# Equivalence of local- and global-best approximations and a simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$

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*Inria Paris & Ecole des Ponts*

USNCCM, Austin, July 29, 2019

# Mary's hospitality



# Outline

# Mixed finite elements for the Laplace equation

**Laplace model problem**

Find $u : \Omega \to \mathbb{R}$ s.t.

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\begin{aligned}
-\Delta u &= f && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
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**Dual mixed weak formulation**

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that $(\boldsymbol{\sigma} = -\nabla u)$

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(\boldsymbol{\sigma}, \boldsymbol{v}) - (u, \nabla{\cdot}\boldsymbol{v}) &= 0 && \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega),\\
(\nabla{\cdot}\boldsymbol{\sigma}, q) &= (f, q) && \forall q \in L^2(\Omega)
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**Mixed finite elements**

Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

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**Notation**

- $\Omega$: computational domain (open polygon/polyhedron)
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## Theorem (Classical *a priori* estimate)

$$\underbrace{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|}_{\textit{MFE error}} = \underbrace{\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \nabla\cdot\boldsymbol{v}_h=\Pi_p f}} \|\boldsymbol{\sigma}-\boldsymbol{v}_h\|}_{\substack{\textit{global-best on } \Omega\\ \textit{normal trace-continuity constraint}\\ \textit{divergence constraint}}} \leq \|\boldsymbol{\sigma}-\underbrace{\boldsymbol{I}_p^{\rm RTN}(\boldsymbol{\sigma})}_{\substack{\in\boldsymbol{V}_h\\ \nabla\cdot=\Pi_p f}}\|$$

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- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Licht (2019): essential boundary conditions on part of $\partial\Omega$

***hp* interpolation estimates**

- Demkowicz and Buffa (2005): $\log(p)$ factors
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})$
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# Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

**Theorem (Equivalence in $H_0^1$, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))**

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$$\min_{\textit{smaller space}} \approx \min_{\textit{bigger space}}$$

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2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
4. Elementwise localized optimal *hp* approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
6. Tools (*p*-robustness)
   - Polynomial extension on a tetrahedron
   - Broken polynomial extension on a patch
7. Conclusions and outlook

# Outline

1. Introduction: *a priori* error estimates for mixed finite element methods and global-best – local best equivalence in $H^1_0(\Omega)$
2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
4. Elementwise localized optimal *hp* approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
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# Simple map $P_p : \boldsymbol{H}(\mathrm{div}, \Omega) \to \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

## Definition (Map $P_p$ by local projection and flux reconstruction)

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary.

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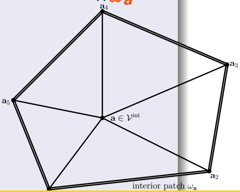

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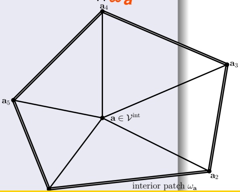

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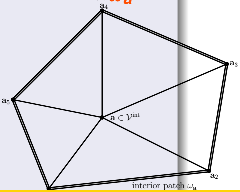

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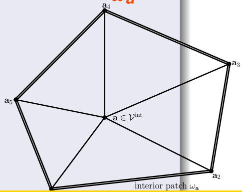

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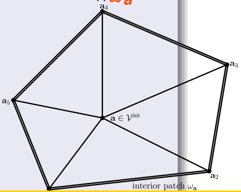

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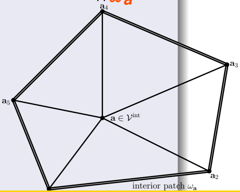

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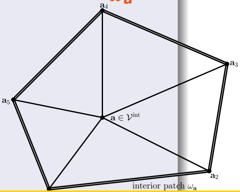

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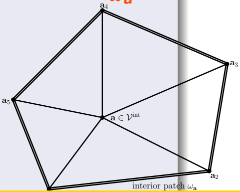

