water well of $f > 0$ and pressure head u

Previous results

General result

- Prager and Synge (1947) ($\underline{\mathbf{K}} = \underline{\mathbf{I}}$):

$$\|\nabla u + \boldsymbol{\sigma}_h\|^2 + \|\nabla(u - u_h)\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2$$

for **any** $u_h \in H_0^1(\Omega)$ and **any** $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla\cdot\boldsymbol{\sigma}_h = f$

- a posteriori estimate: how to practically construct $\boldsymbol{\sigma}_h$?
- Hlaváček, Haslinger, Nečas, and Lovíšek (1979), Repin (1997) . . . global construction: unprecise/costly

Local flux reconstructions

- Ladevèze and Leguillon (1983), equilibrated face fluxes
- Destuynder and Métivet (1999), discrete flux $\boldsymbol{\sigma}_h$
- Luce and Wohlmuth (2004), local efficiency
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Local potential reconstructions ($u_h \notin H_0^1(\Omega)$)

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- Carstensen and Merdon (2013), Ern and Vohralík (2013 preprint), local Neumann problems by MFEs

Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
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Properties of the weak solution

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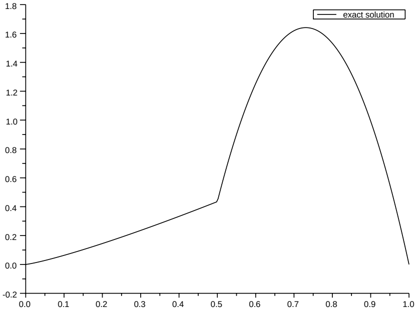
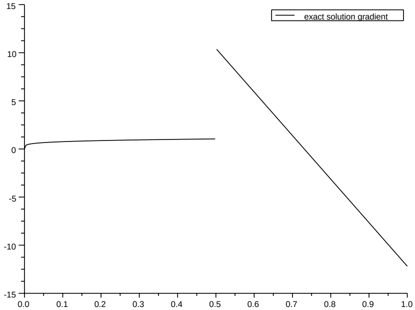
- $u \in H_0^1(\Omega)$ (constraint)
- $\boldsymbol{\sigma} = -\underline{\mathbf{K}}\nabla u$ (constitutive law)
- $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)
- $\nabla{\cdot}\boldsymbol{\sigma} = f$ (equilibrium)

Properties of the weak solution



Solution u (displacement, temperature, pressure . . .) is continuous



Solution gradient ∇u (derivative u' in 1D) is not necessarily continuous

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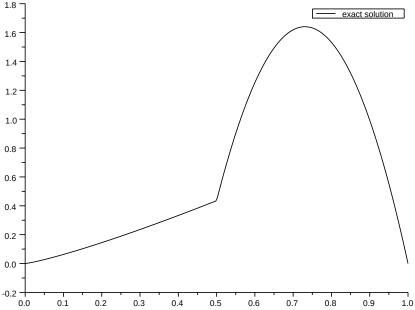
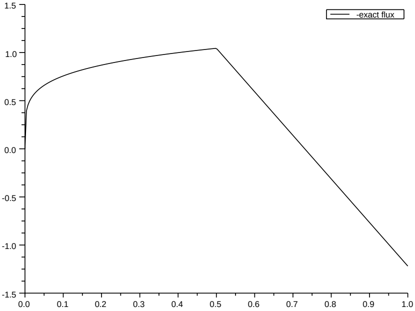
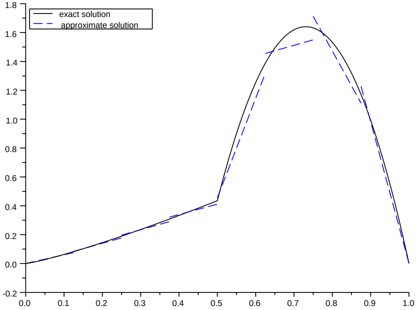


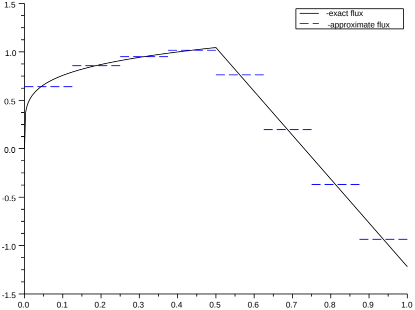
Solution u is continuous



Flux $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u$ (or $-ku'$ in 1D) is continuous

Approximate solution and approximate flux

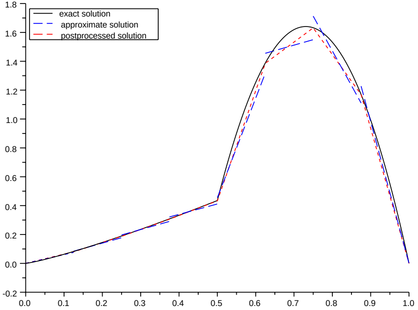




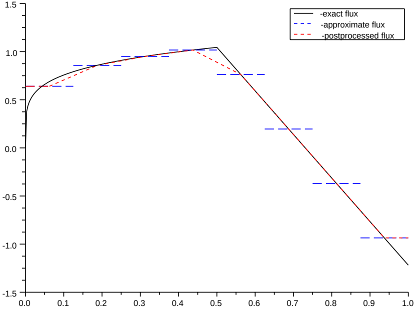
Approximate solution u_h is not necessarily continuous

Approximate flux $-\underline{\mathbf{K}}\nabla u_h$ $(-ku_h')$ is not necessarily continuous

Potential and flux reconstructions



Potential reconstruction

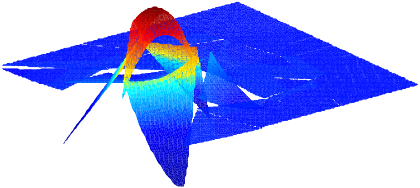


Flux reconstruction

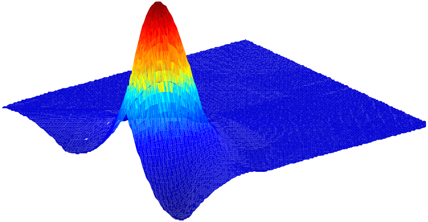
Potential reconstruction

Definition (Potential reconstruction)

A function s_h constructed from u_h verifying $s_h \in H_0^1(\Omega)$.



