A posteriori control of numerical error and stopping criteria for linear and nonlinear solvers

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Outline

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1. Introduction
2. Adaptive inexact Newton method
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Numerical results
3. Application to two-phase flow in porous media
   - A guaranteed a posteriori error estimate
   - Fully implicit cell-centered finite volumes
   - Iteratively coupled implicit pressure–explicit saturation vertex-centered finite volumes
4. Conclusions and future directions

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*
   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$
3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*
      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
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Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

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Previous results

Inexact Newton method

- Eisenstat and Walker (1990's) (conception, convergence, a priori error estimates)
- Moret (1989) (discrete a posteriori error estimates)

Adaptive inexact Newton method

- Bank and Rose (1982), combination with multigrid
- Deuflhard (1990's), adaptive damping and multigrid

Stopping criteria for algebraic solvers

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid

A posteriori error estimates for nonlinear problems

- Han (1994), general framework
- Verfürth (1994), residual estimates
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors

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Outline

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Example

p-Laplacian: $\boldsymbol{\sigma}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$, $p \in (1,+\infty)$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega), v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

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A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} \approx f.$$

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

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Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

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Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

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- $\gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

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- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta^{k,i}_{\mathrm{disc},K} + \eta^{k,i}_{\mathrm{lin},K} + \eta^{k,i}_{\mathrm{alg},K} \lesssim \mathcal{J}^{\mathrm{up}}_{u,\mathfrak{T}_K}(u_h^{k,i}) + \eta^{k,i}_{\mathrm{quad},\mathfrak{T}_K} + \eta^{k,i}_{\mathrm{osc},\mathfrak{T}_K}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

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Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

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- robustness and **local efficiency** for an upper bound on the dual norm

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1. Introduction
2. Adaptive inexact Newton method
   - A guaranteed a posteriori error estimate
   - Stopping criteria and efficiency
   - Numerical results
3. Application to two-phase flow in porous media
   - A guaranteed a posteriori error estimate
   - Fully implicit cell-centered finite volumes
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Numerical experiment I

Model problem

- p-Laplacian

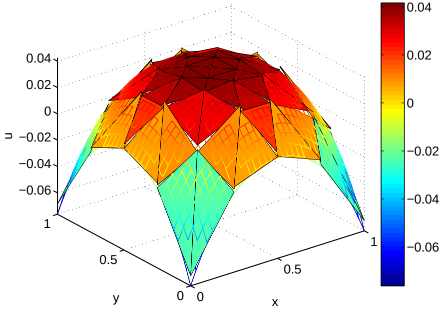
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega, \\ u &= u_0 && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

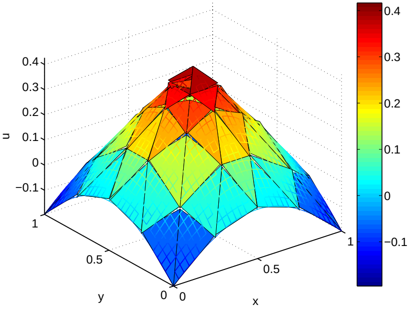
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- nonconforming finite elements

Analytical and approximate solutions

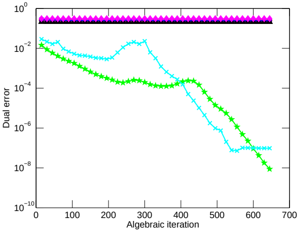


Case $p = 1.5$

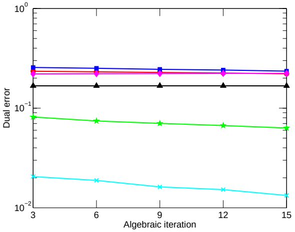


Case $p = 10$

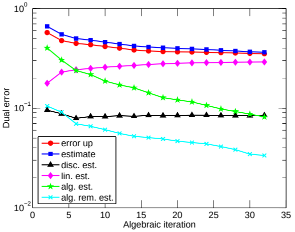
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