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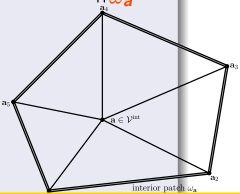
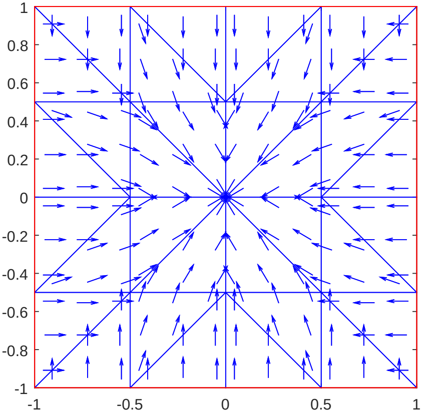


# Equilibrated flux reconstruction in 2D




Flux $\xi_h$

$$\underbrace{\xi_h \in \boldsymbol{RTN}_p(\mathcal{T}),\ \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)}_{(\nabla\cdot\boldsymbol{\sigma},\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}^{\mathrm{int}}}$$

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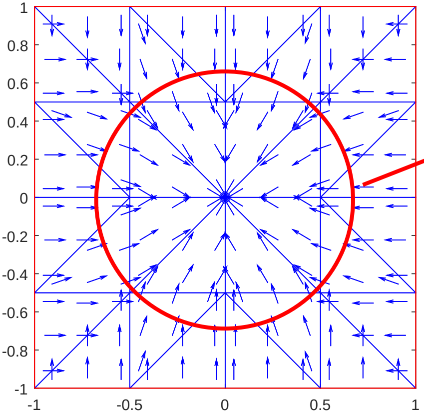


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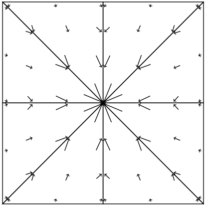

$\psi_{\boldsymbol{a}}\xi_h$

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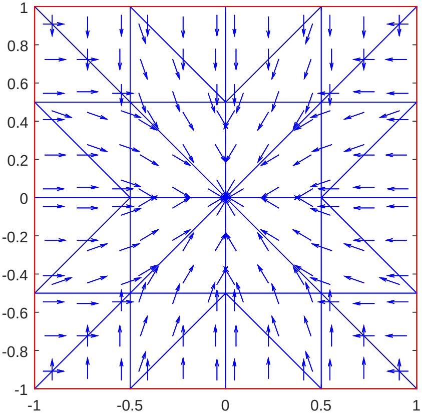


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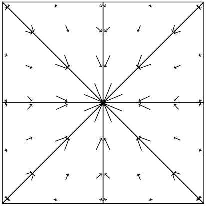

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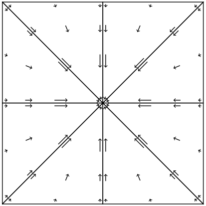

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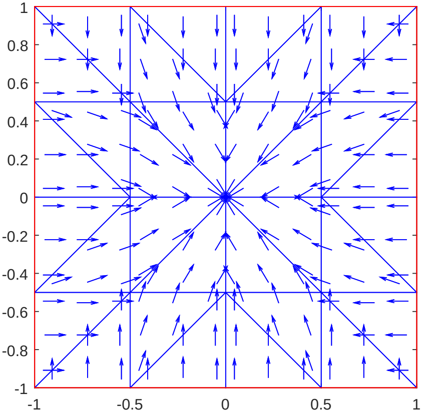


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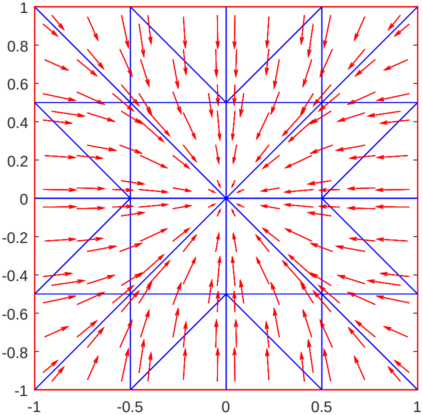


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# Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

**Theorem (Stable local commuting projector,** Ern, Gudi, Smears, & V. (2019)**)**

*Let* $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *from* construction *is locally defined,*

