A posteriori error estimates in numerical approximation of partial differential equations

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Outline

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What is an a posteriori error estimate

A posteriori error estimate

- Let p be a weak solution of a PDE.
- Let p_h be its approximate numerical solution.
- A priori error estimate: $\|p - p_h\|_\Omega \le f(p)h^q$. Dependent on p, not computable. Useful in theory.
- A posteriori error estimate: $\|p - p_h\|_\Omega \lesssim f(p_h)$. Only uses p_h, computable. Great in practice.

Usual form

- $f(p_h)^2 = \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$, where $\eta_K(p_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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What an a posteriori error estimate should fulfill

Guaranteed upper bound (global upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant
- remark (**reliability**): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Global efficiency (global lower bound)

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 \leq C_{\mathrm{eff},\Omega}^2 \|p - p_h\|_\Omega^2$

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$

Local efficiency (local lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

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Continuous finite elements

- Babuška and Rheinboldt (1978), introduction
- Zienkiewicz and Zhu (1987), averaging-based estimates
- Verfürth (1996), residual-based estimates
- Ainsworth and Oden (2000), equilibrated residual estimates
- Repin (2001), functional a posteriori error estimates
- Luce and Wohlmuth (2004), equilibrated fluxes estimates

Discontinuous finite elements

- Karakashian and Pascal (2003), Becker, Hansbo, and Larson (2003), residual-based estimates
- Ainsworth (2007), Kim (2007), Lazarov, Repin, and Tomar (2007), Nicaise (2007), equilibrated fluxes estimates

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- Ohlberger (2001), non-energy norm estimates
- Nicaise (2004), reconstruction-based estimates

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

- Verfürth (1998, 2005), conforming finite elements
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A 1D model problem $-p'' = f$ in Ω, $p = 0$ on $\partial\Omega$

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H_0^1(\Omega)$ by

$$\mathcal{B}(p, \varphi) := (p', \varphi').$$

Definition (Energy norm)

The associated energy norm for $\varphi \in H_0^1(\Omega)$ is given by

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi) = \|\varphi'\|^2.$$

Definition (Weak solution)

Weak solution: $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega).$$

Definition (Finite element approximation)

Finite element approximation: $p_h \in V_h$ such that

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Optimal abstract framework for $-p'' = f$

Theorem (Optimal abstract framework, 1D Laplacian)

Let $p, p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \le |||p - p_h|||.$$

Proof.

We have

$$\begin{aligned}
|||p - p_h||| &= \mathcal{B}\left(p - p_h, \frac{p - p_h}{|||p - p_h|||}\right) \\
&\le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \\
&\le |||p - p_h||| \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} |||\varphi|||.
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Optimal abstract framework for $-p'' = f$

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Optimal abstract estimate for $-p'' = f$

Theorem (Optimal abstract estimate, 1D Laplacian)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \inf_{t \in H^1(\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - t', \varphi) - (p_h' + t, \varphi')\}$$
$$\leq |||p - p_h|||.$$

Proof.

Upper bound: put $\varphi := p - p_h / |||p - p_h|||$ and take $t \in H^1(\Omega)$ arbitrary. Then

$$\begin{aligned}
\mathcal{B}(p - p_h, \varphi) &= (f, \varphi) - (p_h', \varphi') && //\mathcal{B} \text{ lin., weak sol. def.} \\
&= (f, \varphi) - (p_h' + t, \varphi') + (t, \varphi') && // \pm (t, \varphi') \\
&= (f - t', \varphi) - (p_h' + t, \varphi'). && //\text{int. by parts}
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Lower bound: put $t = -p'$ and use the Schwarz inequality.

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Properties

- Guaranteed upper bound (no undetermined constant).
- Exact and robust.
- Not computable (infimum over an infinite-dimensional space).

A first computable estimate for $-p'' = f$

Theorem (A first computable estimate, 1D Laplacian)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $t_h \in H^1(\Omega)$. Then

$$|||p - p_h||| \leq \frac{h_\Omega}{\pi}\|f - t_h'\| + \|p_h' + t_h\|.$$

Proof.

- recall $|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - t_h', \varphi) - (p_h' + t_h, \varphi')\}$;
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Properties

- Guaranteed upper bound ($\frac{1}{\pi}$, Friedrichs constant).
- $\|p_h' + t_h\|$ penalizes $-p_h' \notin H^1(\Omega)$.
- $\|f - t_h'\|$ is a residual term, evaluated for t_h.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).

Numerical experiment for the first computable estimate

Model problem

$$\begin{aligned} -p'' &= \pi^2 sin(\pi x) \quad \text{in }]0,1[, \\ p &= 0 \quad \text{in } 0,1 \end{aligned}$$

Exact solution

$$p(x) = sin(\pi x)$$

Discretization

N given, $h = 1/(N+1)$, $x_k = kh$, $k = 0, \dots, N+1$ ($x_0 = 0$ and $x_{N+1} = 1$), $x_{k+\frac{1}{2}} = (k + \frac{1}{2})h$, $k = 0, \dots, N$, $x_{-\frac{1}{2}} = 0$, $x_{N+1+\frac{1}{2}} = 1$

Choice of t_h

$$\begin{aligned} t_h(x_{k+\frac{1}{2}}) &= -p_h'(x_{k+\frac{1}{2}}) \qquad k = 0, \dots, N, \\ t_h(x_k) &= -(p_h'|_{]x_{k-1},x_k[} + p_h'|_{]x_k,x_{k+1}[})/2 \qquad k = 1, \dots, N, \\ t_h(x_0) &= -p_h'|_{]x_0,x_1[}, \\ t_h(x_{N+1}) &= -p_h'|_{]x_N,x_{N+1}[} \end{aligned}$$

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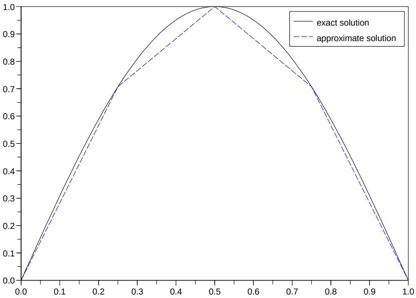
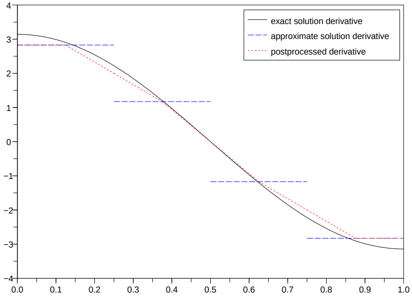
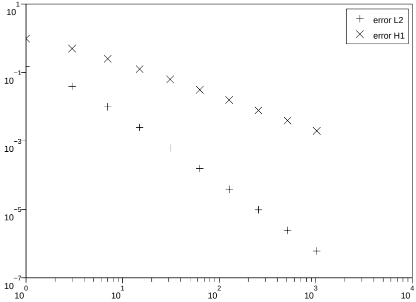


Plot of p and p_h

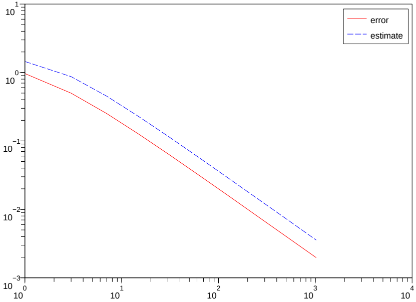


Plot of p', p_h', and $-t_h$

Numerical experiment for the first computable estimate

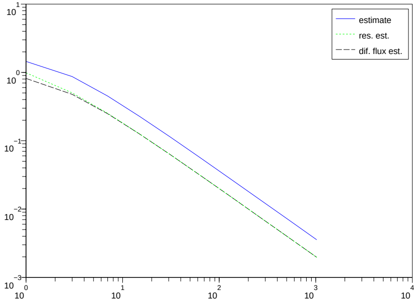


L^2 and H^1 errors

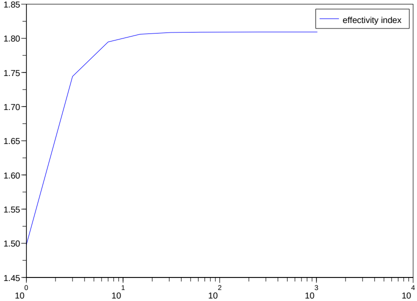


Estimated and actual error

Numerical experiment for the first computable estimate



Estimated error and residual and diffusive flux estimators



Effectivity index

Outline

Optimal estimate for $-p'' = f$

Theorem (Optimal estimate, 1D Laplacian)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take $t_h \in H^1(\Omega)$ such that $(t_h', 1)_{]x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}[} = (f, 1)_{]x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}[}$ for all $k = 1, \ldots, N$. Then

$$|||p - p_h||| \leq \left\{ \sum_{k=0}^{N+1} (\eta_{\mathrm{R},k} + \eta_{\mathrm{DF},k})^2 \right\}^{\frac{1}{2}}.$$

- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},k} := \|p_h' + t_h\|_{]x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}[}$
  - penalizes the fact that $-p_h' \notin H^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},k} := m_k \|f - t_h'\|_{]x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}[}$
  - $m_k := \frac{h}{\pi}$ for $k = 1, \ldots, N$
  - $m_k := \frac{h}{2\sqrt{2}}$ for $k = 0, N+1$
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Optimal estimate for $-p'' = f$

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Proof of the optimal estimate for $-p'' = f$

Proof.

- recall $|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - t_h', \varphi) - (p_h' + t_h, \varphi')\};$
- recall the Poincaré inequality: $\|\varphi - \varphi_k\|_{]x_{k-\frac12}, x_{k+\frac12}[} \leq \frac{h}{\pi}\|\varphi'\|_{]x_{k-\frac12}, x_{k+\frac12}[} = \frac{h}{\pi}|||\varphi|||_{]x_{k-\frac12}, x_{k+\frac12}[}$, where φ_k is the mean value of φ over $]x_{k-\frac12}, x_{k+\frac12}[$;
- use this, the conservativity property of t_h, and the Schwarz inequality when $k = 1, \ldots, N$:
$$(f - t_h', \varphi)_{]x_{k-\frac12}, x_{k+\frac12}[} = (f - t_h', \varphi - \varphi_k)_{]x_{k-\frac12}, x_{k+\frac12}[}$$
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Construction of t_h

Theorem (Construction of t_h)

Let f be piecewise constant and let $t_h \in H^1(\Omega)$ be given by

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Proof.

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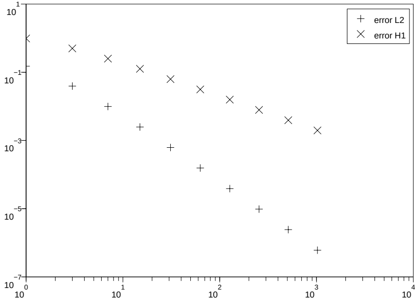
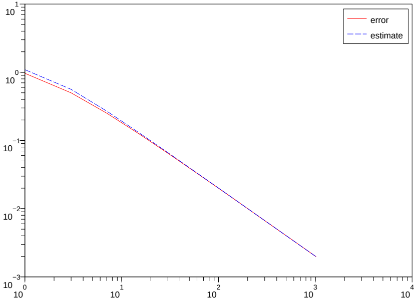
Proof.

- the finite element method: $\int_{x_{k-1}}^{x_{k+1}} p_h'\psi_k' \,\mathrm{d}x = \int_{x_{k-1}}^{x_{k+1}} f\psi_k \,\mathrm{d}x$, $k = 1, \ldots, N$, where ψ_k is the "pyramidal" basis function;
- f piecewise constant: $\int_{x_{k-1}}^{x_{k+1}} f\psi_k \,\mathrm{d}x = \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f \,\mathrm{d}x$;
- construction of t_h: $\int_{x_{k-1}}^{x_{k+1}} p_h'\psi_k' \,\mathrm{d}x = \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} t_h' \,\mathrm{d}x$.