Newton

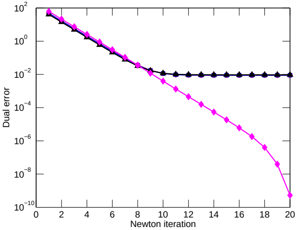


inexact Newton

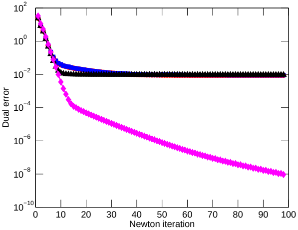


ad. inexact Newton

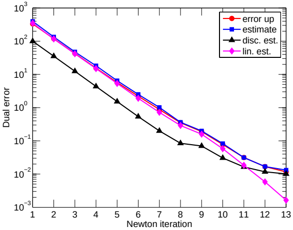
Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



Newton

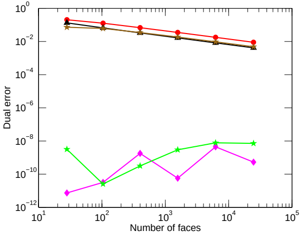


inexact Newton

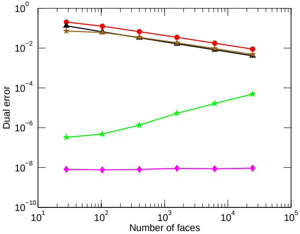


ad. inexact Newton

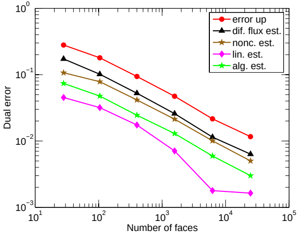
Error and estimators, $p = 10$



Newton

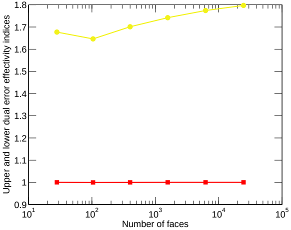


inexact Newton

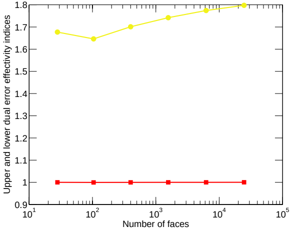


ad. inexact Newton

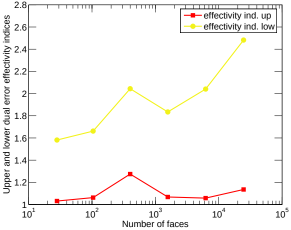
Effectivity indices, $p = 10$



Newton

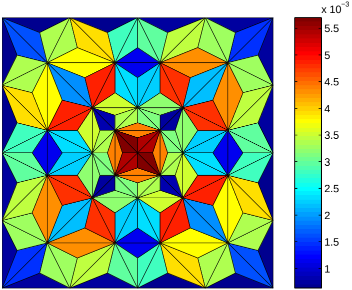


inexact Newton

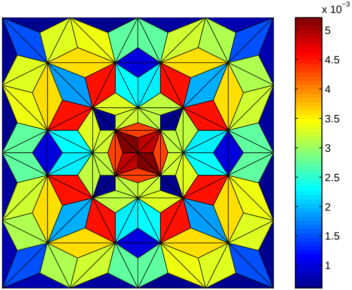


ad. inexact Newton

Error distribution, $p = 10$

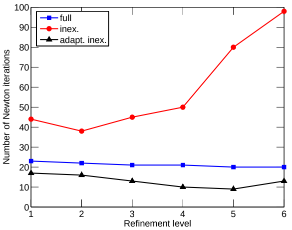


Estimated error distribution

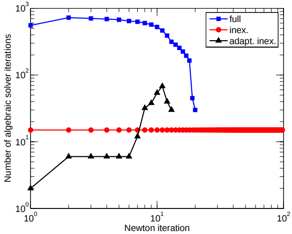


Exact error distribution

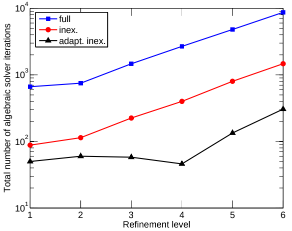
Newton and algebraic iterations, $p = 10$