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$$\|P_p\boldsymbol{\sigma}\| \lesssim \|\boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K^2 \right\}^{1/2}$$ *stable up to data.*

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**Comments**

- $P_p$ defined on $\boldsymbol{H}(\mathrm{div}, \Omega)$
- $\lesssim_p$: only depends on $d$, shape-regularity of $\mathcal{T}$, and $p$
- $h_K\|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K/(p+1)$: data oscillation term common in *hp a posteriori* analysis, disappears when $\nabla{\cdot}\boldsymbol{\sigma}$ is a piecewise $p$-degree polynomial

# Proof: local problems, commutativity

- recall $\xi_h \in \boldsymbol{RTN}_p(\mathcal{T})$ is elementwise $\boldsymbol{L}^2$-orthogonal projection of $\boldsymbol{\sigma}$

$$\xi_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\forall \boldsymbol{a} \in \mathcal{V}_K$, $p \geq 0$,

$$(\boldsymbol{\sigma} - \xi_h, \nabla\psi_{\boldsymbol{a}})_K = 0 \qquad \forall K \in \mathcal{T}$$

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$$(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0$$

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# Proof: stability of the flux reconstruction

**Theorem (Local stability** Braess, Pillwein, Schöberl (2009; 2D), Ern, V. (2016; 3D), using Tools**)**

*There holds*

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**Corollary (Global stability)**

*$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma}_h$ is closer to the elementwise projection $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ up to divergence oscillation:*

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot(\boldsymbol{\xi}_h - \boldsymbol{\sigma})\|_K^2\right\}^{1/2}.$$

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# Outline

1. Introduction: *a priori* error estimates for mixed finite element methods and global-best – local best equivalence in $H_0^1(\Omega)$
2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
4. Elementwise localized optimal *hp* approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
6. Tools (*p*-robustness)
   - Polynomial extension on a tetrahedron
   - Broken polynomial extension on a patch
7. Conclusions and outlook

# Global-best approx. $\approx$ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

**Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))**

*bigger* $\approx$ *smaller*

# Global-best approx. $\approx$ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

**Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))**

$$\min_{\textit{smaller space with constraints}} \approx \min_{\textit{bigger space without constraints}}$$

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$$\approx_p \sum_{K \in \mathcal{T}} \underbrace{\left[ \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p \nabla\cdot\boldsymbol{\sigma}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- the right number (a priori) much smaller than the left one
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# Global-best approx. ≈ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

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*divergence constraint*
*MFE space (much smaller)*

$$\approx_p \sum_{K\in\mathcal{T}} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{\sigma} - \Pi_p\nabla\cdot\boldsymbol{\sigma}\|_K^2\right]}_{\text{local-best on each } K}.$$

*local-best on each $K$*
*no normal trace-continuity constraint*
*no divergence constraint*
*broken MFE space (much bigger)*

- the right number (a priori) much smaller than the left one
- $\approx_p$: only depends on $d$, shape-regularity of $\mathcal{T}$, and $p$
- no need of interpolate for optimal error bounds

# Global-best approx. ≈ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

**Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))**

*Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{\sigma} - \Pi_p\nabla\cdot\boldsymbol{\sigma}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

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# Outline

# Optimal *hp* approximation estimate

**Theorem (Elementwise localized *hp* approx., Ern, Gudi, Smears, & V. (2019))**

*For any* $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on all* $K \in \mathcal{T}$,

$$\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K), \; s > 0,$$

*there holds*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[ \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K \in \mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{\sigma}\|_{\boldsymbol{H}^s(K)}^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{\sigma}\|_K^2 & \text{if } s < 1, \\ \sum_{K \in \mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{\sigma}\|_{\boldsymbol{H}^s(K)}^2 & \text{if } s \geq 1. \end{cases}$$

- $\lesssim$: only depends on $d$, shape-regularity of $\mathcal{T}$, and $s$
- employs flux reconstruction in $\boldsymbol{RTN}_{p+1}$ without $\boldsymbol{I}_p^{\mathrm{RTN}}$ interpolate
- contours known (quasi-)interpolates
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# Laplace model problem: $-\Delta u = f$ in $\Omega$, $u = 0$ on $\partial\Omega$