Potential u_h



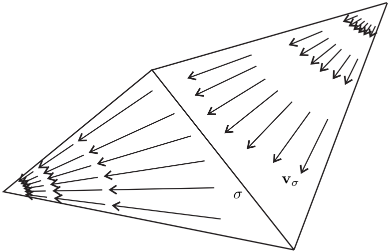
Potential reconstruction s_h

Equilibrated flux reconstruction

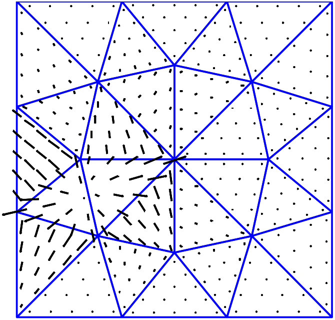
Definition (Equilibrated flux reconstruction)

A function $\boldsymbol{\sigma}_h$ constructed from u_h verifying

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega),$$
$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \qquad \forall K \in \mathcal{T}_h.$$



Raviart–Thomas–Nédélec lowest-order basis function



Flux reconstruction $\boldsymbol{\sigma}_h$

Outline

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2. Steady linear problems: space mesh adaptation
   - Potential and flux reconstructions
   - A guaranteed a posteriori error estimate
   - Polynomial-degree-robust local efficiency
   - Application and numerical results
3. Nonlinear problems: stopping algebraic & linearization solvers
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4. Unsteady problems: time step adaptation
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5. Conclusions and future directions

A posteriori error estimate ($\underline{\mathbf{K}} = \underline{\boldsymbol{I}}$)

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in H_0^1(\Omega)$ *be the weak solution,*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^1(K) \forall K \in \mathcal{T}_h\}$ *be arbitrary,*
- s_h *be a potential reconstruction,* $\boldsymbol{\sigma}_h$ *be an equilibrated flux reconstruction.*

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} \left(\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K \right)^2 + \sum_{K \in \mathcal{T}_h} \|\nabla(u_h - s_h)\|_K^2.$$

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Practical flux reconstruction

Assumption A (Galerkin orthogonality)

There holds

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}.$$

Definition (Construction of σ_h, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\varsigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the local MFE problem

$$\begin{aligned} (\varsigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ (\nabla\cdot\varsigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}}, \end{aligned}$$

with $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ mixed finite element spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, hom. Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$). Set

$$\sigma_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \varsigma_h^{\mathbf{a}}.$$

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- The same problems, only RHS/BC different.

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Flux reconstruction

Theorem (Continuous efficiency, Carstensen & Funken (1999), Braess, Pillwein, and Schöberl (2009))

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ solve

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, v)_{\omega_{\mathbf{a}}} \quad \forall v \in H^1_*(\omega_{\mathbf{a}})$$

with $H^1_*(\omega_{\mathbf{a}}) := \{v \in H^1(\omega_{\mathbf{a}});\ (v,1)_{\omega_{\mathbf{a}}} = 0 / v = 0 \text{ on } \partial\omega_{\mathbf{a}} \cap \partial\Omega\}$. Then there exists a constant $C_{\mathrm{cont,PF}} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Potential reconstruction ($d = 2$)

Assumption B (Weak continuity)

There holds $$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

Theorem (Continuous efficiency)

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ satisfying Assumption B be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ solve

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h), \nabla v)_{\omega_{\mathbf{a}}} + (0, v)_{\omega_{\mathbf{a}}} \quad \forall v \in H^1_*(\omega_{\mathbf{a}})$$

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Mixed finite elements stability ($d = 2$)

Theorem (MFE stability, Braess, Pillwein, and Schöberl (2009))

Let u be the weak solution and let u_h and f be piecewise polynomial. Consider corresponding polynomial degree MFE reconstructions. Then there exists a constant $C_{\rm st} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\varsigma_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\rm st}\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}},$$

with $\boldsymbol{\tau}_h^{\mathbf{a}} = \psi_{\mathbf{a}}\nabla u_h$ for the flux reconstruction and $\boldsymbol{\tau}_h^{\mathbf{a}} = \mathrm{R}_{\frac{\pi}{2}}\nabla(\psi_{\mathbf{a}} u_h)$ for the potential reconstruction.

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution and let u_h and f be piecewise polynomial. Let u_h satisfy Assumptions A and B. Then, for corresponding polynomial degree MFE reconstructions,

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,PF}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,bPF}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- maximal overestimation factor guaranteed
- C_{st} can be bounded by solving the local Neumann problems by a conforming FEs

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Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- $V_h \subset H_0^1(\Omega)$: $s_h := u_h$, no need for Assumption B

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Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \nabla u_h \}\!\} \cdot \mathbf{n}_e, [\![v_h]\!] \rangle_e + \theta \langle \{\!\{ \nabla v_h \}\!\} \cdot \mathbf{n}_e, [\![u_h]\!] \rangle_e \}$$
$$+ \sum_{e \in \mathcal{E}_h} \langle \alpha h_e^{-1} [\![u_h]\!], [\![v_h]\!] \rangle_e = (f, v_h) \qquad \forall v_h \in V_h$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
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Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\boldsymbol{\sigma}_h, q_h) = (f, q_h) \qquad \forall q_h \in Q_h.$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, $v_h \in V_h$ satisfy

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Numerics: finite elements in 1D

Model problem

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\begin{aligned}
-u'' &= \pi^2 sin(\pi x) \quad \text{in } (0,1), \\
u &= 0 \quad \text{in } 0,1
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Exact solution

$$u(x) = sin(\pi x)$$

Discretization

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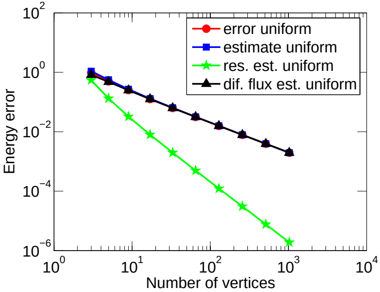
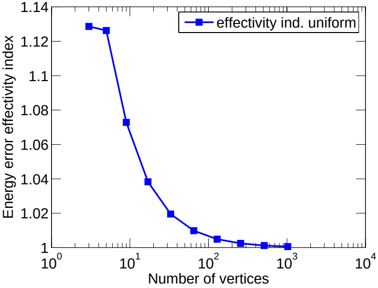
Estimated and actual errors, effectivity index



Actual error and estimator and its components



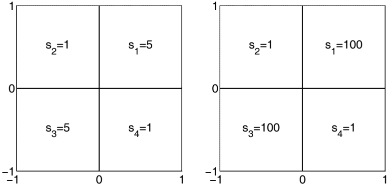
Effectivity index

Numerics: cell-centered finite volumes (adjustments)

Model problem

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla u) = 0 \quad \text{in } \Omega :=]-1,1[\times]-1,1[,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

with discontinuous and inhomogeneous $\underline{\mathbf{K}}$, two cases:



Exact solution (singularity at the origin)

$$u(r,\theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

- (r,θ) polar coordinates in Ω
- a_i, b_i constants depending on Ω_i
- α regularity of the solution

Discretization: cell-centered finite volumes

Numerics: cell-centered finite volumes (adjustments)

Model problem

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla u) = 0 \quad \text{in } \Omega :=]-1,1[\times]-1,1[,$$
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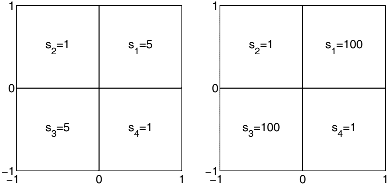