Numerical experiment for the optimal estimate

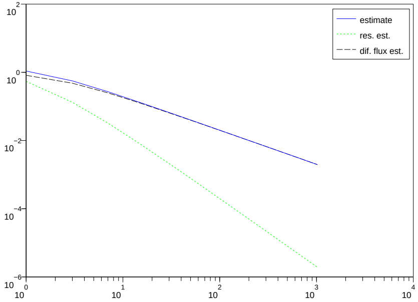


L^2 and H^1 errors

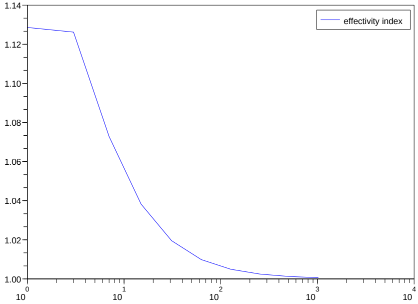


Estimated and actual error

Numerical experiment for the optimal estimate



Estimated error and residual and diffusive flux estimators



Effectivity index

Outline

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$
\begin{aligned}
-\nabla \cdot (a \nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

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Bilinear form, energy norm, and a weak solution

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H_0^1(\Omega)$ by

$$\mathcal{B}(p, \varphi) := (a\nabla p, \nabla \varphi) .$$

Definition (Energy norm)

The associated energy norm for $\varphi \in H_0^1(\Omega)$ is given by

$$|||\varphi|||^2 := \mathcal{B}(\varphi, \varphi) = \|a^{\frac{1}{2}} \nabla \varphi\|^2 .$$

Definition (Weak solution)

Weak solution: $p \in H_0^1(\Omega)$ such that

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Bilinear form, energy norm, and a weak solution

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Outline

Residual a posteriori error estimation for $-\triangle p = f$

Corollary (Classical residual a posteriori error estimate in FEs)

Let $a = 1$. Then there holds (cf. Verfürth 96)

$$
\begin{aligned}
|||p - p_h||| \quad \leq \quad & C_1 \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 \|f + \triangle p_h\|_K^2 \right\}^{1/2} \\
& + C_2 \left\{ \sum_{\sigma \in \mathcal{E}_h} h_\sigma \|[\nabla p_h \cdot \mathbf{n}]\|_\sigma^2 \right\}^{1/2} .
\end{aligned}
$$

Drawbacks

- What are C_1 and C_2?
- If C_1 and C_2 evaluated: overestimation by a factor of 30 (uniform refinement) and 60 (adaptive refinement).
- $\triangle p_h = 0$: $h_K \|f\|_K$ as estimator gives no good sense.
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FEs residual constants C_1 and C_2

Constants C_1 and C_2, Carstensen and Funken 00

$$
\begin{aligned}
c_V &:= \begin{cases} C_{\mathrm{P},\mathcal{T}_V}^{\frac{1}{2}} h_{\mathcal{T}_V} & V \in \mathcal{V}_h^{\mathrm{int}}, \\ C_{\mathrm{F},\mathcal{T}_V,\partial\Omega}^{\frac{1}{2}} h_{\mathcal{T}_V} & V \in \mathcal{V}_h^{\mathrm{ext}}, \end{cases} \\
C_1 &:= \max_{K \in \mathcal{T}_h} \left\{ \sum_{V \in \mathcal{V}_K} c_V^2 / \min_{K \in \mathcal{T}_V} h_K^2 \right\}^{\frac{1}{2}}, \\
C_2^2 &:= 3 C_1 \max_{K \in \mathcal{T}_h} \max_{\sigma \in \mathcal{E}_K} \{ h_K / h_\sigma h_K^2 / |K| \} \\
&\quad + \frac{1}{2} 3^{\frac{3}{2}} C_1^2 \max_{K \in \mathcal{T}_h} \max_{\sigma \in \mathcal{E}_K} \{ h_K / h_\sigma h_K^2 / |K| (3 + h_K^2 / |K|) \}.
\end{aligned}
$$

Zienkiewicz–Zhu averaging a posteriori error estimation for $-\triangle p = f$

Corollary (Zienkiewicz–Zhu averaging a posteriori error estimate in FEs)

There holds (cf. Zienkiewicz–Zhu 87)

$$|||p - p_h||| \lesssim \|\nabla p_h + \mathbf{t}_h\|,$$

where $\mathbf{t}_h$ *is an averaged smooth flux.*

Drawbacks

- No error upper bound (neither guaranteed, nor reliable).
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Outline

Optimal abstract framework for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract framework, conf. & pure dif. case)

Let $p, p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \leq |||p - p_h|||.$$

Proof.

We have

$$\begin{aligned} |||p - p_h||| &= \mathcal{B}\left(p - p_h, \frac{p - p_h}{|||p - p_h|||}\right) \\ &\leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \\ &\leq |||p - p_h||| \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} |||\varphi|||. \end{aligned}$$

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Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$\begin{aligned} |||p - p_h||| &\leq \inf_{\mathbf{t}\in\mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\} \\ &\leq |||p - p_h|||. \end{aligned}$$

Proof.

Upper bound: put $\varphi := p - p_h/|||p - p_h|||$ and take $\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)$ arbitrary. Then

$$\begin{aligned} \mathcal{B}(p - p_h, \varphi) &= (f, \varphi) - (a\nabla p_h, \nabla\varphi) && //\mathcal{B} \text{ lin., weak sol. def.} \\ &= (f, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi) + (\mathbf{t}, \nabla\varphi) && // \pm (\mathbf{t}, \nabla\varphi) \\ &= (f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi). && //\text{Green th.} \end{aligned}$$

Lower bound: put $\mathbf{t} = -a\nabla p$ and use the Schwarz inequality.

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

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$$\leq |||p - p_h|||.$$

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$$\le |||p - p_h|||.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- Exact and robust.
- Not computable (infimum over an infinite-dimensional space).

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\}$;
- Friedrichs inequality: $\|\varphi\| \leq C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla\varphi\| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||$;
- use this and the Schwarz inequality:
  $$(f - \nabla \cdot \mathbf{t}_h, \varphi) \leq \|f - \nabla \cdot \mathbf{t}_h\| \|\varphi\| \leq \|f - \nabla \cdot \mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$$
- use the Schwarz inequality for the second term:
  $-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi) \leq \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\| |||\varphi|||$.

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- use the Schwarz inequality for the second term:
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- use this and the Schwarz inequality:

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A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

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A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

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Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).

Outline

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal a posteriori error estimate)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Let $\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ be a partition of Ω and take $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ for all $D \in \mathcal{D}_h^{\mathrm{int}}$. Then

$$|||p - p_h||| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2}.$$

- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},D} := m_{D,a} \|f - \nabla \cdot \mathbf{t}_h\|_D$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,a}^2 := C_{\mathrm{F},D} h_D^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{a,D}$ is the smallest value of a on D
  - residue evaluated for $\mathbf{t}_h$

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Proof of the optimal estimate for $-\nabla\cdot(a\nabla p) = f$

Proof.

- recall $|||p - p_h||| \leq \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\}$;
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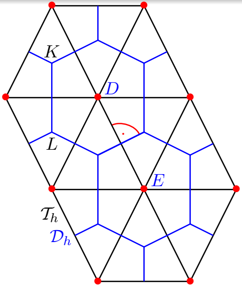
Outline

Finite element and cell-centered finite volume methods



$$
\begin{aligned}
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$$

Finite elements

$$(a \nabla p_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall \varphi_h \in V_h$$

- $-\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega) \Rightarrow$ not locally conservative
- $p_h \in H_0^1(\Omega) \Rightarrow$ conforming
- Galerkin orthogonality
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Cell-centered finite volumes

$$-\sum_{E \in \mathcal{N}(D)} \{a\}_\omega \frac{|\sigma_{D,E}|}{d_{D,E}} (p_E - p_D) = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}$$

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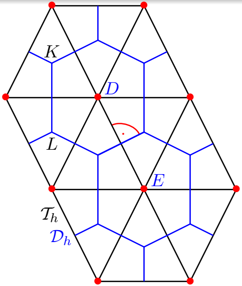

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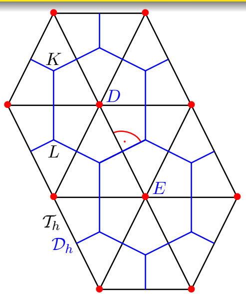

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Equivalence between FEs and FVs

Theorem (Equivalence between FEs and FVs, EGH 00)

Let $d = 2$, let $a = 1$, let $\mathcal{T}_h$ be Delaunay and let $\mathcal{D}_h$ be its Voronoï dual (given by the orthogonal bisectors of the edges from $\mathcal{T}_h$). Let next f be piecewise constant on $\mathcal{T}_h$. Then FEs and FVs produce the same discrete systems.

Consequences:

- interpretation of the results
- local conservativity of FEs on $\mathcal{D}_h$
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Finite elements for $-\nabla \cdot (a\nabla p) = f$

Finite element method

- Find $p_h \in V_h$ such that

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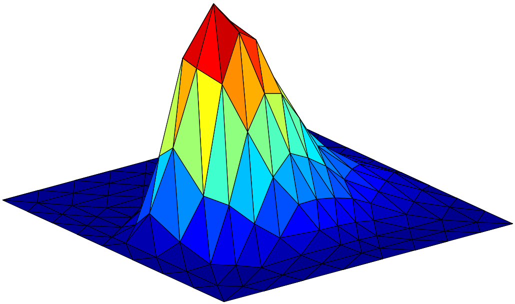
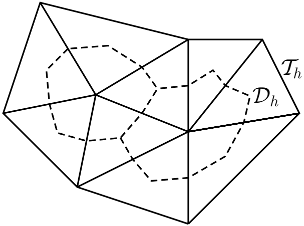
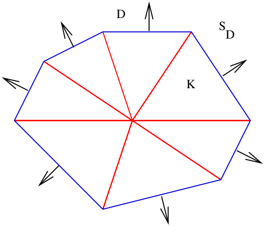
- $p_h \in H_0^1(\Omega)$:



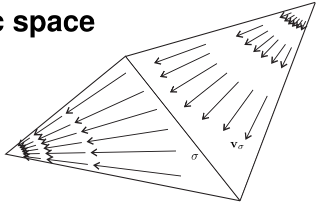
Choice of $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$

Recall the equivalence with finite volumes





Raviart–Thomas–Nédélec space



Choice of $\mathbf{t}_h$ based on the equivalence with FVs

- using the FV fluxes on $\mathcal{D}_h$, construct $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$;
  $\langle \mathbf{t}_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}$.

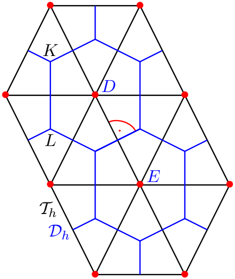
Cell-centered finite volumes for $-\nabla \cdot (a\nabla p) = f$

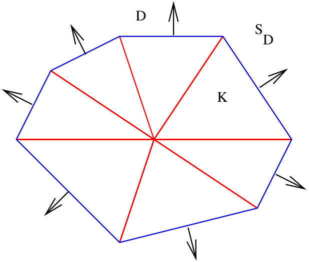
Cell-centered finite volume method

- Find $\{p_D\}_{D\in\mathcal{D}_h^{\rm int}}$ such that

$$-\{a\}_\omega \sum_{E\in\mathcal{N}(D)} \frac{|\sigma_{D,E}|}{d_{D,E}}(p_E - p_D) = (f,1)_D \quad \forall D \in \mathcal{D}_h^{\rm int}.$$

- $\{a\}_\omega$: harmonic averaging of the diffusion tensor.
- We immediately have $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$ which verifies $\langle \mathbf{t}_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (\nabla \cdot \mathbf{t}_h, 1)_D = (f,1)_D \quad \forall D \in \mathcal{D}_h^{\rm int}$.





Interpretation of $\{p_D\}_{D\in\mathcal{D}_h^{\mathrm{int}}}$ as $p_h \in V_h$

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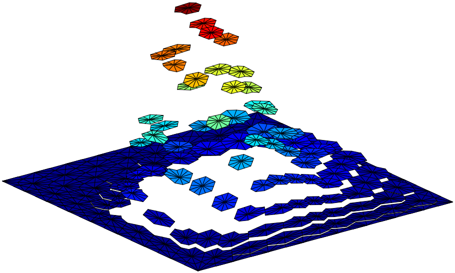
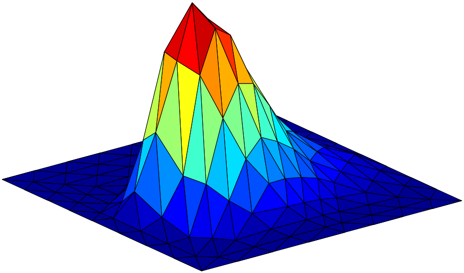


p_D piecewise constant on $\mathcal{D}_h$



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Outline

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal a posteriori error estimate)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Let $\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ be a partition of Ω and take $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ for all $D \in \mathcal{D}_h^{\mathrm{int}}$. Then

$$|||p - p_h||| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2}.$$

- **diffusive flux estimator**
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},D} := m_{D,a} \|f - \nabla \cdot \mathbf{t}_h\|_D$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
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Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Theorem (Local efficiency)

Let $\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{a\nabla p_h \cdot \mathbf{n}_\sigma\}_\omega$ for all $\sigma \in \mathcal{G}_h$. Then

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C|||p - p_h|||_{\mathcal{T}_{V_D}},$$

where C depends only on the space dimension d, on the shape regularity parameter $\kappa_{\mathcal{T}}$, and on the polynomial degree m of f.