Newton it. / refinement

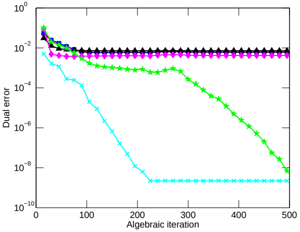


alg. it. / Newton step

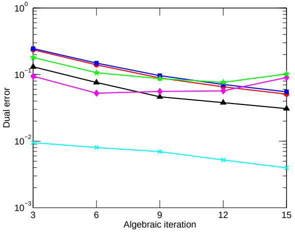


alg. it. / refinement

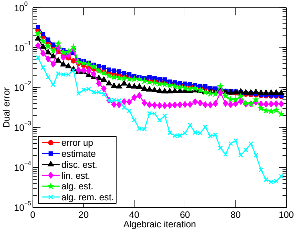
Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.



Newton

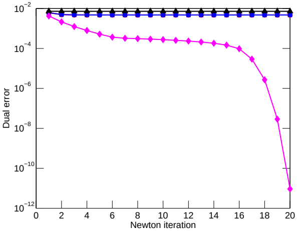


inexact Newton

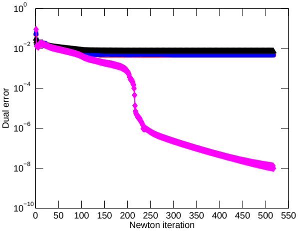


ad. inexact Newton

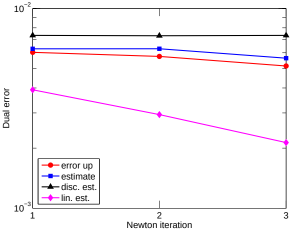
Error and estimators as a function of Newton iterations, $p = 1.5$, 6th level mesh