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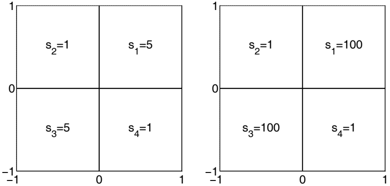

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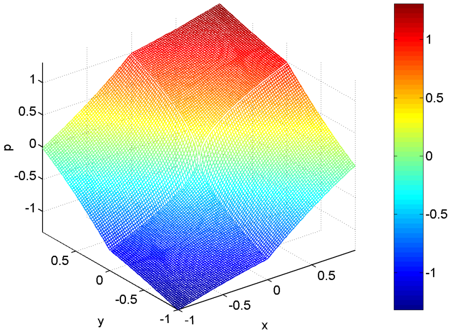
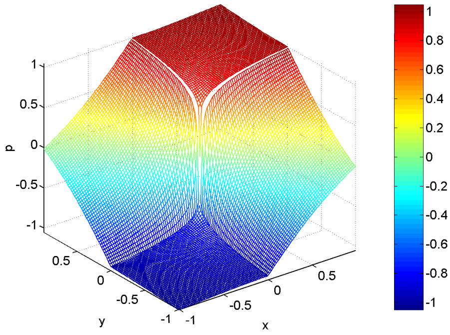
Discretization: cell-centered finite volumes

Analytical solutions

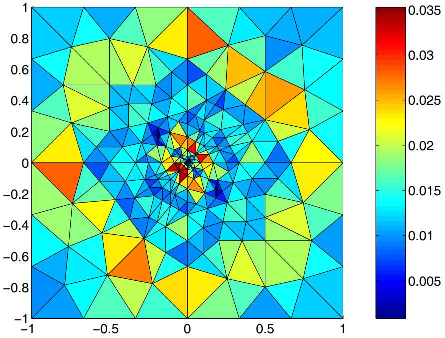


Case 1

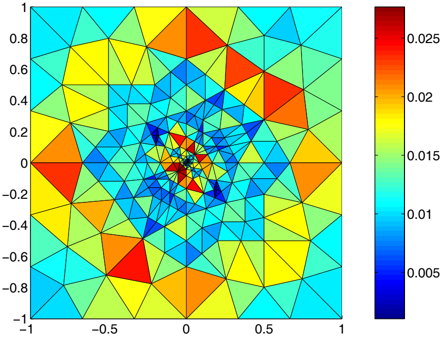


Case 2

Error distribution on an adaptively refined mesh, case 1

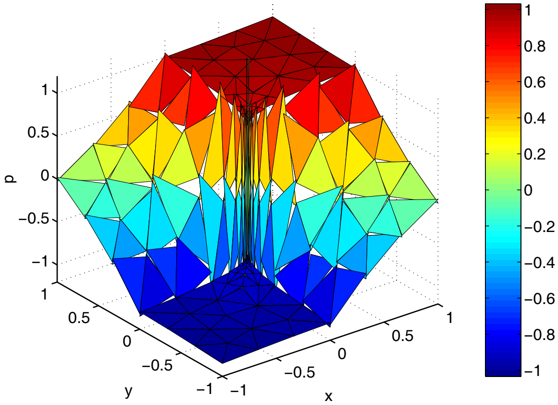


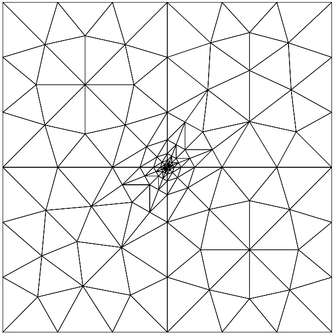
Estimated error distribution



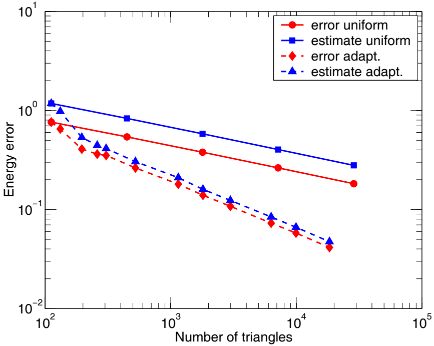
Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, case 2

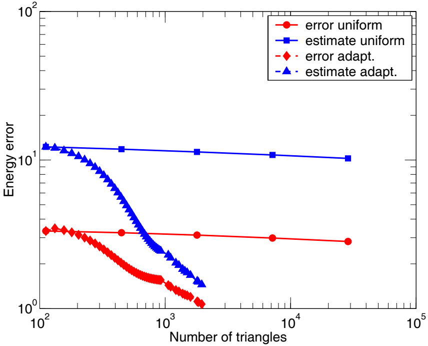




Estimated and actual errors in uniformly/adaptively refined meshes

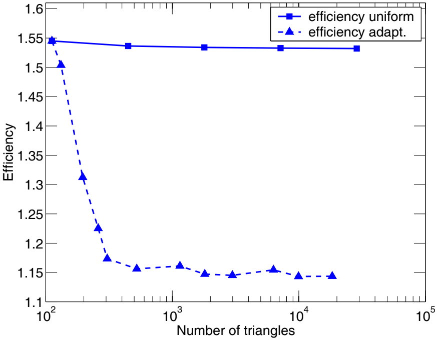


Case 1



Case 2

Effectivity indices in uniformly/adaptively refined meshes



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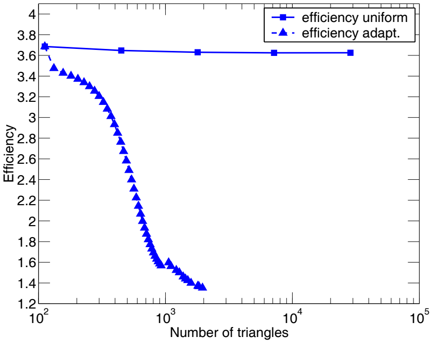

Case 2

Numerics: discontinuous Galerkin

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega :=]0,1[\times]0,1[, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

incomplete interior penalty discontinuous Galerkin method

Numerics: discontinuous Galerkin

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Numerics: discontinuous Galerkin

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u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