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \leq Ch_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
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Properties

- **guaranteed upper bound**
- local and global efficiency
- **full robustness**
- negligible evaluation cost
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Changes with respect to classical FEs

- of course $\tilde{a} = a$ when a piecewise constant on $\mathcal{T}_h$
- a **piecewise constant on** $\mathcal{D}_h$: harmonic averaging of a

Cell-centered finite volumes

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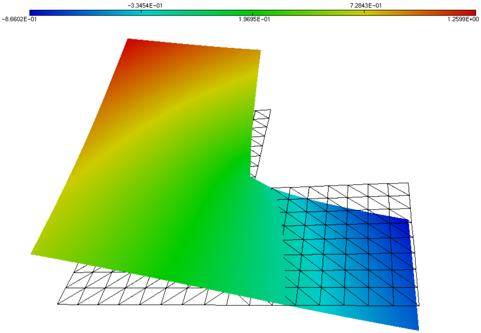
L-shape domain example and finite elements

Problem

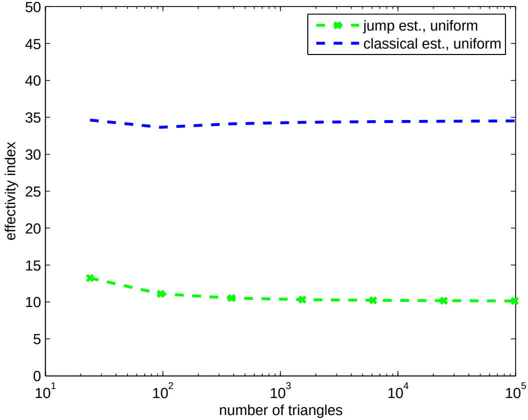
$$
\begin{aligned}
-\triangle p &= 0, && \text{in } \Omega \\
p &= p_0, && \text{on } \partial\Omega
\end{aligned}
$$

Exact solution
(polar coordinates)

$$p_0(r,\varphi) = r^{-\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right)$$



Effectivity index – comparison, uniform refinement

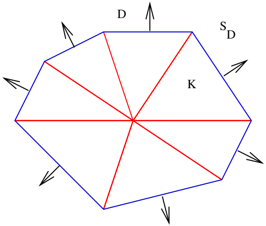


Effectivity indices for the jump and classical estimators

Improvement by local minimization

Observation

- Fluxes of $\mathbf{t}_h$ need to be prescribed on the boundary of dual volumes only to get $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$.
- We can choose them on other edges.



Local minimization (for each vertex)

- compute local minimization matrix for the internal fluxes
- solve local linear problem (size = number od sides sharing the given vertex)
- compute the estimators
- the whole estimate still has a linear cost

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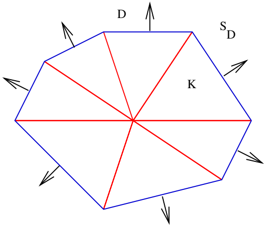

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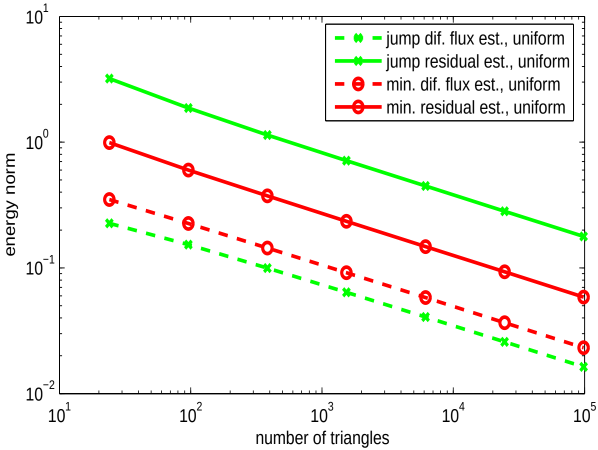
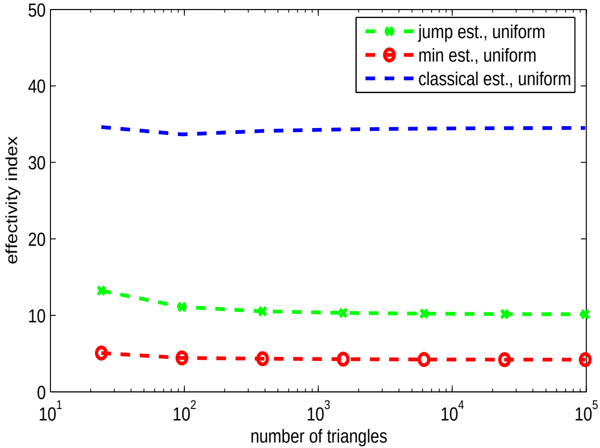
Residual and diffusive flux estimators, uniform refinement



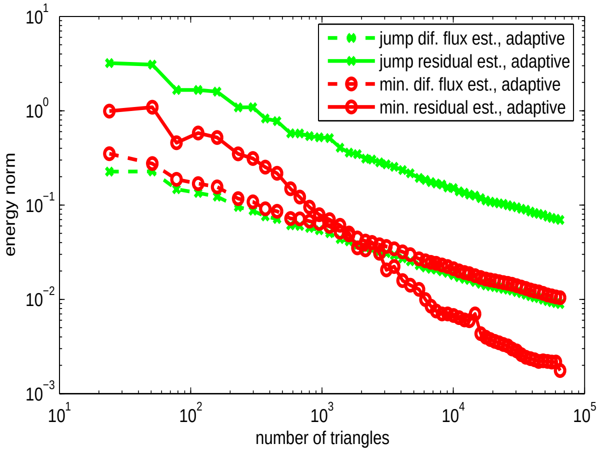
Residual and diffusive flux estimators comparison

Effectivity index – comparison, uniform refinement



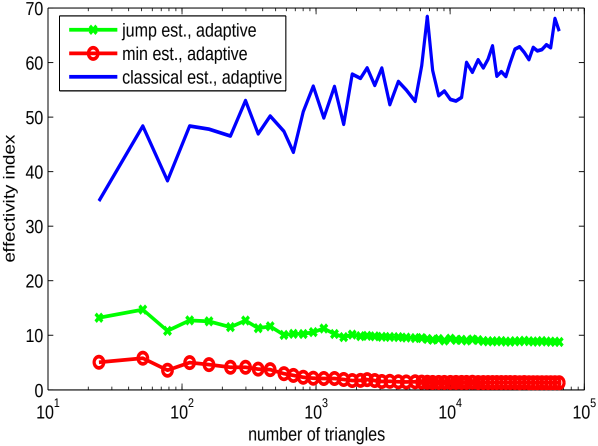
Effectivity indices for the jump, minimization, and classical estimators

Residual and diffusive flux estimators, uniform refinement



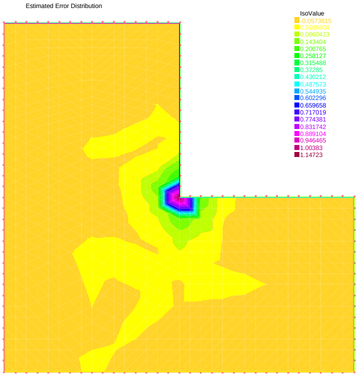
Residual and diffusive flux estimators comparison

Effectivity index – comparison, adaptive refinement

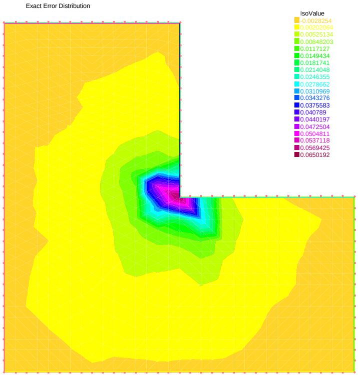


Effectivity indices for the jump, minimization, and classical estimators

Error distribution on a uniformly refined mesh

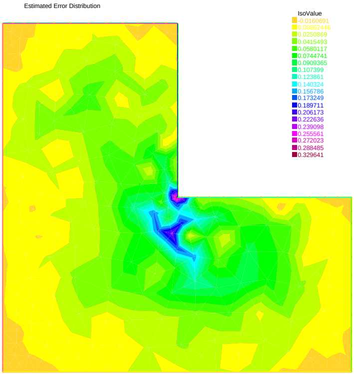


Estimated error distribution



Exact error distribution

Error distribution on an adaptively refined mesh



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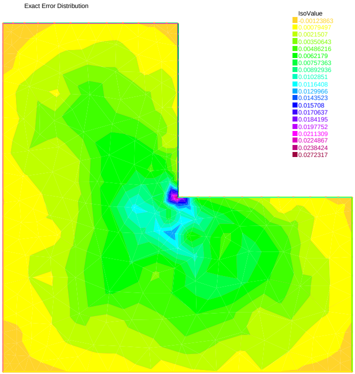
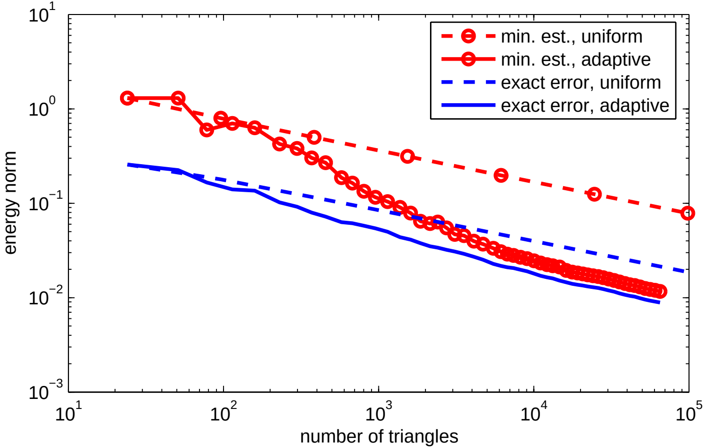


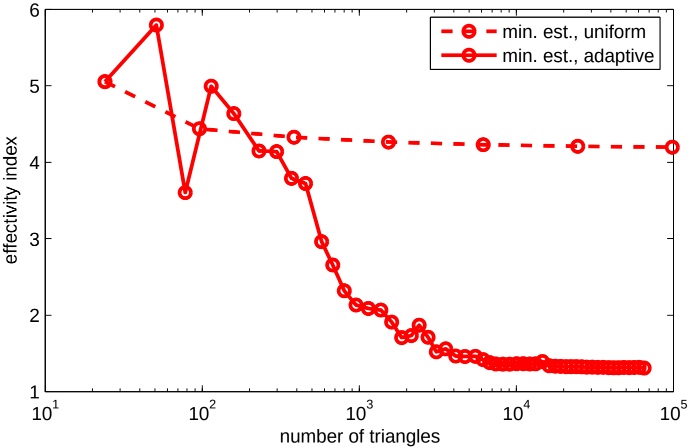
Exact error distribution

Energy error



Estimated and actual energy error, uniformly/adaptively refined meshes

Effectivity index



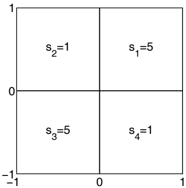
Effectivity index, uniformly/adaptively refined meshes

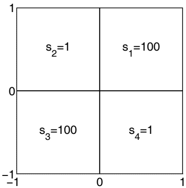
Discontinuous diffusion tensor and vertex-centered finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot (a\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous a, two cases:



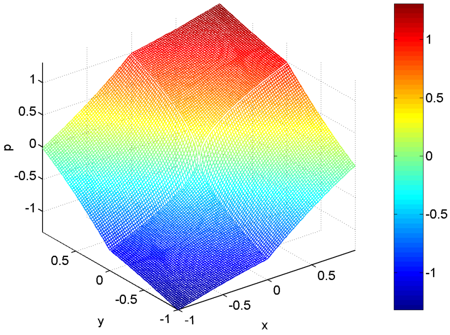


- analytical solution: singularity at the origin

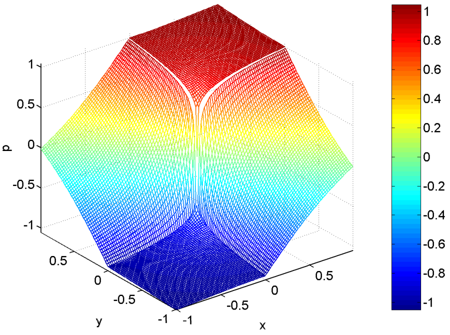
$$p(r,\theta)|_{\Omega_i} = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