Newton

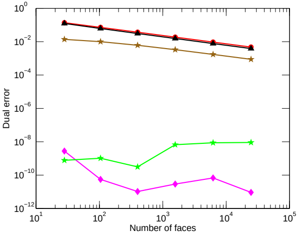


inexact Newton

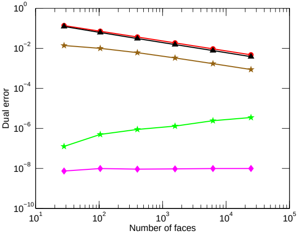


ad. inexact Newton

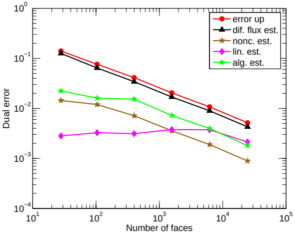
Error and estimators, $p = 1.5$



Newton

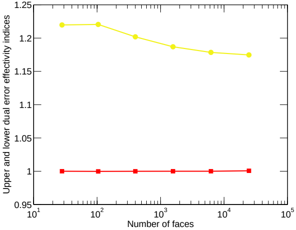


inexact Newton

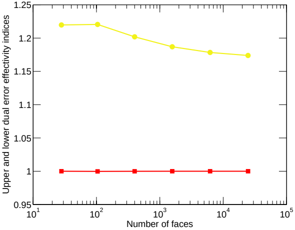


ad. inexact Newton

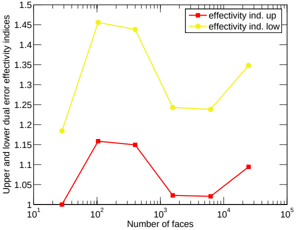
Effectivity indices, $p = 1.5$



Newton

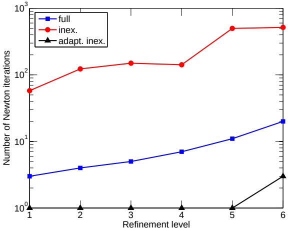


inexact Newton

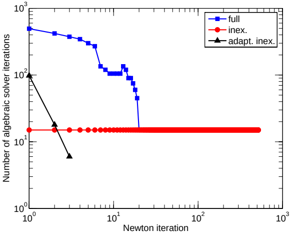


ad. inexact Newton

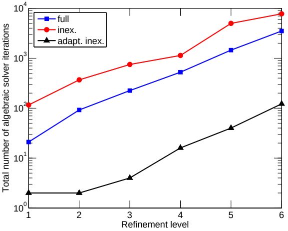
Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement



alg. it. / Newton step



alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

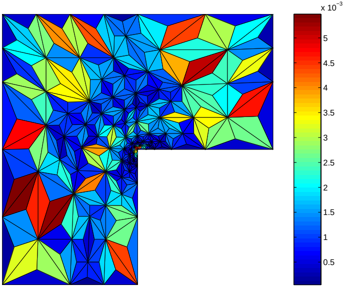
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega, \\ u &= u_0 && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

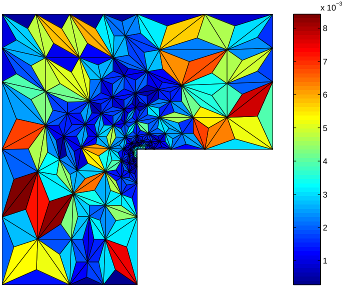
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

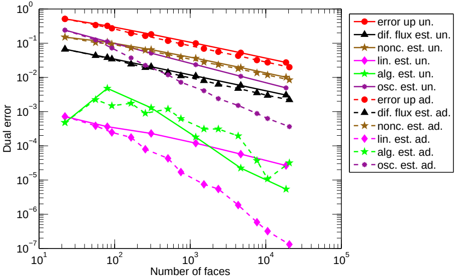


Estimated error distribution

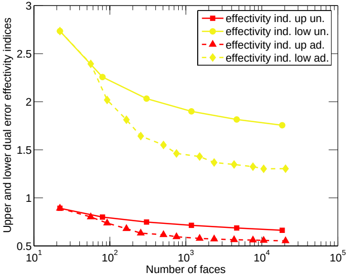


Exact error distribution

Estimated and actual errors and the effectivity index

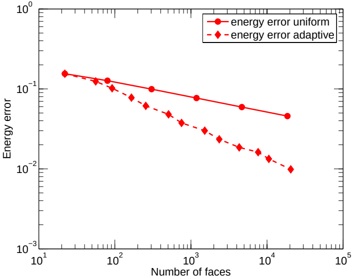


Estimated and actual errors

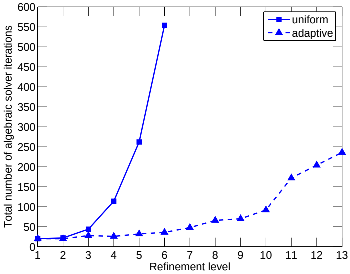


Effectivity index

Energy error and overall performance



Energy error



Overall performance

Outline

Two-phase flow

Horizontal two-phase flow in porous media

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = \pi(s_\mathrm{w})$$

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

Brooks–Corey model, $s_\mathrm{e} := \frac{s_\mathrm{w} - s_\mathrm{rw}}{1 - s_\mathrm{rw} - s_\mathrm{rn}}$

- relative permeabilities

$$k_{\mathrm{r,w}}(s_\mathrm{w}) = s_\mathrm{e}^4, \quad k_{\mathrm{r,n}}(s_\mathrm{w}) = (1 - s_\mathrm{e})^2(1 - s_\mathrm{e}^2)$$

- capillary pressure

$$\pi(s_\mathrm{w}) = p_\mathrm{d} s_\mathrm{e}^{-\frac{1}{2}}$$

Two-phase flow

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Two-phase flow in porous media

Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s^{n,k,i}_{\mathrm{w},h\tau}, p^{n,k,i}_{\mathrm{w},h\tau})$. Then

$$|||(s_{\mathrm{w}} - s^{n,k,i}_{\mathrm{w},h\tau}, p_{\mathrm{w}} - p^{n,k,i}_{\mathrm{w},h\tau})|||_{I_n} \le \eta^{n,k,i}_{\mathrm{sp}} + \eta^{n,k,i}_{\mathrm{tm}} + \eta^{n,k,i}_{\mathrm{lin}} + \eta^{n,k,i}_{\mathrm{alg}}.$$

Error components

- $\eta^{n,k,i}_{\mathrm{sp}}$: spatial discretization
- $\eta^{n,k,i}_{\mathrm{tm}}$: temporal discretization
- $\eta^{n,k,i}_{\mathrm{lin}}$: linearization
- $\eta^{n,k,i}_{\mathrm{alg}}$: algebraic solver

Two-phase flow in porous media

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Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
- $\eta_{\mathrm{tm}}^{n,k,i}$: temporal discretization
- $\eta_{\mathrm{lin}}^{n,k,i}$: linearization
- $\eta_{\mathrm{alg}}^{n,k,i}$: algebraic solver

Local estimators

- *spatial estimators*

$$\eta_{\mathrm{sp},K}^{n,k,i}(t) := \Bigg\{ \sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}} \big(\|\mathbf{d}_{\alpha,h}^{n,k,i} - \mathbf{v}_\alpha(p_{\mathrm{w},h}^{n,k,i}, s_{\mathrm{w},h}^{n,k,i})\|_K + h_K/\pi\|q_\alpha^n - \partial_t^n(\phi s_{\alpha,h\tau}^{n,k,i}) - \nabla\cdot\mathbf{u}_{\alpha,h}^{n,k,i}\|_K\big)^2 + \big(\|\underline{\mathbf{K}}(\lambda_{\mathrm{w}}(s_{\mathrm{w},h\tau}^{n,k,i}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h\tau}^{n,k,i}))\nabla(\mathfrak{p}(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{p}}_{h\tau}^{n,k,i})\|_K(t)\big)^2 + \big(\|\underline{\mathbf{K}}\nabla(\mathfrak{q}(s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{q}}_{h\tau}^{n,k,i})\|_K(t)\big)^2 \Bigg\}^{\frac{1}{2}}$$

- *temporal estimators*

$$\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t) := \|\mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t) - \mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t^n)\|_K \quad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *linearization estimators*

$$\eta_{\mathrm{lin},K,\alpha}^{n,k,i} := \|\mathbf{l}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg},K,\alpha}^{n,k,i} := \|\mathbf{a}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

Global estimators

Global estimators

$$\eta_{\mathrm{sp}}^{n,k,i} := \left\{ 3 \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{sp},K}^{n,k,i}(t))^2 \, \mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{tm}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t))^2 \, \mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{lin}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{lin},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{alg}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{alg},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}}$$

Quarter five spot test problem

Data from Klieber & Rivière (2006)

$$\Omega = (0, 300)\text{m} \times (0, 300)\text{m}, \quad T = 4{\cdot}10^6\text{s},$$

$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$

$$\mu_\text{w} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_\text{n} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$

$$s_\text{rw} = s_\text{rn} = 0, \quad p_\text{d} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m × 18m lower left corner block)

$$s_\text{w}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$

$$s_\text{w}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m × 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_\text{w} = 0.95$, $p_\text{w} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_\text{w} = 0.2$, $p_\text{w} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

Quarter five spot test problem

Data from Klieber & Rivière (2006)

$$\Omega = (0,300)\text{m} \times (0,300)\text{m}, \quad T = 4{\cdot}10^6\text{s},$$
$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$
$$\mu_{\rm w} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_{\rm n} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$
$$s_{\rm rw} = s_{\rm rn} = 0, \quad p_{\rm d} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m $\times$ 18m lower left corner block)

$$s_{\rm w}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$
$$s_{\rm w}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m $\times$ 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_{\rm w} = 0.95$, $p_{\rm w} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_{\rm w} = 0.2$, $p_{\rm w} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