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Discretization

incomplete interior penalty discontinuous Galerkin method

Estimates, errors, effectivity indices (calc. V. Dolejší)

h	p	$\|u-u_h\|$	$\|\nabla(u-u_h)\|$	$\|\nabla u_h+\boldsymbol{\sigma}_h\|$	$\frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_h\|$	$\|\nabla(u_h-s_h)\|$	η	$I_{\rm eff}$
1.3E-01	1	1.39E-02	4.98E-01	5.32E-01	4.34E-02	3.52E-02	5.71E-01	1.15
6.3E-02		3.85E-03	2.67E-01	2.80E-01	6.29E-03	1.96E-02	2.86E-01	1.07
(EOC)		(1.85)	(0.90)	(0.93)	(2.79)	(0.85)	(1.00)	
3.1E-02		9.90E-04	1.37E-01	1.42E-01	8.19E-04	1.01E-02	1.43E-01	1.04
(EOC)		(1.96)	(0.97)	(0.98)	(2.94)	(0.96)	(1.00)	
1.6E-02		2.49E-04	6.89E-02	7.12E-02	1.03E-04	5.01E-03	7.15E-02	1.04
(EOC)		(1.99)	(0.99)	(0.99)	(2.99)	(1.01)	(1.00)	
1.3E-01	2	2.08E-03	9.38E-02	9.48E-02	8.83E-03	1.27E-02	1.03E-01	1.10
6.3E-02		4.02E-04	2.61E-02	2.62E-02	6.37E-04	4.07E-03	2.71E-02	1.04
(EOC)		(2.37)	(1.84)	(1.85)	(3.79)	(1.65)	(1.93)	
3.1E-02		8.26E-05	6.75E-03	6.76E-03	4.13E-05	1.19E-03	6.90E-03	1.02
(EOC)		(2.28)	(1.95)	(1.96)	(3.95)	(1.77)	(1.97)	
1.6E-02		1.85E-05	1.71E-03	1.71E-03	2.61E-06	3.22E-04	1.74E-03	1.02
(EOC)		(2.16)	(1.98)	(1.99)	(3.99)	(1.88)	(1.99)	
1.3E-01	3	2.29E-04	1.55E-02	1.51E-02	1.38E-03	2.36E-03	1.66E-02	1.07
6.3E-02		1.65E-05	2.20E-03	2.15E-03	4.98E-05	3.85E-04	2.23E-03	1.01
(EOC)		(3.80)	(2.81)	(2.81)	(4.79)	(2.62)	(2.89)	
3.1E-02		1.08E-06	2.85E-04	2.80E-04	1.62E-06	5.25E-05	2.86E-04	1.01
(EOC)		(3.94)	(2.95)	(2.94)	(4.94)	(2.87)	(2.97)	
1.6E-02		6.77E-08	3.58E-05	3.54E-05	5.10E-08	6.62E-06	3.60E-05	1.01
(EOC)		(3.99)	(2.99)	(2.98)	(4.99)	(2.99)	(2.99)	
1.3E-01	4	2.64E-05	2.28E-03	2.17E-03	1.69E-04	3.46E-04	2.37E-03	1.04
6.3E-02		1.08E-06	1.63E-04	1.57E-04	3.05E-06	3.25E-05	1.63E-04	1.00
(EOC)		(4.61)	(3.80)	(3.79)	(5.79)	(3.41)	(3.86)	
3.1E-02		4.70E-08	1.05E-05	1.02E-05	4.96E-08	2.46E-06	1.05E-05	1.00
(EOC)		(4.52)	(3.95)	(3.94)	(5.94)	(3.73)	(3.95)	
1.3E-01	5	2.69E-06	2.78E-04	2.60E-04	1.69E-05	4.39E-05	2.81E-04	1.01
6.3E-02		4.92E-08	9.98E-06	9.46E-06	1.53E-07	1.91E-06	9.80E-06	0.98
(EOC)		(5.77)	(4.80)	(4.78)	(6.78)	(4.53)	(4.84)	
3.1E-02		7.58E-10	3.22E-07	3.10E-07	1.77E-09	6.46E-08	3.18E-07	0.99
(EOC)		(6.02)	(4.95)	(4.93)	(6.44)	(4.88)	(4.94)	

Outline

1. Introduction
2. Steady linear problems: space mesh adaptation
   - Potential and flux reconstructions
   - A guaranteed a posteriori error estimate
   - Polynomial-degree-robust local efficiency
   - Application and numerical results
3. Nonlinear problems: stopping algebraic & linearization solvers
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Application and numerical results
4. Unsteady problems: time step adaptation
   - A guaranteed a posteriori error estimate
   - Application and numerical results
5. Conclusions and future directions

Inexact iterative linearization

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. $U^{k-1} \Rightarrow$ *matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*
   $$\mathbb{A}^{k-1}U^k \approx F^{k-1}.$$
3. 1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*
      $$\mathbb{A}^{k-1}U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
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Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

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Previous results

Inexact Newton method

- Eisenstat and Walker (1990's) (conception, convergence, a priori error estimates)
- Moret (1989) (discrete a posteriori error estimates)

Adaptive inexact Newton method

- Bank and Rose (1982), combination with multigrid
- Hackbusch and Reusken (1989), damping and multigrid
- Deuflhard (1990's, 2004 book), adaptive damping and multigrid

Stopping criteria for algebraic solvers

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid stopping criterion
- Arioli (2000's), comparison of the algebraic and discretization errors by a priori arguments

Previous results

Inexact Newton method

- Eisenstat and Walker (1990's) (conception, convergence, a priori error estimates)
- Moret (1989) (discrete a posteriori error estimates)

Adaptive inexact Newton method

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Previous results

A posteriori error estimates for numerical discretizations of nonlinear problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors
- Kim (2007), guaranteed estimates, locally conservative methods

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

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- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

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Approximate solution

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Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

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   - Application and numerical results
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A posteriori error estimate

Assumption A (Total quasi-equilibrated flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \;\; \forall K \in \mathcal{T}_h$.

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

A posteriori error estimate

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Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization **error** flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic **error** flux reconstruction*

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- $\mathbf{l}_h^{k,i}$: *linearization* ***error*** *flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic* ***error*** *flux reconstruction*

Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
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- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}.$$

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Estimators

- *discretization* estimator
$$\eta_{\mathrm{disc},K}^{k,i} := 2^{\frac{1}{p}} \left(\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{ \sum_{e\in\mathcal{E}_K} h_e^{1-q} \| [\![u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q} \right)$$
- *linearization* estimator
$$\eta_{\mathrm{lin},K}^{k,i} := \|\mathbf{l}_h^{k,i}\|_{q,K}$$
- *algebraic* estimator
$$\eta_{\mathrm{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_{q,K}$$
- *algebraic remainder estimator*
$$\eta_{\mathrm{rem},K}^{k,i} := h_\Omega \|\rho_h^{k,i}\|_{q,K}$$
- *quadrature estimator*
$$\eta_{\mathrm{quad},K}^{k,i} := \|\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) - \overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K}$$
- *data oscillation estimator*
$$\eta_{\mathrm{osc},K}^{k,i} := C_{\mathrm{P},p} h_K \|f - f_h\|_{q,K}$$
- $$\eta_{\cdot}^{k,i} := \left\{ \sum_{K\in\mathcal{T}_h} \left(\eta_{\cdot,K}^{k,i}\right)^q \right\}^{1/q}$$

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Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{k,i} \leq \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},$$
$$\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem},K}^{k,i} \leq \gamma_{\mathrm{rem},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}, \eta_{\mathrm{alg},K}^{k,i}\} \quad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{alg},K}^{k,i} \leq \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \quad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \leq \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \quad \forall K \in \mathcal{T}_h$$

- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

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Assumption for efficiency

Assumption C (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} \lesssim \eta_{\sharp,\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i},$$

where

$$\eta_{\sharp,\mathfrak{T}_K}^{k,i} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f_h + \nabla \cdot \overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K'}^q + \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\overline{\boldsymbol{\sigma}}_h^{k,i} \cdot \mathbf{n}_e]\!]\|_{q,e}^q + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}.$$

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

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Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta^{k,i}_{\mathrm{disc},K} + \eta^{k,i}_{\mathrm{lin},K} + \eta^{k,i}_{\mathrm{alg},K} + \eta^{k,i}_{\mathrm{rem},K} \lesssim \mathcal{J}^{\mathrm{up}}_{u,\mathfrak{T}_K}(u^{k,i}_h) + \eta^{k,i}_{\mathrm{quad},\mathfrak{T}_K} + \eta^{k,i}_{\mathrm{osc},\mathfrak{T}_K}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

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Outline

1. Introduction
2. Steady linear problems: space mesh adaptation
   - Potential and flux reconstructions
   - A guaranteed a posteriori error estimate
   - Polynomial-degree-robust local efficiency
   - Application and numerical results
3. Nonlinear problems: stopping algebraic & linearization solvers
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Application and numerical results
4. Unsteady problems: time step adaptation
   - A guaranteed a posteriori error estimate
   - Application and numerical results
5. Conclusions and future directions

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

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Nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f_h, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- $f_h := \Pi_0 f$
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

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Linearization

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$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f_h, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2} \boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2} \boldsymbol{\xi} + (p-2) |\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

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Algebraic solution

Algebraic solution
Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f_h, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\mathrm{int}}}$
- discrete system

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Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

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Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{\mathrm{quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

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Verification of the assumptions – upper bound

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_0(\mathcal{S}_h) \Rightarrow \mathbf{a}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$.

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Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\sigma(\nabla u_h^{k,i})$
- approximation properties of Raviart–Thomas–Nédélec spaces

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Discontinuous Galerkin for the quasi-linear diffusion

Discretization

Find $u_h \in V_h := \mathbb{P}_m(\mathcal{T}_h)$, $m \geq 1$, such that, for all $v_h \in V_h$,

$$(\boldsymbol{\sigma}(u_h, \nabla u_h), \nabla v_h) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}(u_h, \nabla u_h) \}\!\} \cdot \mathbf{n}_e, [\![v_h]\!] \rangle_e$$

$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}(u_h) \nabla v_h \}\!\} \cdot \mathbf{n}_e, [\![u_h]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h]\!], [\![v_h]\!] \rangle_e = (f, v_h).$$

- $\theta \in \{-1, 0, 1\}$
- $\bar{\alpha}_e := \|\underline{\mathbf{A}}\|_{L^\infty(\mathbb{R})} \chi_e$, χ_e large enough
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

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- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization $\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi$
- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi + (v - u_h^{k-1}) \partial_v \underline{\mathbf{A}}(u_h^{k-1}) \nabla u_h^{k-1},$$

$$\underline{\mathbf{A}}^{k-1}(v) := \underline{\mathbf{A}}(u_h^{k-1}) + \partial_v \underline{\mathbf{A}}(u_h^{k-1})(v - u_h^{k-1})$$

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Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

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$$= (f, \psi_{K,j}) - R_{K,j}^{k,i}.$$

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Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}) \in \mathbf{RTN}_l(\mathcal{T}_h)$, $l := m-1/m$)

For all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_K$,

$$\langle(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})\cdot\mathbf{n}_e, q_h\rangle_e := \langle -\{\!\{\boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i})\}\!\}\cdot\mathbf{n}_e + \bar{\alpha}_e h_e^{-1}[\![u_h^{k,i}]\!], q_h\rangle_e,$$

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Verification of the assumptions – upper bound

Definition (Construction of f_h, $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $f_h := \Pi_l f$ and $\overline{\boldsymbol{\sigma}}_h^{k,i} := \mathbf{I}_l^{\mathrm{RTN}}(\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}))$.

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_l(\mathcal{T}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$.

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Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

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- $\mathbf{d}_h^{k,i}$ close to $\overline{\sigma}_h^{k,i}$
- approximation properties of Raviart–Thomas–Nédélec spaces

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Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

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Numerical experiment I

Model problem

- p-Laplacian

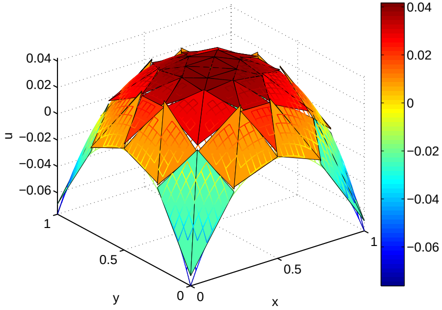
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

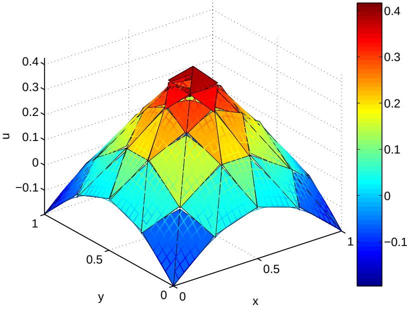
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- nonconforming finite elements

Analytical and approximate solutions

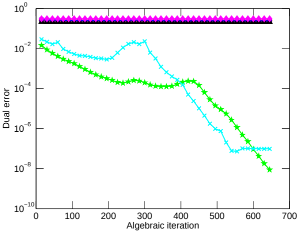


Case $p = 1.5$

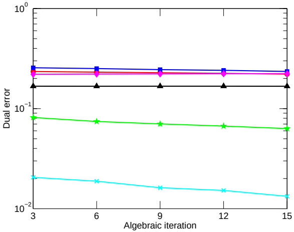


Case $p = 10$

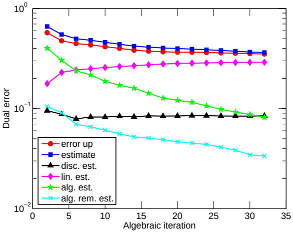
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



Newton

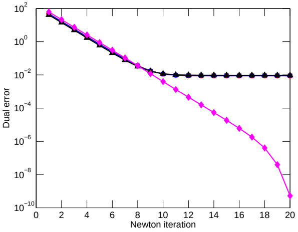


inexact Newton

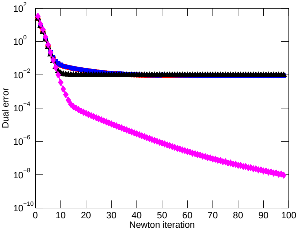


ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



Newton



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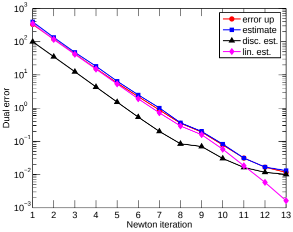
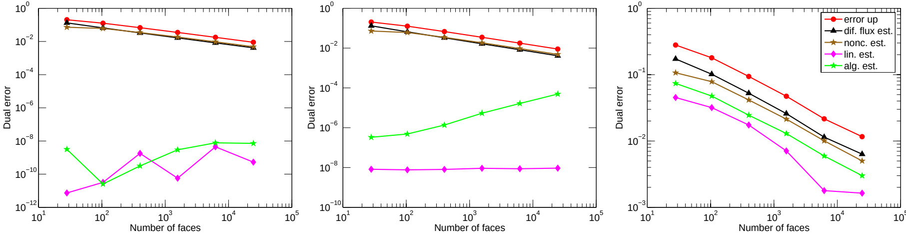