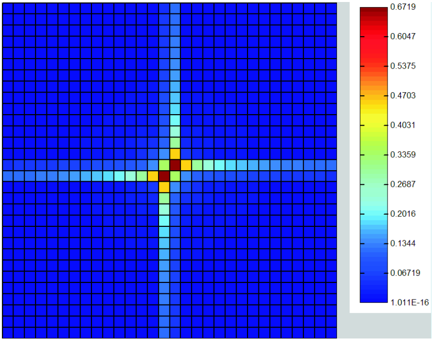


Case 1

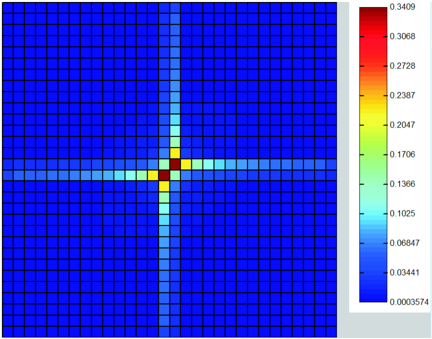


Case 2

Error distribution on a uniformly refined mesh, case 1

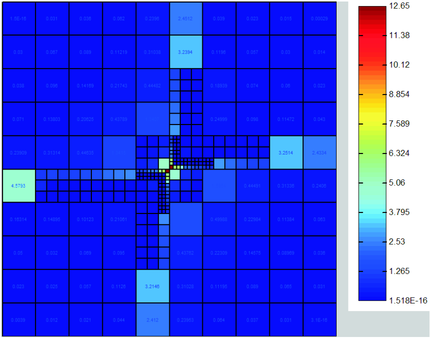


Estimated error distribution

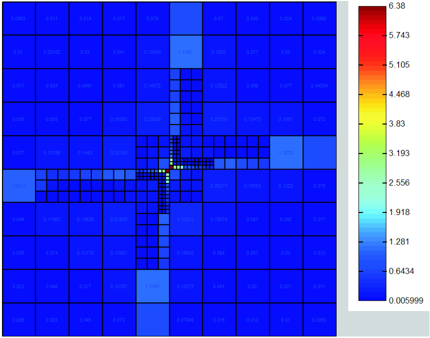


Exact error distribution

Error distribution on an adaptively refined mesh, case 2

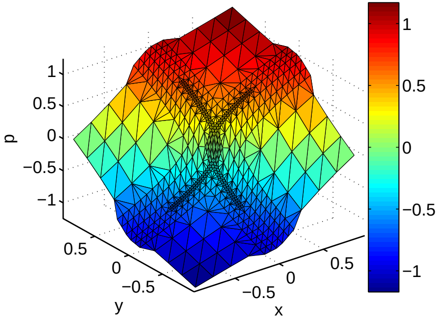


Estimated error distribution

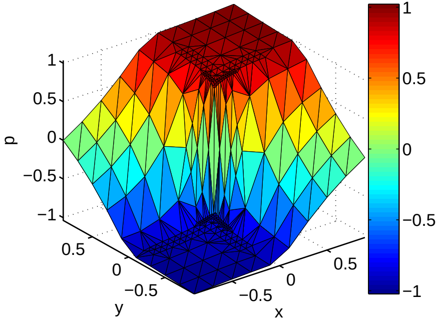


Exact error distribution

Approximate solutions on adaptively refined meshes

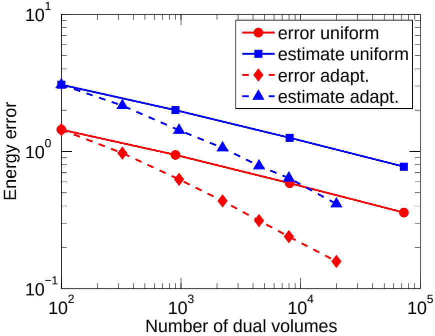


Case 1

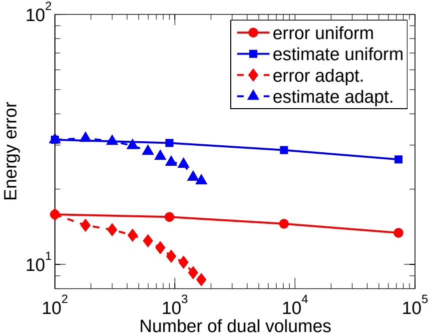


Case 2

Estimated and actual error in uniformly/adaptively refined meshes

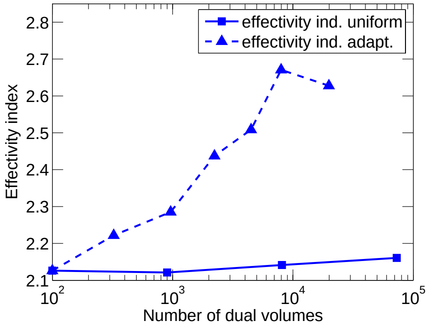


Case 1

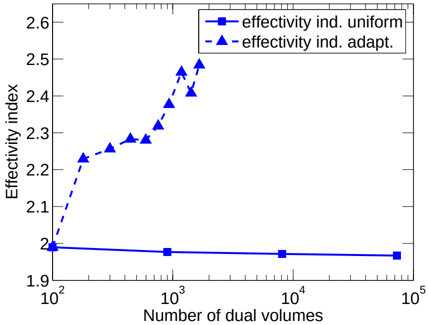


Case 2

Original effectivity indices in uniformly/adaptively refined meshes

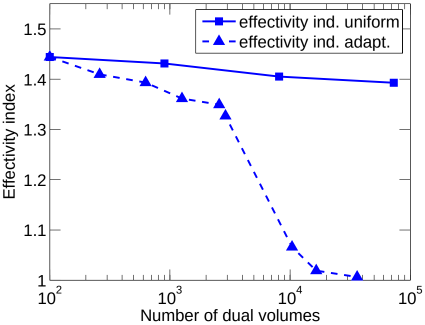


Case 1

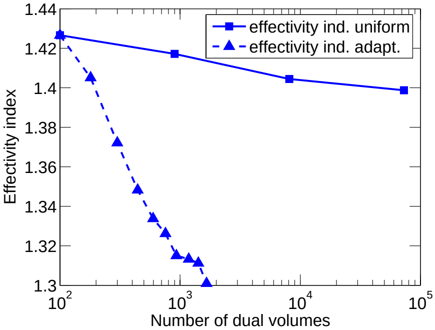


Case 2

Effectivity indices in uniformly/adaptively refined meshes using a simple local minimization



Case 1



Case 2

Outline

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p)+\mathbf{w}\cdot\nabla p+rp &= f \quad \text{in } \Omega,\\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $\mathbf{S}|_K$ is a constant SPD matrix, $c_{\mathbf{S},K}$ its smallest, and $C_{\mathbf{S},K}$ its largest eigenvalue on each $K \in \mathcal{T}_h$
- $(r - \frac{1}{2}\nabla\cdot\mathbf{w})|_K \geq c_{\mathbf{w},r,K} \geq 0$ on each $K \in \mathcal{T}_h$ (from pure diffusion to convection–diffusion–reaction cases)

Difficulties

- $\mathbf{S}$ is a piecewise constant matrix, inhomogeneous and anisotropic
- $\mathbf{w}$ is dominating

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla p) + \mathbf{w} \cdot \nabla p + rp &= f \quad \text{in } \Omega,\\
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$$

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- $\mathbf{S}$ is a piecewise constant matrix, inhomogeneous and anisotropic
- $\mathbf{w}$ is dominating

Bilinear form, weak solution, and energy norm

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H^1(\mathcal{T}_h)$ by

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} \left\{ (\mathbf{S}\nabla p, \nabla\varphi)_K + (\mathbf{w}\cdot\nabla p, \varphi)_K + (rp,\varphi)_K \right\}.$$

Definition (Weak solution)

Weak solution: $p \in H^1_0(\Omega)$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi) \quad \forall \varphi \in H^1_0(\Omega).$$

Definition (Energy (semi-)norm)

We define the energy (semi-)norm for $\varphi \in H^1(\mathcal{T}_h)$ by

$$|||\varphi|||^2 := \sum_{K\in\mathcal{T}_h} |||\varphi|||_K^2, \; |||\varphi|||_K^2 := \left\|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\right\|_K^2 + \left\|\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)^{\frac{1}{2}}\varphi\right\|_K^2.$$

Bilinear form, weak solution, and energy norm

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Definition (Weak solution)

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Bilinear form, weak solution, and energy norm

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Outline

Optimal abstr. fr. for $-\nabla \cdot (\mathbf{S}\nabla p) + \mathbf{w} \cdot \nabla p + rp = f$

Theorem (Optimal abstract framework, nonconf. & gen. case)

Let $p \in H_0^1(\Omega), p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\begin{aligned} |||p - p_h||| \leq & \inf_{s \in H_0^1(\Omega)} \Big\{ |||p_h - s||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \Big| \mathcal{B}(p - p_h, \varphi) \\ & + \big(\mathbf{w} \cdot \nabla(p_h - s) + \tfrac{1}{2}(\nabla \cdot \mathbf{w})(p_h - s), \varphi\big) \Big| \Big\}. \\ \leq & \; 2|||p - p_h|||. \end{aligned}$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.
- Not computable (includes p).

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- Not computable (includes p).

Optimal abstr. est. for $-\nabla\cdot(\mathbf{S}\nabla p)+\mathbf{w}\cdot\nabla p+rp=f$

Theorem (Optimal abstract estimate, nonconf. & gen. case)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||p-p_h||| \leq & \inf_{s\in H_0^1(\Omega)} \Bigg\{ |||p_h - s||| \\
& + \inf_{\mathbf{t}\in\mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi\in H_0^1(\Omega),\,|||\varphi|||=1} \Big| (f-\nabla\cdot\mathbf{t}-\mathbf{w}\cdot\nabla s - rs,\varphi) \\
& -(\mathbf{S}\nabla p_h+\mathbf{t},\nabla\varphi) + \big((r-\tfrac{1}{2}\nabla\cdot\mathbf{w})(s-p_h),\varphi\big)\Big| \Bigg\} \\
\leq & \; 2|||p-p_h|||.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh and polynomial degree.
- The solution p has been eliminated using $\mathcal{B}(p,\varphi)=(f,\varphi)$.
- Not computable (infimum over an infinite-dim. space).

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& -(\mathbf{S}\nabla p_h+\mathbf{t},\nabla\varphi) + \big(\big(r-\tfrac{1}{2}\nabla\cdot\mathbf{w}\big)(s-p_h),\varphi\big) \Big| \Big\} \\
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$$
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|||p-p_h||| \leq & \inf_{s\in H_0^1(\Omega)} \Big\{ |||p_h - s||| \\
& + \inf_{\mathbf{t}\in\mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi\in H_0^1(\Omega),\,|||\varphi|||=1} \Big| (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
& - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s-p_h), \varphi\big) \Big| \Big\} \\
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$$
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A first comp. est. for $-\nabla \cdot (\mathbf{S}\nabla p) + \mathbf{w} \cdot \nabla p + rp = f$

Theorem (A first computable estimate, nonconf. & gen. case)

Let p be the weak solution and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ and any $s_h \in H^1_0(\Omega)$. Then $|||p - p_h|||$

$$
\begin{aligned}
&\leq |||p_h - s_h||| + \min\left\{ \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{\min_{K\in\mathcal{T}_h} c_{\mathbf{S},K}^{1/2}}, \frac{1}{\min_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}^{1/2}} \right\} \\
&\quad \times \|f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla s_h - r s_h\| \\
&\quad + \left(\|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|^2 + \|(r - \tfrac{1}{2}\nabla\cdot\mathbf{w})^{1/2}(p_h - s_h)\|^2 \right)^{1/2}.
\end{aligned}
$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$).
- $\|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|$ penalizes $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $|||p_h - s_h|||$ penalizes $p_h \notin H^1_0(\Omega)$.

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$$\leq |||p_h - s_h||| + \min\left\{\frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{\min_{K\in\mathcal{T}_h} c_{\mathbf{S},K}^{1/2}}, \frac{1}{\min_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}^{1/2}}\right\}$$

$$\times\|f-\nabla\cdot\mathbf{t}_h-\mathbf{w}\cdot\nabla s_h - rs_h\|$$

$$+\left(\left\|\mathbf{S}^{\frac{1}{2}}\nabla p_h+\mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\right\|^2+\left\|\left(r-\tfrac{1}{2}\nabla\cdot\mathbf{w}\right)^{1/2}(p_h-s_h)\right\|^2\right)^{1/2}.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega}\leq 1$).
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- $|||p_h - s_h|||$ penalizes $p_h \notin H_0^1(\Omega)$.