**Theorem (Optimal *hp a priori* error estimate for MFEs, Ern, Gudi, Smears, & V. (2019))**

*From* ▸ $\boldsymbol{H}(\mathrm{div},\Omega)$ *hp* approx. *, there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \lesssim_{s,\boldsymbol{\sigma}} \frac{h^{\min(s,p+1)}}{(p+1)^s}.$$

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**Mixed least-squares weak formulation**

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma} + \nabla u, \nabla v) &= 0 & \forall v \in H_0^1(\Omega), \\ h_\Omega^2(\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{p}) + (\boldsymbol{\sigma} + \nabla u, \boldsymbol{p}) &= h_\Omega^2(f, \nabla\cdot\boldsymbol{p}) & \forall \boldsymbol{p} \in \boldsymbol{H}(\mathrm{div}, \Omega). \end{aligned}$$

**Least-squares mixed finite elements**

Let $\boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$, $V_h := \mathbb{P}_q(\mathcal{T}) \cap H_0^1(\Omega)$, $q \geq 1$. Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h \times V_h$ such that

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# Laplace model problem: $-\Delta u = f$ in $\Omega$, $u = 0$ on $\partial\Omega$

## Lemma (A priori bound for least-squares mixed finite elements)

*There exists a positive constant* $C = C(\Omega) \leq 1/8$ *s.t.*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\nabla(u - u_h)\| \leq C\left( \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| + \min_{v_h \in \mathbb{P}_q(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \right),$$

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## Corollary (Localized *a priori* estimate for least-squares MFEs)

*Let* $\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K)$, $s>0$, *and* $u|_K \in H^{1+t}(K)$, $t>0$, $\forall K \in \mathcal{T}$. *Then*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| + \|\nabla(u-u_h)\| + h_\Omega\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\| \lesssim_{p,\boldsymbol{\sigma},u} h^{\min\{s,p+1\}} + h^{\min\{t,q\}}.$$

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# Polynomial extension on a tetrahedron

**Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron** Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

*Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying the compatibility $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

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**Context**

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

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$$\|\varphi_{h,K}\|_K \stackrel{MFEs}{=} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

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$$-\Delta \zeta_K = r_K \quad \text{in } K,$$

$$-\nabla \zeta_K \cdot \boldsymbol{n}_K = r_F \quad \text{on all } F \in \mathcal{F}_K^{\mathrm{N}},$$

$$\zeta_K = 0 \quad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}.$$

Set $\varphi_K := -\nabla \zeta_K$.

# Outline

1. Introduction: *a priori* error estimates for mixed finite element methods and global-best – local best equivalence in $H_0^1(\Omega)$
2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
4. Elementwise localized optimal *hp* approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
6. Tools (*p*-robustness)
   - Polynomial extension on a tetrahedron
   - **Broken polynomial extension on a patch**
7. Conclusions and outlook

# Broken polynomial extension on a patch

**Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a patch** Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2016; 3D)**)**

*For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility*

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

*Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

# Outline

# Conclusions and outlook

## Conclusions

- a simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
- global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
- optimal localized *hp* approximation estimates under minimal regularity
- optimal *a priori* error estimates for mixed finite elements and least-squares mixed finite elements

## Ongoing work

- extensions to other settings

# Conclusions and outlook

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## Ongoing work

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**Thank you for your attention!**

# Potentials

**Lemma ($H^1$ polynomial extension on a tetrahedron** Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

*Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then*

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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**Context**

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

# Potentials

**Lemma ($H^1$ polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))**

*Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then*

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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$$\|\nabla \zeta_{h,K}\|_K \stackrel{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

**Context**

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

**Theorem (Broken $H^1$ polynomial extension on a patch Ern & V. (2015, 2016))**

*For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility*

$$
\begin{aligned}
r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}, \\
\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.
\end{aligned}
$$

*Then*

$$
\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}}) \\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.
$$