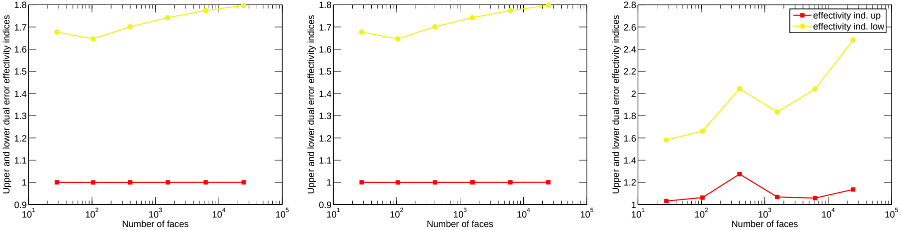
ad. inexact Newton

Error and estimators, $p = 10$



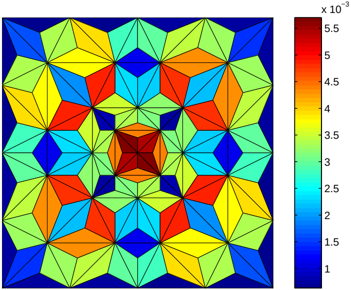
Newton   inexact Newton   ad. inexact Newton

Effectivity indices, $p = 10$

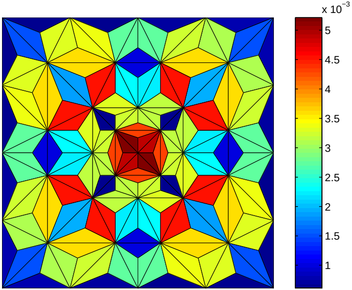


Newton inexact Newton ad. inexact Newton

Error distribution, $p = 10$

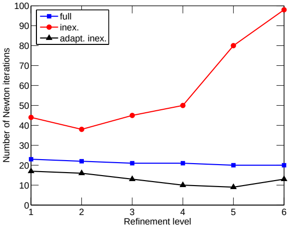


Estimated error distribution

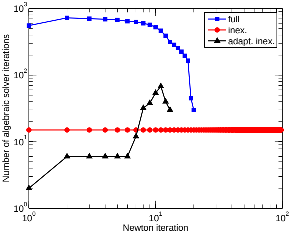


Exact error distribution

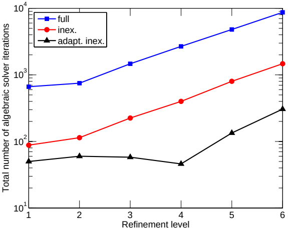
Newton and algebraic iterations, $p = 10$



Newton it. / refinement

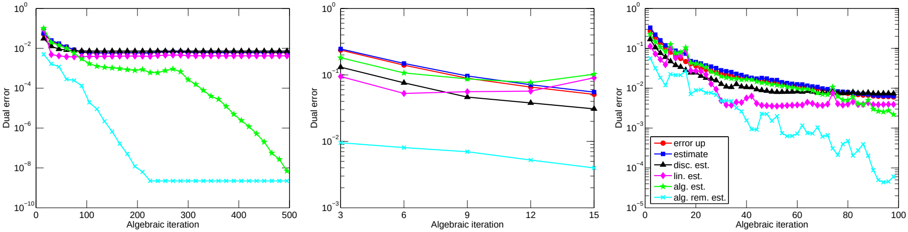


alg. it. / Newton step



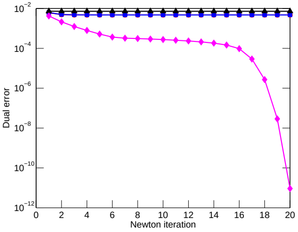
alg. it. / refinement

Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.

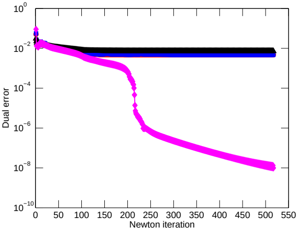


Newton inexact Newton ad. inexact Newton

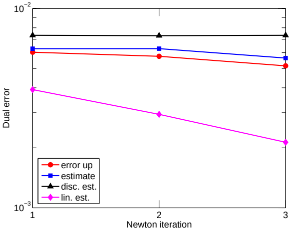
Error and estimators as a function of Newton iterations, $p = 1.5$, 6th level mesh



Newton



inexact Newton



ad. inexact Newton

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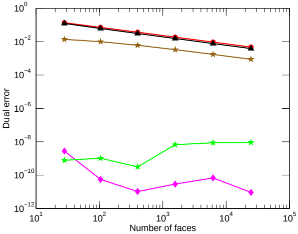

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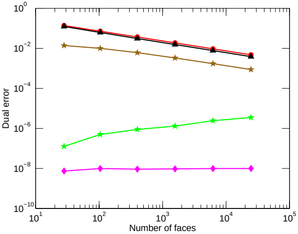

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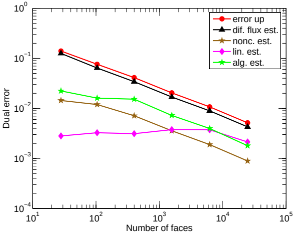
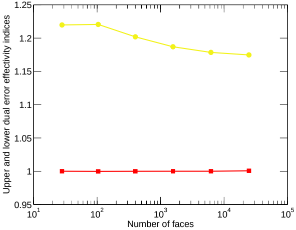
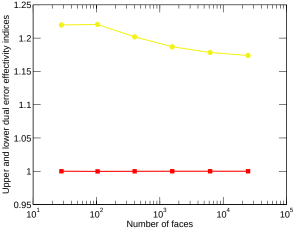


ad. inexact Newton

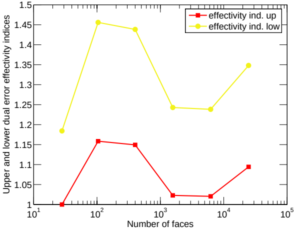
Effectivity indices, $p = 1.5$



Newton

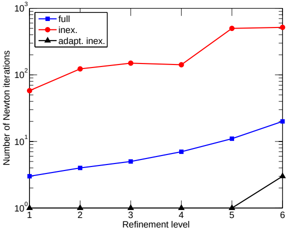


inexact Newton

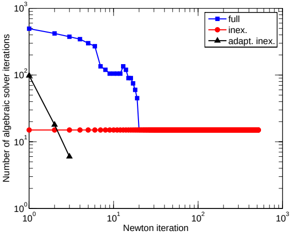


ad. inexact Newton

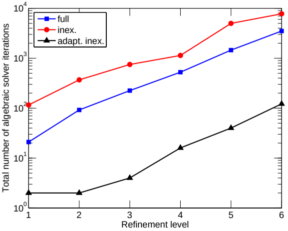
Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement



alg. it. / Newton step



alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

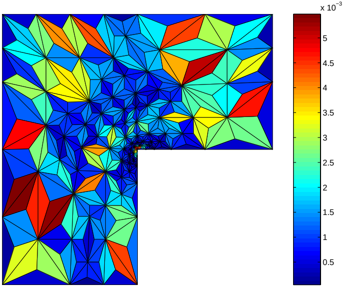
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= u_{\mathrm{D}} \qquad \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