A first comp. est. for $-\nabla\cdot(\mathbf{S}\nabla p)+\mathbf{w}\cdot\nabla p+rp=f$

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$$
\begin{aligned}
&\le |||p_h-s_h|||+\min\left\{\frac{C_{\mathrm{F},\Omega}^{1/2}h_\Omega}{\min_{K\in\mathcal{T}_h}c_{\mathbf{S},K}^{1/2}},\frac{1}{\min_{K\in\mathcal{T}_h}c_{\mathbf{w},r,K}^{1/2}}\right\}\\
&\quad\times\|f-\nabla\cdot\mathbf{t}_h-\mathbf{w}\cdot\nabla s_h-rs_h\|\\
&\quad+\left(\left\|\mathbf{S}^{\frac12}\nabla p_h+\mathbf{S}^{-\frac12}\mathbf{t}_h\right\|^2+\left\|\left(r-\tfrac12\nabla\cdot\mathbf{w}\right)^{1/2}(p_h-s_h)\right\|^2\right)^{1/2}.
\end{aligned}
$$

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- Guaranteed upper bound ($C_{\mathrm{F},\Omega}\le 1$).
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Let p be the weak solution and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)$ and any $s_h \in H^1_0(\Omega)$. Then $|||p-p_h|||$

$$
\begin{aligned}
&\leq |||p_h-s_h||| + \min\left\{\frac{C_{\mathrm{F},\Omega}^{1/2}h_\Omega}{\min_{K\in\mathcal{T}_h} c_{\mathbf{S},K}^{1/2}}, \frac{1}{\min_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}^{1/2}}\right\}\\
&\quad\times\|f-\nabla\cdot\mathbf{t}_h-\mathbf{w}\cdot\nabla s_h-rs_h\|\\
&\quad+\left(\left\|\mathbf{S}^{\frac12}\nabla p_h+\mathbf{S}^{-\frac12}\mathbf{t}_h\right\|^2+\left\|\left(r-\tfrac12\nabla\cdot\mathbf{w}\right)^{1/2}(p_h-s_h)\right\|^2\right)^{1/2}.
\end{aligned}
$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega}\leq 1$).
- $\left\|\mathbf{S}^{\frac12}\nabla p_h+\mathbf{S}^{-\frac12}\mathbf{t}_h\right\|$ penalizes $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$.
- $|||p_h-s_h|||$ penalizes $p_h \notin H^1_0(\Omega)$.

Outline

Discontinuous Galerkin method

Discontinuous Galerkin method

- Find $p_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $\varphi_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$
\begin{aligned}
&(\mathbf{S}\nabla p_h, \nabla\varphi_h) + ((r - \nabla\cdot\mathbf{w})p_h, \varphi_h) - (p_h, \mathbf{w}\cdot\nabla\varphi_h)\\
&- \sum_{\sigma\in\mathcal{E}_h} \{\langle \mathbf{n}_\sigma^t \{\mathbf{S}\nabla p_h\}_\omega, [\![\varphi_h]\!]\rangle_\sigma + \theta\langle \mathbf{n}_\sigma^t\{\mathbf{S}\nabla\varphi_h\}_\omega, [\![p_h]\!]\rangle_\sigma\}\\
&+ \sum_{\sigma\in\mathcal{E}_h} \{\langle \gamma_\sigma [\![p_h]\!], [\![\varphi_h]\!]\rangle_\sigma + \langle \mathbf{w}\cdot\mathbf{n}_\sigma \{p_h\}, [\![\varphi_h]\!]\rangle_\sigma\} = (f, \varphi_h)
\end{aligned}
$$

- jump operator $[\![v]\!]_\sigma = v^- - v^+$
- average operator $\{v\}_\sigma = \frac{1}{2}(v^- + v^+)$
- harmonic-weighted average operator $\{v\}_\omega = (\omega^- v^- + \omega^+ v^+)$
- $p_h \notin H_0^1(\Omega)$, $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$

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Scalar and diffusive/convective flux reconstructions

Choice of $s_h \in H_0^1(\Omega)$

- $s_h = \mathcal{I}_{\mathrm{Os}}(p_h)$ is the so-called Oswald interpolate of p_h

Choice of $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{H}(\mathrm{div}, \Omega)$

- $\mathbf{t}_h$: diffusive flux reconstruction
- $\mathbf{q}_h$: convective flux reconstruction
- both given on $\mathcal{T}_h$ in the Raviart–Thomas–Nédélec spaces
- defined using the properties of the DG scheme
- satisfy in general

$$(\nabla \cdot \mathbf{t}_h + \nabla \cdot \mathbf{q}_h + (r - \nabla \cdot \mathbf{w}) p_h)|_K = \Pi_K(f)|_K \quad \forall K \in \mathcal{T}_h$$

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Diffusive and convective flux reconstructions

Diffusive flux reconstruction ($l = k$ or $l = k - 1$)

$$\begin{aligned}\langle \mathbf{t}_h \cdot \mathbf{n}_\sigma, q_h \rangle_\sigma &= \langle -\mathbf{n}_\sigma^t \{\mathbf{S}\nabla p_h\}_\omega + \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1} [\![p_h]\!], q_h \rangle_\sigma \\ &\qquad\qquad \forall q_h \in \mathbb{P}_l(\sigma),\ \forall \sigma \in \mathcal{E}_K, \\ (\mathbf{t}_h, \mathbf{r}_h)_K &= -(\mathbf{S}\nabla p_h, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K} \omega_{K,\sigma} \langle \mathbf{n}_\sigma^t \mathbf{S} \mathbf{r}_h, [\![p_h]\!] \rangle_\sigma \\ &\qquad\qquad \forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)\end{aligned}$$

Convective flux reconstruction ($l = k$ or $l = k - 1$)

$$\begin{aligned}\langle \mathbf{q}_h \cdot \mathbf{n}_\sigma, q_h \rangle_\sigma &= \langle \mathbf{w} \cdot \mathbf{n}_\sigma \{p_h\} + \gamma_{\mathbf{w},\sigma} [\![p_h]\!], q_h \rangle_\sigma \\ &\qquad\qquad \forall q_h \in \mathbb{P}_l(\sigma),\ \forall \sigma \in \mathcal{E}_K, \\ (\mathbf{q}_h, \mathbf{r}_h)_K &= (p_h, \mathbf{w} \cdot \mathbf{r}_h)_K \qquad \forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)\end{aligned}$$

Diffusive and convective flux reconstructions

Diffusive flux reconstruction ($l = k$ or $l = k - 1$)

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(\mathbf{q}_h, \mathbf{r}_h)_K &= (p_h, \mathbf{w} \cdot \mathbf{r}_h)_K \qquad \forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)
\end{aligned}
$$

A post. estimate for $-\nabla\cdot(\mathbf{S}\nabla p)+\mathbf{w}\cdot\nabla p+rp=f$

Theorem (A posteriori error estimate)

There holds

$$|||p-p_h|||\leq\left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{\frac12}$$

$$+\left\{\sum_{K\in\mathcal{T}_h}\left(\eta_{\mathrm{R},K}+(\eta_{\mathrm{DF},K}^2+\eta_{\mathrm{C},2,K}^2)^{\frac12}+\eta_{\mathrm{C},1,K}+\eta_{\mathrm{U},K}\right)^2\right\}^{\frac12},$$

where

- $\eta_{\mathrm{NC},K}=|||p_h-\mathcal{I}_{\mathrm{Os}}(p_h)|||_K$ *(nonconformity)*
- $\eta_{\mathrm{DF},K}=\|\mathbf{S}^{\frac12}\nabla p_h+\mathbf{S}^{-\frac12}\mathbf{t}_h\|_K$ *(diffusive flux)*
- $\eta_{\mathrm{R},K}=m_K\|f-\nabla\cdot\mathbf{t}_h-\nabla\cdot\mathbf{q}_h-(r-\nabla\cdot\mathbf{w})p_h\|_K$ *(residual)*
- $\eta_{\mathrm{C},1,K}=m_K\|\nabla\cdot(\mathbf{q}_h-\mathbf{w}s_h)-\Pi_0(\nabla\cdot(\mathbf{q}_h-\mathbf{w}s_h))\|_K$ *(convection)*
- $\eta_{\mathrm{C},2,K}=\frac{1}{c_{\mathbf{w},r,K}^{1/2}}\left\|\frac12(\nabla\cdot\mathbf{w})(p_h-s_h)\right\|_K$ *(convection)*
- $\eta_{\mathrm{U},K}=\sum_{\sigma\in\mathcal{E}_K}m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h-\mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$ *(upwinding).*

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- $\eta_{\mathrm{C},2,K}=\frac{1}{c_{\mathbf{w},r,K}^{1/2}}\left\|\frac{1}{2}(\nabla\cdot\mathbf{w})(p_h-s_h)\right\|_K$ *(convection)*
- $\eta_{\mathrm{U},K}=\sum_{\sigma\in\mathcal{E}_K}m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h-\mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$ *(upwinding).*

Loc. efficiency for $-\nabla \cdot (\mathbf{S}\nabla p) + \mathbf{w} \cdot \nabla p + rp = f$

Theorem (Local efficiency)

There holds

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{DF},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{C},1,K} + \eta_{\mathrm{C},2,K} + \eta_{\mathrm{U},K} \leq C_{\mathrm{eff},K} |||p - p_h|||_{*,\widetilde{\mathcal{E}}_K}.$$

Properties

- guaranteed upper bound
- local and global efficiency
- negligible evaluation cost
- residual estimator $\eta_{\mathrm{R},K}$ is a higher-order term
- valid also on anisotropic meshes
- valid uniformly with respect to polynomial degree
- semi-robust ($C_{\mathrm{eff},K}$ depends on local inhomogeneities and anisotropies and affinely on Pe_K)

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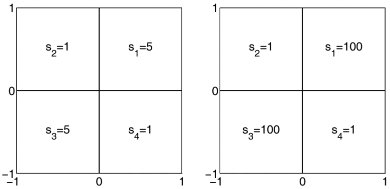
Discontinuous diffusion tensor and discontinuous Galerkin methods

- consider the pure diffusion equation

$$-\nabla \cdot (\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1, 1) \times (-1, 1)$$

- discontinuous and inhomogeneous **S**, two cases:

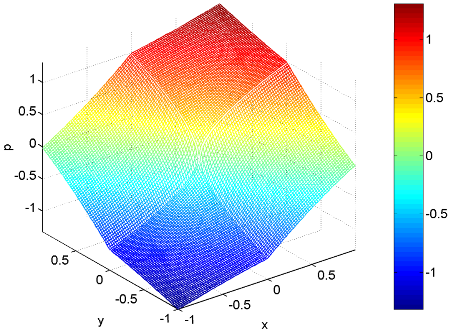


- analytical solution: singularity at the origin

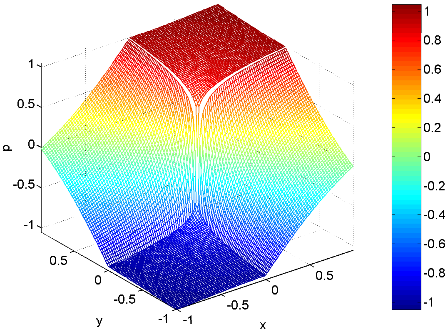
$$p(r, \theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r, θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

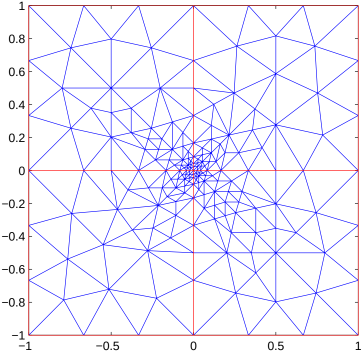


Case 1

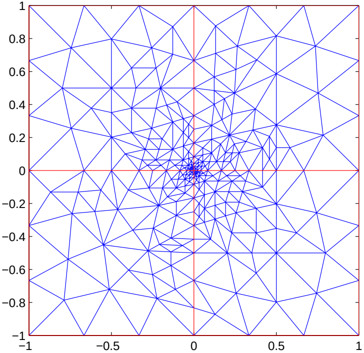


Case 2

Series of refined meshes, case 1



Mesh with 342 elements



Mesh with 494 elements

Estimated and actual error, case 1

			$l=0$		$l=1$	
N	$\lvert\lvert\lvert p-p_h \rvert\rvert\rvert$	η_{NC}	η_{DF}	eff.	η_{DF}	eff.
112	6.11e-01	8.70e-1	7.43e-1	1.9	6.00e-1	1.7
448	4.28e-01	6.09e-1	5.35e-1	1.9	4.32e-1	1.7
1792	2.97e-01	4.23e-1	3.74e-1	1.9	3.05e-1	1.8
7168	2.01e-01	2.92e-1	2.60e-1	1.9	2.12e-1	1.8
order	0.53	0.53	0.53	-	0.52	-