Quarter five spot test problem

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$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$

$$\mu_{\rm w} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_{\rm n} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$

$$s_{\rm rw} = s_{\rm rn} = 0, \quad p_{\rm d} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m $\times$ 18m lower left corner block)

$$s_{\rm w}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$

$$s_{\rm w}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m $\times$ 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_{\rm w} = 0.95$, $p_{\rm w} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_{\rm w} = 0.2$, $p_{\rm w} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

Outline

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,w}}(s^n_{\mathrm{w},L})}{2} |\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},L} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_L|} |\sigma_{KL}|,$$

$$F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},L})}{2} |\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},L} + \pi(s^n_{\mathrm{w},L}) - (\bar{p}^n_{\mathrm{w},K} + \pi(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_L|} |\sigma_{KL}|$$

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \le n \le N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

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$$F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},L})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},L} + \pi(s^n_{\mathrm{w},L}) - (\bar{p}^n_{\mathrm{w},K} + \pi(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_L|}|\sigma_{KL}|$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL}\in\mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL}\in\mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned}F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M\in\{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1})\cdot(s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M\in\{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1})\cdot(\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}).\end{aligned}$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL}\in\mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL}\in\mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M\in\{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1})\cdot(s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M\in\{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1})\cdot(\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{\sigma_{KL}} &:= F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{\sigma_{KL}} &:= F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$
\begin{aligned}
-\eta_{\mathrm{r,w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,\\
p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) &= \bar{p}_{\mathrm{w},K}^{n,k,i},\\
-\eta_{\mathrm{r,n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,\\
p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) &= \pi(s_{\mathrm{w},K}^{n,k,i})+\bar{p}_{\mathrm{w},K}^{n,k,i}
\end{aligned}
$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$\begin{aligned}(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{\sigma_{KL}} &:= F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\ ((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{\sigma_{KL}} &:= F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\ \mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\end{aligned}$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$\begin{aligned}-\eta_{\mathrm{r,w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,\\ p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) &= \bar{p}_{\mathrm{w},K}^{n,k,i},\end{aligned}$$

$$\begin{aligned}-\eta_{\mathrm{r,n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,\\ p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) &= \pi(s_{\mathrm{w},K}^{n,k,i})+\bar{p}_{\mathrm{w},K}^{n,k,i}\end{aligned}$$

Global pressure and Kirchhoff transform

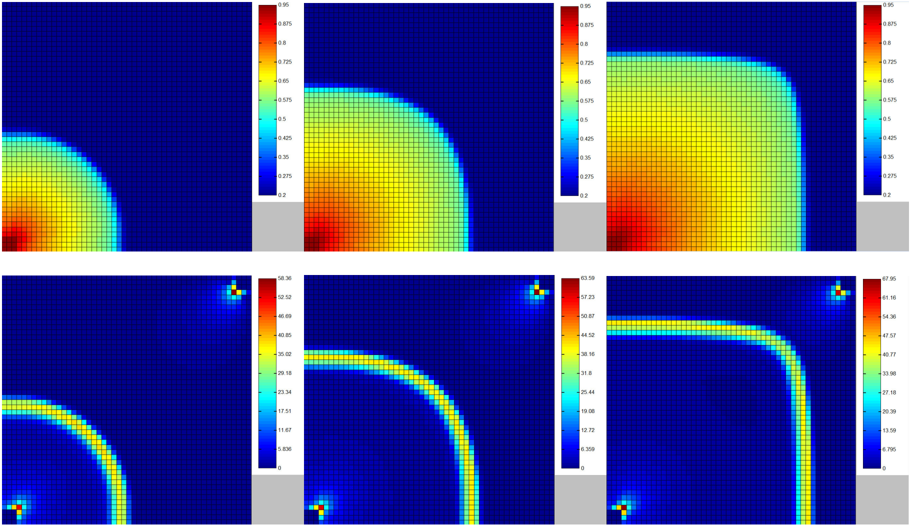
Global pressure and Kirchhoff transform postprocessing