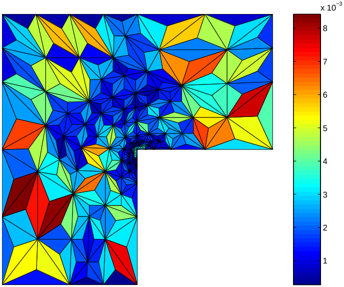
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

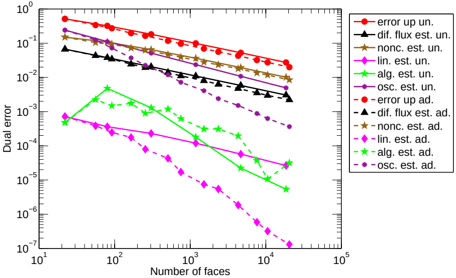


Estimated error distribution



Exact error distribution

Estimated and actual errors and the effectivity index



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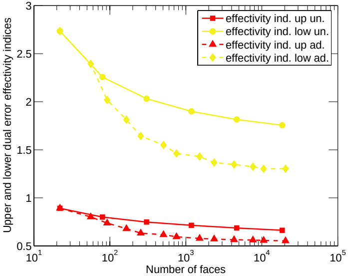
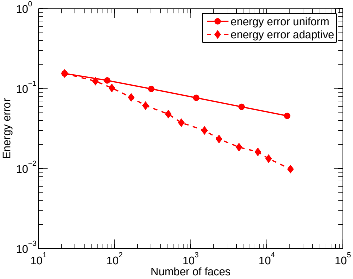
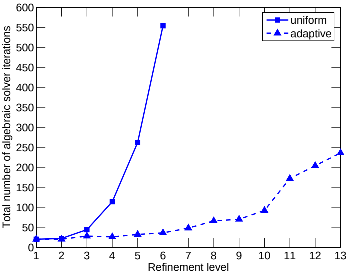


Effectivity index

Energy error and overall performance



Energy error



Overall performance

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1 Introduction
2 Steady linear problems: space mesh adaptation
- Potential and flux reconstructions
- A guaranteed a posteriori error estimate
- Polynomial-degree-robust local efficiency
- Application and numerical results

3 Nonlinear problems: stopping algebraic & linearization solvers
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- Stopping criteria and efficiency
- Application and numerical results

4 Unsteady problems: time step adaptation
- A guaranteed a posteriori error estimate
- Application and numerical results

5 Conclusions and future directions

Numerical approximation of a nonlinear, unsteady PDE

Exact and approximate solution

- let u be the weak solution of $A(u) = f$, A **nonlinear**, **unsteady** PDE on $\Omega \times (0, T)$
- let $u_{h\tau}$ be its approximate numerical solution, $\mathcal{A}_{h\tau}(u_{h\tau}) = F_{h\tau}$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n and $\mathcal{T}_h^n$, solve a system of nonlinear algebraic equations $\mathcal{A}_h^n(u_h^n) = F_h^n$

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Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
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Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

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- Bieterman and Babuška (1982), introduction
- Picasso (1998), energy error estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Nonlinear unsteady problems

- Verfürth (1998), framework for energy norm control
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Degenerate parabolic problems

- Nochetto, Schmidt, Verdi (2000), the Stefan problem
- Dolejší, Ern, Vohralík (2013), advection–diffusion–reaction equation, the Richards problem, robustness in a space–time dual mesh-dependent norm

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Two-phase flow in porous media

Two-phase flow in porous media

$$\begin{aligned} \partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, & \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ -\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, & \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ s_\mathrm{n} + s_\mathrm{w} &= 1, \\ p_\mathrm{n} - p_\mathrm{w} &= p_\mathrm{c}(s_\mathrm{w}) \end{aligned}$$

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

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Two-phase flow in porous media

Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \le \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
- $\eta_{\mathrm{tm}}^{n,k,i}$: temporal discretization
- $\eta_{\mathrm{lin}}^{n,k,i}$: linearization
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- $\eta_{\mathrm{alg}}^{n,k,i}$: algebraic solver

Local estimators

- *spatial estimators*

$$\eta_{\mathrm{sp},K}^{n,k,i}(t) := \Bigg\{ \sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}} \big(\|\mathbf{d}_{\alpha,h}^{n,k,i} - \mathbf{v}_\alpha(p_{\mathrm{w},h}^{n,k,i}, s_{\mathrm{w},h}^{n,k,i})\|_K$$
$$+ h_K/\pi \|q_\alpha^n - \partial_t^n(\phi s_{\alpha,h\tau}^{n,k,i}) - \nabla\cdot\mathbf{u}_{\alpha,h}^{n,k,i}\|_K\big)^2$$
$$+ \big(\|\underline{\mathbf{K}}(\lambda_\mathrm{w}(s_{\mathrm{w},h\tau}^{n,k,i}) + \lambda_\mathrm{n}(s_{\mathrm{w},h\tau}^{n,k,i}))\nabla(\mathfrak{p}(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{p}}_{h\tau}^{n,k,i})\|_K(t)\big)^2$$
$$+ \big(\|\underline{\mathbf{K}}\nabla(\mathfrak{q}(s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{q}}_{h\tau}^{n,k,i})\|_K(t)\big)^2 \Bigg\}^{\frac{1}{2}}$$

- *temporal estimators*

$$\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t) := \|\mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t) - \mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t^n)\|_K \quad \alpha\in\{\mathrm{n},\mathrm{w}\}$$

- *linearization estimators*

$$\eta_{\mathrm{lin},K,\alpha}^{n,k,i} := \|\mathbf{l}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha\in\{\mathrm{n},\mathrm{w}\}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg},K,\alpha}^{n,k,i} := \|\mathbf{a}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha\in\{\mathrm{n},\mathrm{w}\}$$

Global estimators

Global estimators

$$\eta_{\mathrm{sp}}^{n,k,i} := \left\{ 3 \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{sp},K}^{n,k,i}(t))^2 \,\mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{tm}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t))^2 \,\mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{lin}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{lin},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{alg}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{alg},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}}$$

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Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{w}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{w}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|,$$