Convergence rates of error estimators for test case 1, unstructured meshes

Estimated and actual error, case 2

			$l=0$		$l=1$	
N	$\|\|\|p-p_h\|\|\|$	η_{NC}	η_{DF}	eff.	η_{DF}	eff.
112	3.27	11.8	2.39	3.7	1.89	3.7
448	3.11	11.3	2.33	3.7	1.84	3.7
1792	2.93	10.8	2.23	3.8	1.77	3.7
7168	2.75	10.3	2.12	3.8	1.68	3.8
order	0.09	0.08	0.08	-	0.07	-

Convergence rates of error estimators for test case 2, unstructured meshes

Outline

A convection–diffusion–reaction problem with general boundary conditions

Problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p)+\nabla\cdot(p\mathbf{w})+rp &= f \quad \text{in } \Omega,\\
p &= g \quad \text{on } \Gamma_{\mathrm{D}},\\
-\mathbf{S}\nabla p\cdot\mathbf{n} &= u \quad \text{on } \Gamma_{\mathrm{N}}
\end{aligned}
$$

Assumptions

- $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a polygonal domain
- $\mathbf{S}|_K$ is a constant SPD matrix, $c_{\mathbf{S},K}$ its smallest, and $C_{\mathbf{S},K}$ its largest eigenvalue on each $K\in\mathcal{T}_h$
- $\left(\frac{1}{2}\nabla\cdot\mathbf{w}+r\right)|_K\geq c_{\mathbf{w},r,K}\geq 0$ on each $K\in\mathcal{T}_h$ (from pure diffusion to convection–diffusion–reaction cases)

Difficulties

- $\mathbf{S}$ is a piecewise constant matrix, inhomogeneous and anisotropic
- $\mathbf{w}$ is dominating

A convection–diffusion–reaction problem with general boundary conditions

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Bilinear form, weak solution, and energy norm

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H^1(\mathcal{T}_h)$ by

$$\mathcal{B}(p,\varphi) := \sum_{K \in \mathcal{T}_h} \left\{ (\mathbf{S}\nabla p, \nabla\varphi)_K + (\nabla\cdot(\mathbf{w}p), \varphi)_K + (rp,\varphi)_K \right\}.$$

Definition (Weak solution)

Weak solution: $p \in H^1(\Omega)$ with $p|_{\Gamma_{\mathrm{D}}} = g$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi) - \langle u, \varphi\rangle_{\Gamma_{\mathrm{N}}} \qquad \forall \varphi \in H^1_{\mathrm{D}}(\Omega).$$

Definition (Energy (semi-)norm)

We define the energy (semi-)norm for $\varphi \in H^1(\mathcal{T}_h)$ by

$$|||\varphi|||^2 := \sum_{K \in \mathcal{T}_h} |||\varphi|||_K^2, \quad |||\varphi|||_K^2 := \left\|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\right\|_K^2 + \left\|\left(\tfrac{1}{2}\nabla\cdot\mathbf{w} + r\right)^{\frac{1}{2}}\varphi\right\|_K^2.$$

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General finite volume scheme

Definition (FV scheme for $-\nabla \cdot (\mathbf{S}\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f$)

Find p_K, $K \in \mathcal{T}_h$, such that

$$\sum_{\sigma\in\mathcal{E}_K} S_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} W_{K,\sigma} + r_K p_K|K| = f_K|K| \qquad \forall K \in \mathcal{T}_h\,.$$

- $\left.\begin{array}{ll} S_{K,\sigma}: & \text{diffusive flux} \\ W_{K,\sigma}: & \text{convective flux} \end{array}\right\}$ no specific form, just conservativity needed
- $r_K := (r, 1)/|K|$
- $f_K := (f, 1)/|K|$

Example

- $S_{K,\sigma} = -s_{K,L}\frac{|\sigma_{K,L}|}{d_{K,L}}(p_L - p_K)$
- $W_{K,\sigma} = p_\sigma \langle \mathbf{w} \cdot \mathbf{n}, 1\rangle_\sigma$: weighted-upwind

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Example

- $S_{K,\sigma}=-s_{K,L}\frac{|\sigma_{K,L}|}{d_{K,L}}(p_L-p_K)$
- $W_{K,\sigma}=p_\sigma\langle\mathbf{w}\cdot\mathbf{n},1\rangle_\sigma$: weighted-upwind

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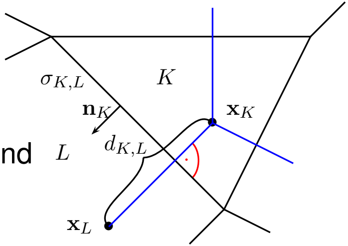
Find p_K, $K \in \mathcal{T}_h$, such that

$$\sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma} + \sum_{\sigma \in \mathcal{E}_K} W_{K,\sigma} + r_K p_K |K| = f_K |K| \qquad \forall K \in \mathcal{T}_h .$$

- $\left.\begin{array}{ll} S_{K,\sigma}: & \text{diffusive flux} \\ W_{K,\sigma}: & \text{convective flux} \end{array}\right\}$ no specific form, just conservativity needed
- $r_K := (r, 1)/|K|$
- $f_K := (f, 1)/|K|$

Example

- $S_{K,\sigma} = -s_{K,L} \frac{|\sigma_{K,L}|}{d_{K,L}} (p_L - p_K)$
- $W_{K,\sigma} = p_\sigma \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma$: weighted-upwind



A locally postprocessed scalar variable $\tilde{p}_h$

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) &= \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} S_{K,\sigma}, \\ (1-\mu_K)(\tilde{p}_h,1)_K/|K| + \mu_K\tilde{p}_h(\mathbf{x}_K) &= p_K, \\ -\mathbf{S}\nabla\tilde{p}_h|_K\cdot\mathbf{n} &= S_{K,\sigma}/|\sigma| \quad \forall\sigma\in\mathcal{E}_K\,. \end{aligned}$$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- flux of $\tilde{p}_h$ is given by $S_{K,\sigma}$, point or mean value by p_K
- $\tilde{p}_h \notin H^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- $-\mathbf{S}\nabla\tilde{p}_h \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow$ put $\mathbf{t}_h = -\mathbf{S}\nabla\tilde{p}_h$ in the gen. fram.
- given on $\mathcal{T}_h$, no need for a dual mesh
- for simplices or rectangular parallelepipeds when $\mathbf{S}$ is diagonal: $\tilde{p}_h$ is a piecewise second-order polynomial

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A post. estimate for $-\nabla\cdot(\mathbf{S}\nabla p)+\nabla\cdot(p\mathbf{w})+rp=f$

Theorem (A posteriori error estimate)

There holds

$$|||p-\tilde{p}_h|||\le\left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{\frac{1}{2}}+\left\{\sum_{K\in\mathcal{T}_h}(\eta_{\mathrm{R},K}+\eta_{\mathrm{C},K}+\eta_{\mathrm{U},K}+\eta_{\mathrm{RQ},K}+\eta_{\Gamma_{\mathrm{N}},K})^2\right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K}:=|||\tilde{p}_h-\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$: Oswald int. operator (Burman and Ern '07)
- **residual estimator**
  - $\eta_{\mathrm{R},K}:=m_K\|f+\nabla\cdot(\mathbf{S}_K\nabla\tilde{p}_h)-\nabla\cdot(\tilde{p}_h\mathbf{w})-r\tilde{p}_h\|_K$
  - $m_K^2:=\min\left\{C_{\mathrm{P}}\frac{h_K^2}{c_{\mathbf{S},K}},\frac{1}{c_{\mathbf{w},r,K}}\right\}$
- **convection estimator**
  - $\eta_{\mathrm{C},K}:=\min\left\{\frac{\|\nabla\cdot(v\mathbf{w})-\frac{1}{2}v\nabla\cdot\mathbf{w}\|_K+\|\nabla\cdot(v\mathbf{w})\|_K}{\sqrt{c_{\mathbf{w},r,K}}},\left(\frac{C_{\mathrm{P}}h_K^2\|\nabla v\cdot\mathbf{w}\|_K^2}{c_{\mathbf{S},K}}+\frac{9\|v\nabla\cdot\mathbf{w}\|_K^2}{4c_{\mathbf{w},r,K}}\right)^{\frac{1}{2}}\right\}$
  - $v=\tilde{p}_h-\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$

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  - $\eta_{\mathrm{U},K} := \sum_{\sigma \in \mathcal{E}_K \setminus \mathcal{E}_h^{\mathrm{N}}} m_\sigma \|(W_{K,\sigma} - \langle \mathcal{I}_{\mathrm{Os}}^{\Gamma}(\tilde{p}_h)\mathbf{w} \cdot \mathbf{n}, 1\rangle_\sigma)|\sigma|^{-1}\|_\sigma$
  - $W_{K,\sigma} = p_\sigma \langle \mathbf{w} \cdot \mathbf{n}, 1\rangle_\sigma$: weighted-upwind
  - m_σ: function of $c_{\mathbf{S},K}$, $c_{\mathbf{w},r,K} = \left(\frac{1}{2}\nabla \cdot \mathbf{w} + r\right)|_K$, d, h_K, $|\sigma|$, $|K|$
  - all dependencies evaluated explicitly
- **reaction quadrature estimator**
  - $\eta_{\mathrm{RQ},K} := \frac{1}{\sqrt{c_{\mathbf{w},r,K}}} \|r_K p_K - (r\tilde{p}_h, 1)_K |K|^{-1}\|_K$
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- **Neumann boundary estimator**
  - $\eta_{\Gamma_{\mathrm{N}},K} := 0 + \frac{\sqrt{h_K}}{\sqrt{c_{\mathbf{S},K}}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{N}}} \sqrt{C_{\mathrm{t},K,\sigma}} \|u_\sigma - u\|_\sigma$

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Loc. efficiency for $-\nabla\cdot(\mathbf{S}\nabla p)+\nabla\cdot(p\mathbf{w})+rp=f$

Theorem (Local efficiency of the residual estimator)

There holds $\eta_{\mathrm{R},K}\leq$

$$|||p-\tilde{p}_h|||_K C\left\{\sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}}\max\left\{1,\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\}+\min\left\{\mathrm{Pe}_K,\sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}}\varrho_K\right\}\right\}.$$

- residual estimator is **locally efficient** (lower bound for error on K) and **semi-robust** ($C_{\mathrm{eff},K}$ depends on local anisotropies and affinely on Pe_K)
- $C_{\mathrm{eff},K}$:
  - C independent of h_K, $\mathbf{S}$, $\mathbf{w}$, and r
  - no dependency on inhomogeneities
  - $\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\leq 2$ for r nonnegative
  - $C_{\mathrm{eff},K}$ depends affinely on Pe_K
  - $\varrho_K:=\frac{|\mathbf{w}|_K|}{\sqrt{c_{\mathbf{w},r,K}}\sqrt{c_{\mathbf{S},K}}}$ prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

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Loc. efficiency for $-\nabla \cdot (\mathbf{S}\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f$

Theorem (Local efficiency of the nonconformity and convection estimators)

There holds

$$\eta_{\mathrm{NC},K}^2 + \eta_{\mathrm{C},K}^2 \leq \alpha \sum_{L;\, L\cap K \neq \emptyset} |||p - \tilde{p}_h|||_L^2 + \beta \inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)} \sum_{L;\, L\cap K\neq\emptyset} \|p - s_h\|_L^2 .$$

- nonconformity and convection estimators are **locally efficient** (up to higher-order terms if $c_{\mathbf{w},r,K} \neq 0$) and **semi-robust** ($C_{\mathrm{eff},K}$ depends on local inhomogeneities and anisotropies and affinely on Pe_K)
- $C_{\mathrm{eff},K}$:
  - depends on maximal ratio of inhomogeneities around K
  - depends on anisotropy in each L around K by $\frac{C_{\mathbf{S},L}}{c_{\mathbf{S},L}}$
  - $C_{\mathrm{eff},K}$ depends affinely on Pe_K
  - again $\min\{\mathrm{Pe}_L, \varrho_L\}$ in each L around K prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