- Piecewise quadratic global pressure and Kirchhoff transform used in the estimators:

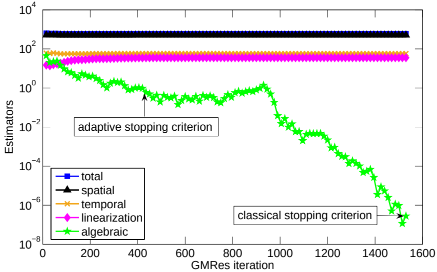
$$
\begin{aligned}
-\left(\eta_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i}) + \eta_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\right)\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) &= (\mathbf{d}_{\mathrm{w},h}^{n,k,i} + \mathbf{d}_{\mathrm{n},h}^{n,k,i})|_K, \\
\mathfrak{p}_h^{n,k,i}(\mathbf{x}_K) &= P(\bar{p}_{\mathrm{w},K}^{n,k,i}, s_{\mathrm{w},K}^{n,k,i}),
\end{aligned}
$$

$$
\begin{aligned}
\underline{\mathbf{K}}\nabla(\mathfrak{q}_h^{n,k,i}|_K) &= \eta_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) + \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K, \\
\mathfrak{q}_h^{n,k,i}(\mathbf{x}_K) &= \varphi(s_{\mathrm{w},K}^{n,k,i})
\end{aligned}
$$

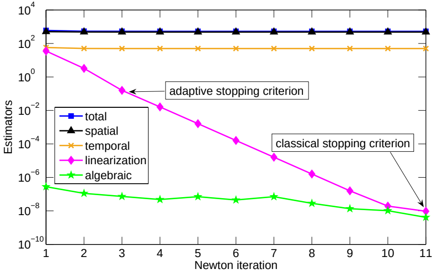
Water saturation/estimators evolution



Estimators and stopping criteria

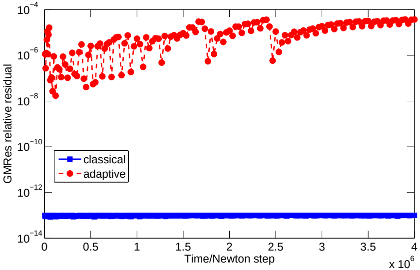


Estimators in function of GMRes iterations

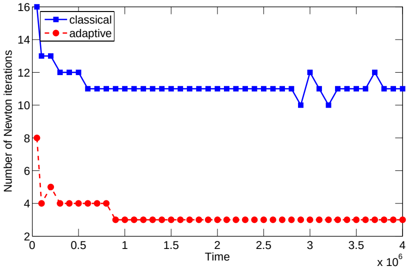


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

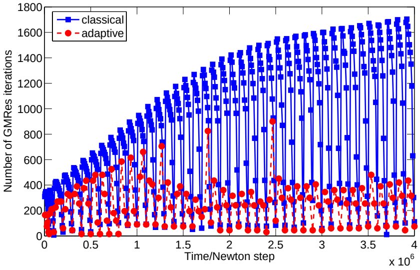


GMRes relative residual

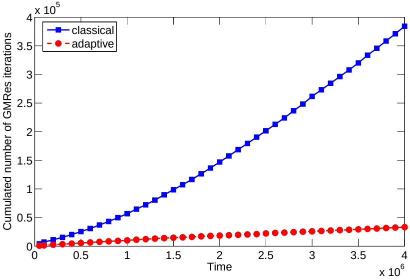


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\left(\left(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\right)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\right)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k} := \frac{\tau^n}{\phi|D|}\left(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D, 1\right)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D$$
$$+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\Big)_{\partial D\setminus\partial\Omega}=0\quad\forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k}:=\frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D,1\big)_{\partial D\setminus\partial\Omega}+s_{\mathrm{w},D}^{n-1}\quad\forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\big((\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1}))\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Fluxes reconstructions