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Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K, 1)_{e_{KK'}} := F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K, 1)_{e_{KK'}} := F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$-\lambda_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,$$
$$p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) = \bar{p}_{\mathrm{w},K}^{n,k,i},$$

$$-\lambda_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,$$
$$p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) = p_{\mathrm{c}}(s_{\mathrm{w},K}^{n,k,i}) + \bar{p}_{\mathrm{w},K}^{n,k,i}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i} \cdot \mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i}) \cdot \mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

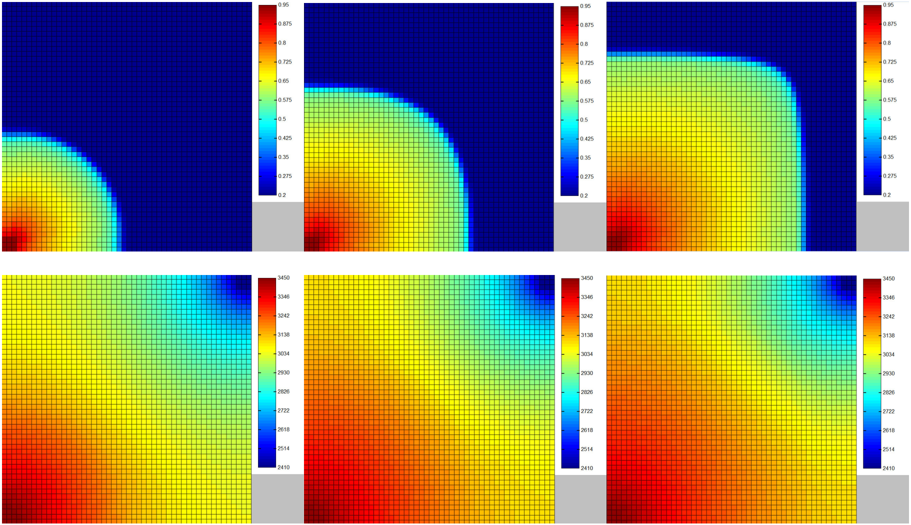
Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

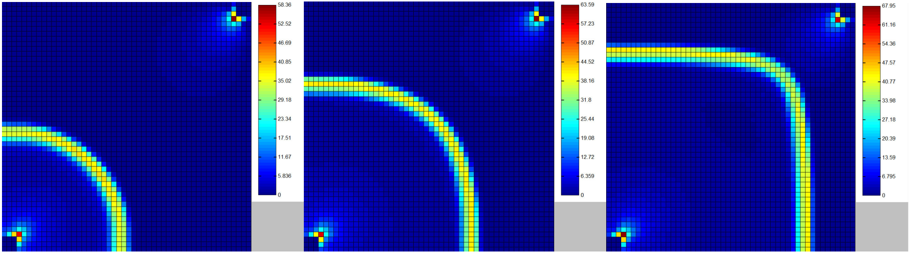
$$
\begin{aligned}
-\lambda_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i}) \underline{\mathbf{K}} \nabla (p_{\mathrm{w},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,\\
p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) &= \bar{p}_{\mathrm{w},K}^{n,k,i},
\end{aligned}
$$

$$
\begin{aligned}
-\lambda_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i}) \underline{\mathbf{K}} \nabla (p_{\mathrm{n},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,\\
p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) &= p_{\mathrm{c}}(s_{\mathrm{w},K}^{n,k,i}) + \bar{p}_{\mathrm{w},K}^{n,k,i}
\end{aligned}
$$

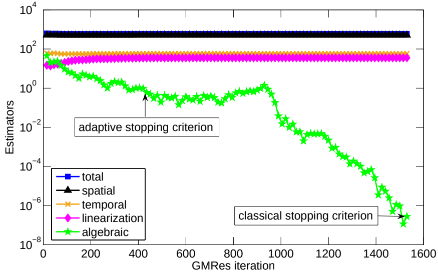
Water saturation/water pressure evolution



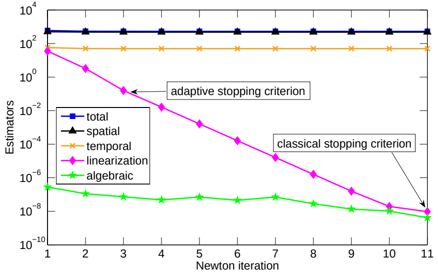
Estimators evolution



Estimators and stopping criteria

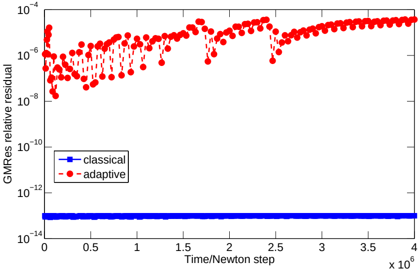


Estimators in function of GMRes iterations

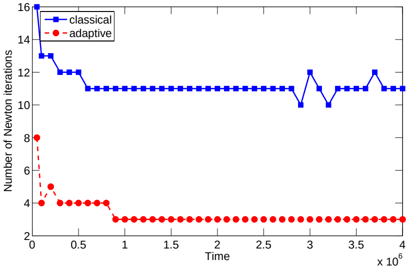


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

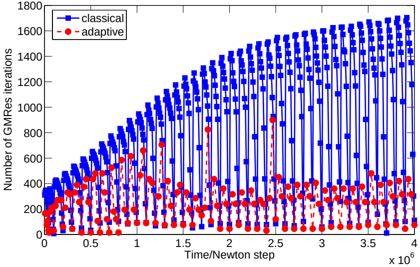


GMRes relative residual

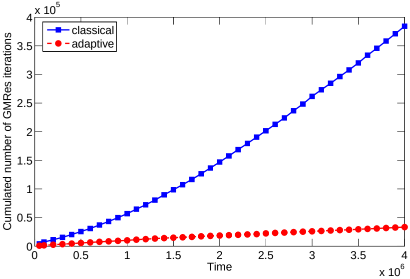


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

1. Introduction
2. Steady linear problems: space mesh adaptation
   - Potential and flux reconstructions
   - A guaranteed a posteriori error estimate
   - Polynomial-degree-robust local efficiency
   - Application and numerical results
3. Nonlinear problems: stopping algebraic & linearization solvers
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Application and numerical results
4. Unsteady problems: time step adaptation
   - A guaranteed a posteriori error estimate
   - Application and numerical results
5. Conclusions and future directions

Conclusions

Entire adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **"smart online decisions"**: algebraic step / linearization step / space mesh refinement / time step modification
- important **computational savings**
- guaranteed and robust error upper bound via **a posteriori estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

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Bibliography

Bibliography

- ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, HAL Preprint 00921583, submitted for publication.
- ERN A., VOHRALÍK M., Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, *SIAM J. Sci. Comput.* **35** (2013), A1761–A1791.
- VOHRALÍK M., WHEELER M. F., A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows, *Comput. Geosci.* **17** (2013), 789–812.

Thank you for your attention!