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There holds

$$\eta_{\mathrm{NC},K}^2 + \eta_{\mathrm{C},K}^2 \leq \alpha \sum_{L;\,L\cap K\neq\emptyset} |||p - \tilde{p}_h|||_L^2 + \beta \inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)} \sum_{L;\,L\cap K\neq\emptyset} \|p - s_h\|_L^2 .$$

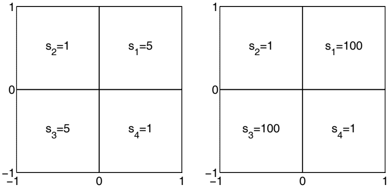
- nonconformity and convection estimators are **locally efficient** (up to higher-order terms if $c_{\mathbf{w},r,K} \neq 0$) and **semi-robust** ($C_{\mathrm{eff},K}$ depends on local inhomogeneities and anisotropies and affinely on Pe_K)
- $C_{\mathrm{eff},K}$:
  - depends on maximal ratio of inhomogeneities around K
  - depends on anisotropy in each L around K by $\frac{C_{\mathbf{S},L}}{c_{\mathbf{S},L}}$
  - $C_{\mathrm{eff},K}$ depends affinely on Pe_K
  - again $\min\{\mathrm{Pe}_L, \varrho_L\}$ in each L around K prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

Discontinuous diffusion tensor and finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot (\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

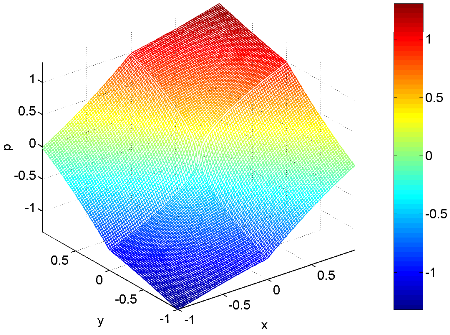


- analytical solution: singularity at the origin

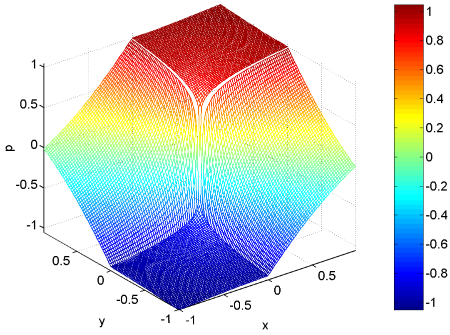
$$p(r,\theta)|_{\Omega_i} = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

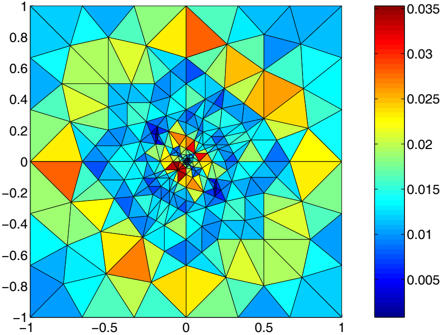


Case 1

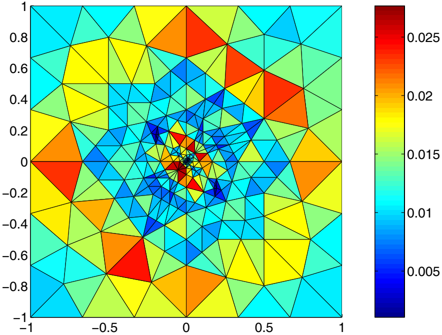


Case 2

Error distribution on an adaptively refined mesh, case 1

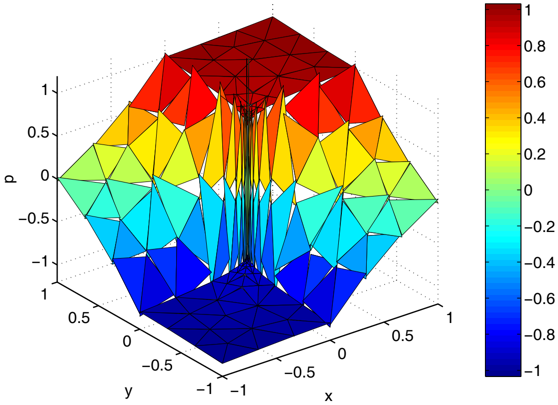


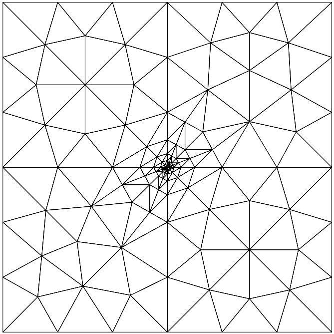
Estimated error distribution



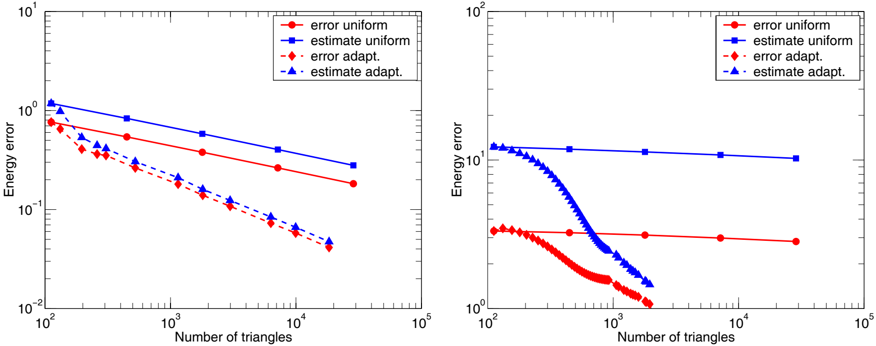
Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, case 2



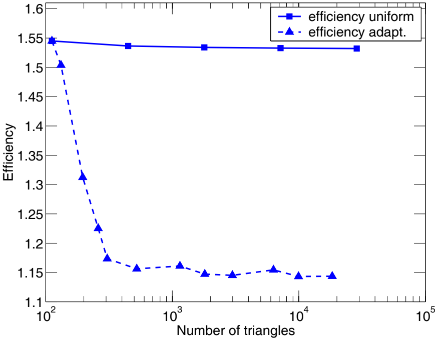


Estimated and actual error in uniformly/adaptively refined meshes

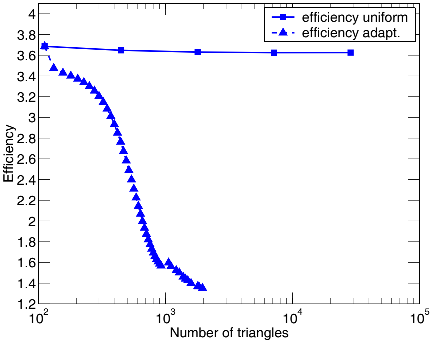


Case 1 Case 2

Effectivity indices in uniformly/adaptively refined meshes



Case 1



Case 2

Convection-dominated problem

- consider the convection–diffusion–reaction equation

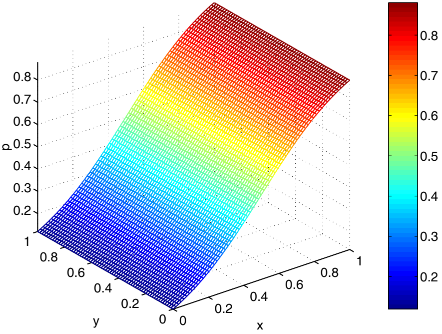
$$-\varepsilon \triangle p + \nabla \cdot (p(0,1)) + p = f \quad \text{in} \quad \Omega = (0,1) \times (0,1)$$

- analytical solution: layer of width a

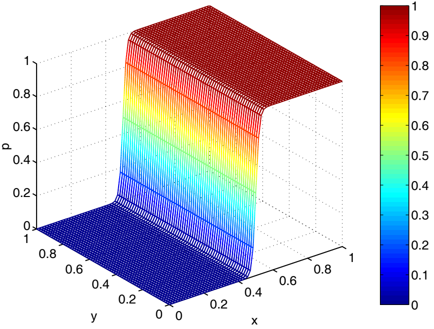
$$p(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)$$

- consider
  - $\varepsilon = 1$, $a = 0.5$
  - $\varepsilon = 10^{-2}$, $a = 0.05$
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- unstructured grid of 46 elements given, uniformly/adaptively refined

Analytical solutions

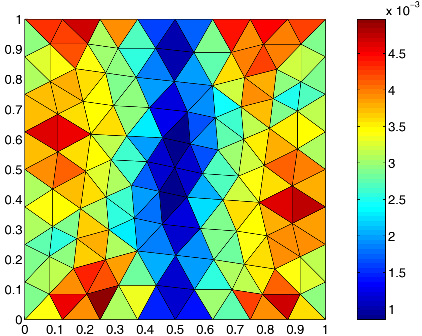


Case $\varepsilon = 1$, $a = 0.5$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

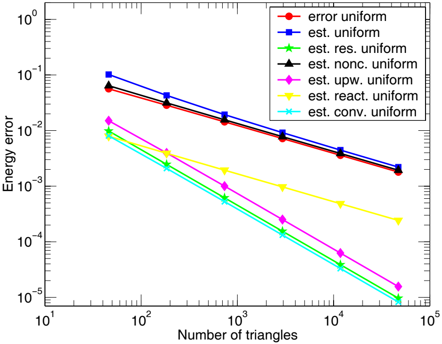
Error distribution on a uniformly refined mesh, $\varepsilon = 1$, $a = 0.5$



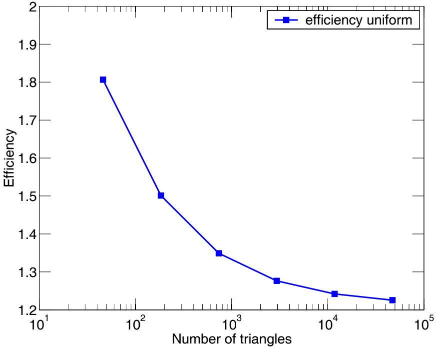
Estimated error distribution

Exact error distribution

Estimated and actual error and the effectivity index, $\varepsilon = 1$, $a = 0.5$

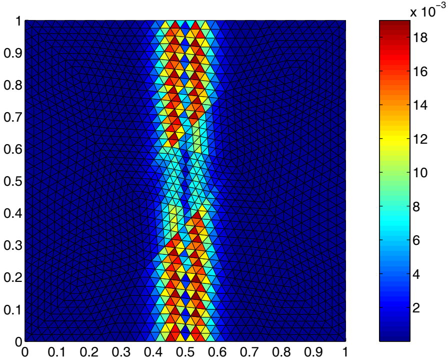


The different estimators

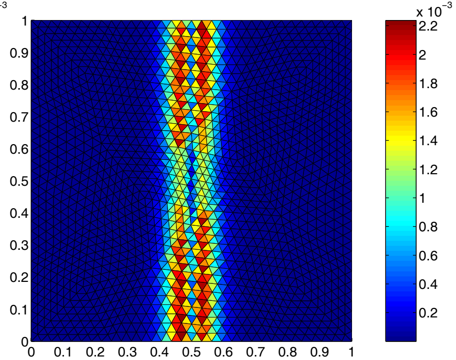


Effectivity index

Error distribution on a uniformly refined mesh, $\varepsilon = 10^{-2}$, $a = 0.05$