Total fluxes

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\Big)_\sigma,$$

$$((\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\Big)_\sigma,$$

$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu}+\mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu}-(\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

Wetting fluxes

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_\sigma,$$

$$((\mathbf{d}_{\mathrm{w},h}^{n,k,i}+\mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_\sigma,$$

$$\mathbf{a}_{\mathrm{w},h}^{n,k,i} := 0$$

Fluxes reconstructions

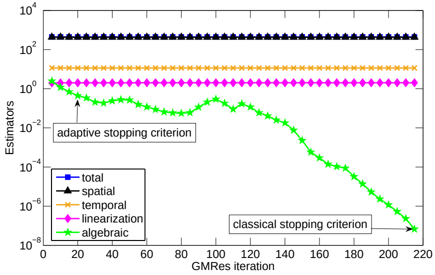
Total fluxes

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\quad + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\Big)_\sigma, \\
((\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\quad + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\Big)_\sigma, \\
\mathbf{a}_{\mathrm{t},h}^{n,k,i} &:= \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu}+\mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu}-(\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})
\end{aligned}
$$

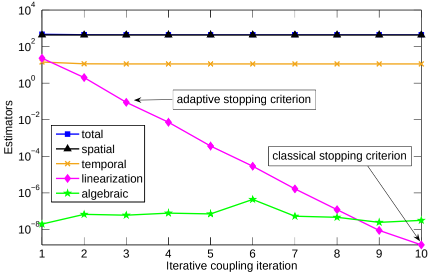
Wetting fluxes

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_\sigma, \\
((\mathbf{d}_{\mathrm{w},h}^{n,k,i}+\mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_\sigma, \\
\mathbf{a}_{\mathrm{w},h}^{n,k,i} &:= 0
\end{aligned}
$$

Estimators and stopping criteria

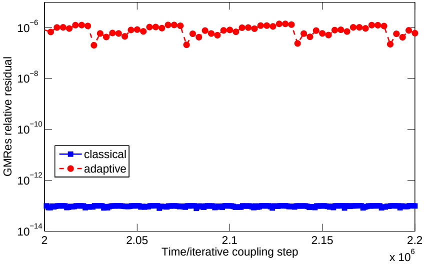


Estimators in function of GMRes iterations

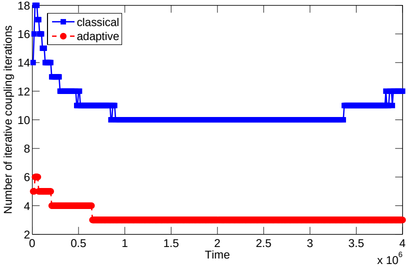


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

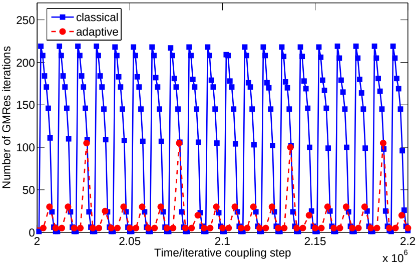


GMRes relative residual

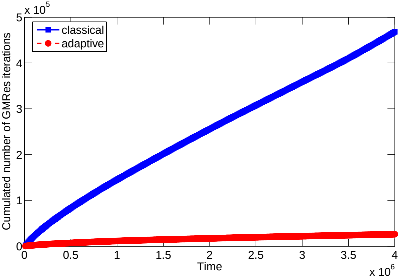


Iterative coupling iterations

GMRes iterations



Per time and iterative coupling step



Cumulated

Outline

Conclusions

Entire adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **"smart online decisions"**: algebraic step / linearization step / space mesh refinement / time step modification
- important **computational savings**
- guaranteed and robust error upper bound via **a posteriori estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

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Bibliography

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Thank you for your attention!