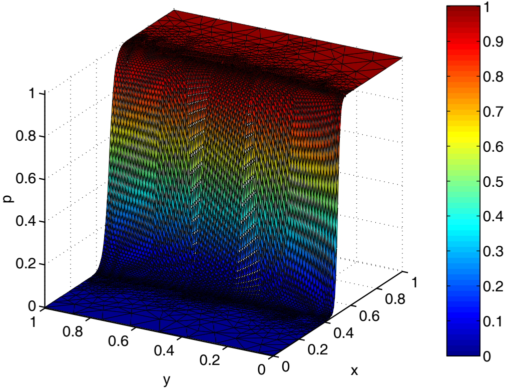


Estimated error distribution

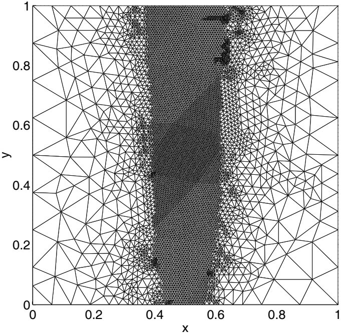


Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$

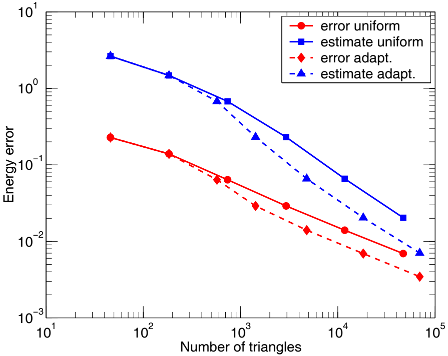


Approximate solution

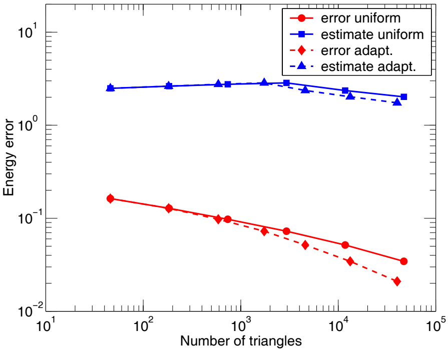


Adaptively refined mesh

Estimated and actual error in uniformly/adaptively refined meshes

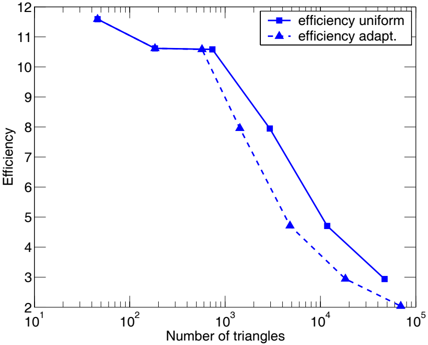


Case $\varepsilon = 10^{-2}$, $a = 0.05$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

Effectivity indices in uniformly/adaptively refined meshes



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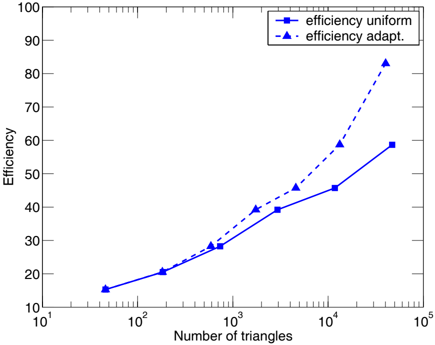

Case $\varepsilon = 10^{-4}$, $a = 0.02$

Outline

A reaction–diffusion problem

Problem

$$\begin{aligned} -\triangle p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $r \in L^\infty(\Omega)$ such that for each $K \in \mathcal{T}_h$, $0 \leq c_{r,K} \leq r$, a.e. in K

Bilinear form, energy norm, and weak solution

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H_0^1(\Omega)$ by

$$\mathcal{B}(p, \varphi) := (\nabla p, \nabla \varphi)_\Omega + (r^{1/2} p, r^{1/2} \varphi)_\Omega \,.$$

Definition (Energy norm)

The associated energy norm for $\varphi \in H_0^1(\Omega)$ is given by

$$|||\varphi|||_\Omega^2 := \mathcal{B}(\varphi, \varphi) \,.$$

Definition (Weak solution)

Weak solution: $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p, \varphi) = (f, \varphi)_\Omega \qquad \forall \varphi \in H_0^1(\Omega).$$

Bilinear form, energy norm, and weak solution

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Residual and diffusive flux estimators

Define:

- **residual estimator**

$$\eta_{\mathrm{R},D} := m_D \| f - \nabla \cdot \mathbf{t}_h - r p_h \|_D$$

- **diffusive flux estimator**

$$\eta_{\mathrm{DF},D} := \min \left\{ \eta^{(1)}_{DF,D}, \eta^{(2)}_{DF,D} \right\},$$

where

$$\eta^{(1)}_{\mathrm{DF},D} := \| \nabla p_h + \mathbf{t}_h \|_D$$

$$\eta^{(2)}_{\mathrm{DF},D} := \left\{ \sum_{K \in \mathcal{S}_D} \left(m_K \| \triangle p_h + \nabla \cdot \mathbf{t}_h \|_K + \tilde{m}_K^{\frac{1}{2}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{G}_h^{\mathrm{int}}} C_t^{\frac{1}{2}} \| (\nabla p_h + \mathbf{t}_h) \cdot \mathbf{n} \|_\sigma \right)^2 \right\}^{\frac{1}{2}}$$

Robust a posteriori error estimates for $-\triangle p + rp = f$

Theorem (A posteriori error estimate)

There holds

$$|||p - p_h|||_\Omega \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{\frac{1}{2}} .$$

Theorem (Local efficiency)

There holds

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C |||p - p_h|||_{\mathcal{T}_{V_D}} ,$$

where C depends only on d, $\kappa_{\mathcal{T}}$, and m.

Properties

- **guaranteed upper bound**
- local and global efficiency
- **robustness**
- negligible evaluation cost

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Properties

- **guaranteed upper bound**
- local and global efficiency
- **robustness**
- negligible evaluation cost

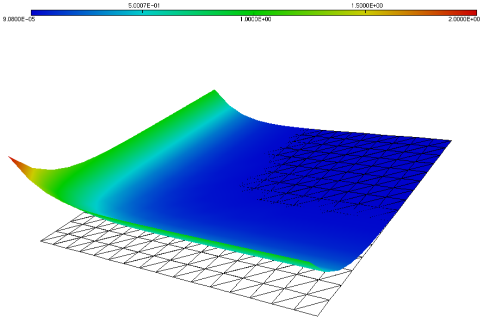
Problem and exact solution

Problem

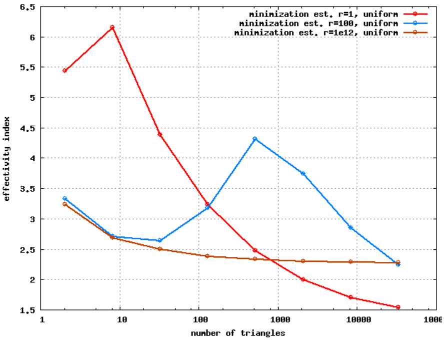
$$
\begin{aligned}
-\triangle p + rp &= 0, \qquad &\text{in } \Omega \\
p &= p_0, \qquad &\text{on } \partial\Omega
\end{aligned}
$$

Solution

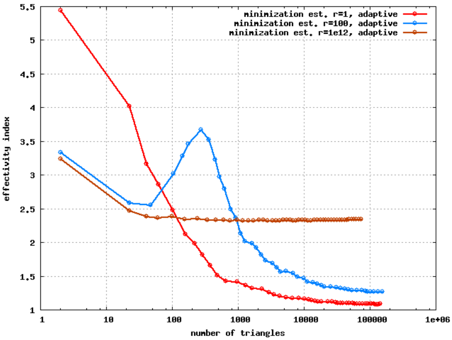
$$p_0(x,y) = e^{-\sqrt{r}x} + e^{-\sqrt{r}y}$$



Effectivity indices

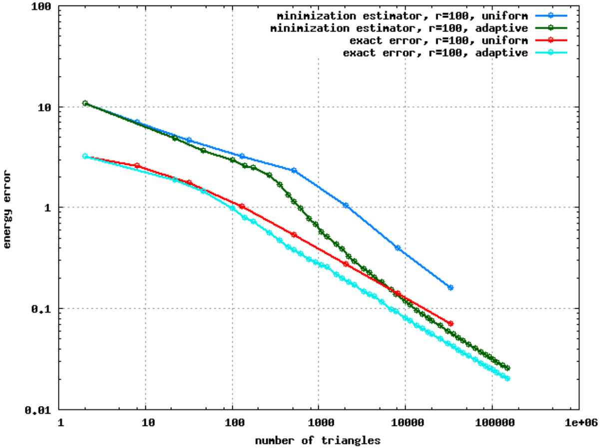


Uniform refinement



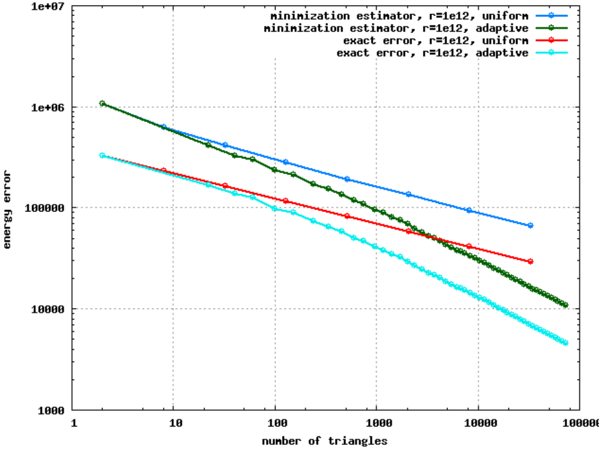
Adaptive refinement

Estimated and actual errors, $r = 100$



Estimated and actual errors, $r = 100$

Estimated and actual errors, $r = 10^{12}$



Estimated and actual errors, $r = 10^{12}$

A model pure diffusion problem

A model pure diffusion problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

A model pure diffusion problem

A model pure diffusion problem

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\begin{aligned}
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A model pure diffusion problem

A model pure diffusion problem

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Estimate including inexact linear systems error

Theorem (A posteriori error estimate including inexact linear systems solution error, cell-centered FVs or MFEs)

There holds

$$|||p - \tilde{p}_h^*||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^* - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^*)|||_K$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla \cdot (\mathbf{S}_K \nabla \tilde{p}_h^*)\|_K$
  - $m_K^2 := C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}}$
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \|\mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
  - $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{T}_h)$ is such that $\nabla \cdot \mathbf{t}_h|_K = \frac{R_K}{|K|}$
  - R is the residual vector

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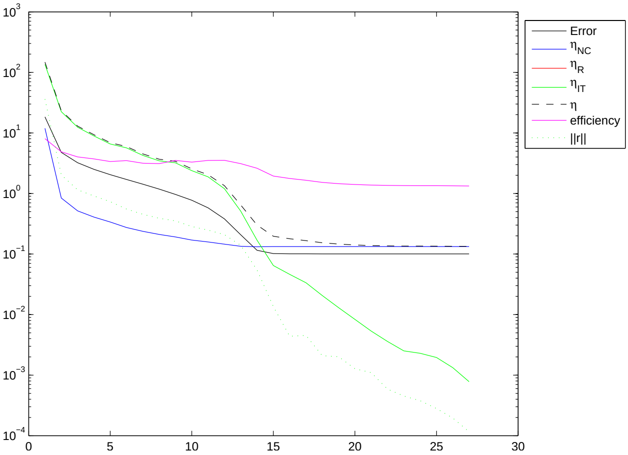
Finite volume estimates including inexact linear systems solution



Different estimators, error, and effectivity index

Time-dependent and nonlinear problems

Time-dependent problems

VERFÜRTH, R., A posteriori error estimates for finite element discretizations of the heat equation, *Calcolo* **40** (2003), 195–212.

- divide the estimate into time and space estimators
- use the time estimator to refine the time step
- use the space estimator to refine the space mesh
- mesh refinement and coarsening (“moving meshes”)

Nonlinear problems

KIM, K. Y., A posteriori error estimators for locally conservative methods of nonlinear elliptic problems, *Appl. Numer. Math.* **57** (2007), 1065–1080.

- a posteriori error estimates for monotone elliptic operators
- applies directly to all our results

Time-dependent and nonlinear problems

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Outline

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no convexity of Ω needed
- no saturation assumption
- no Helmholtz decomposition
- no shape-regularity needed for the upper bounds (only for the efficiency proofs)
- polynomial degree-independent upper bound
- no “monotonicity” hypothesis on inhomogeneities distribution
- the only important tool: optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases

Essentials of the estimates

Essentials of the estimates

- nonconformity estimate: compare the approximate solution p_h to a $H^1(\Omega)$-conforming potential s_h
- diffusive flux estimate: compare the flux of the approximate solution $-\mathbf{S}\nabla p_h$ to a $\mathbf{H}(\mathrm{div},\Omega)$-conforming flux $\mathbf{t}_h$
- evaluate the residue for $\mathbf{t}_h$
- for optimality, $\mathbf{t}_h$ has to be locally conservative
- in conforming methods ($p_h \in H^1(\Omega)$), there is no nonconformity estimate
- in flux-conforming methods ($-\mathbf{S}\nabla p_h \in \mathbf{H}(\mathrm{div},\Omega)$), there is no diffusive flux estimate
- additional nonsymmetric term for convection
- use problem-dependent energy norms

Conclusions and future work

Conclusions

- guaranteed, locally efficient, and robust (in some cases) a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes
- based on local conservativity

Future work

- asymptotic exactness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Maxwell)

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Thank you for your attention!