Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach

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in collaboration with J. Papež, U. Rüde, and B. Wohlmuth

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Outline

Setting: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$, $\Omega \subset \mathbb{R}^d$, $d \geq 1$

Exact solution

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Linear algebraic system

Find $U_h \in \mathbb{R}^N$, $N = |V_h|$, such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)

On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N \Leftrightarrow$ **inexact FE approximation** $u_h^i \in V_h$

$$\mathbb{A}_h U_h^i = F_h - R_h^i \qquad (R_h^i := F_h - \mathbb{A}_h U_h^i)$$

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Context & goals: **a posteriori estimates** for **any** $i \geq 1$

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Algebraic error

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Further goals

- estimate the **distribution** of the errors
- prove (local) **efficiency**
- design reliable (local) **stopping criteria**

One-dimensional example

Setting

- $d = 1$, $\Omega = (0, 1)$, $f = 2$, $p = 1$
- $u = u(x) = x(1 - x)$
- u_h pointwise exact on mesh $\mathcal{T}_h$, unknown sol. of $\mathbb{A}_h U_h = F_h$
- $u_h^i := u_H$ pointwise exact on mesh $\mathcal{T}_H$, $R_h := F_h - \mathbb{A}_h U_H$
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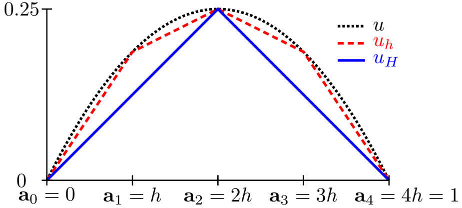

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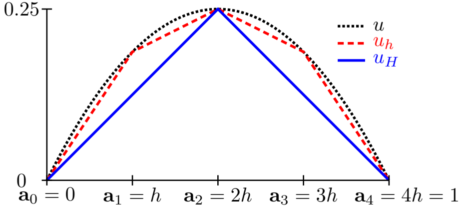

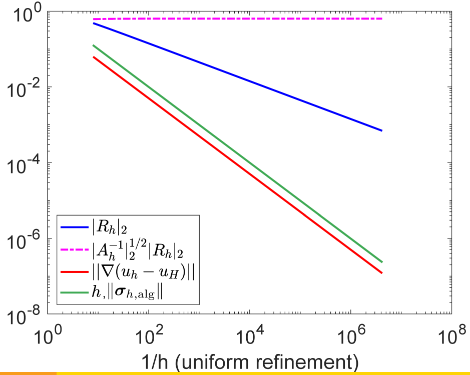

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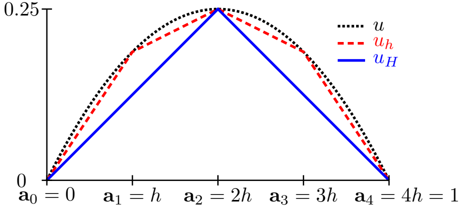

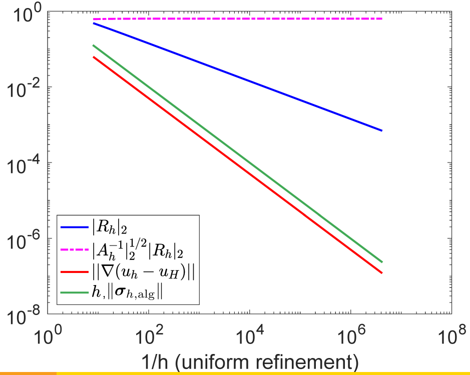

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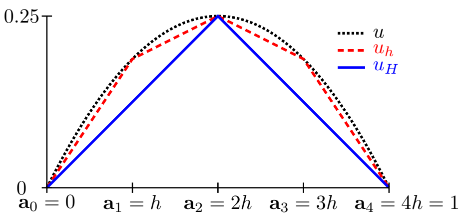

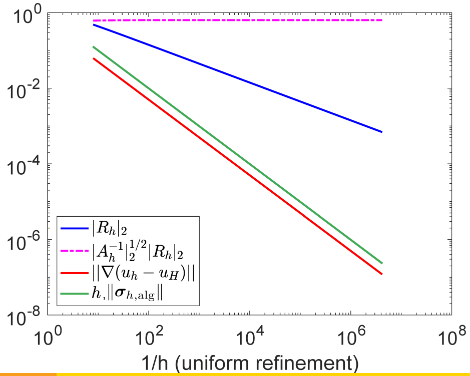

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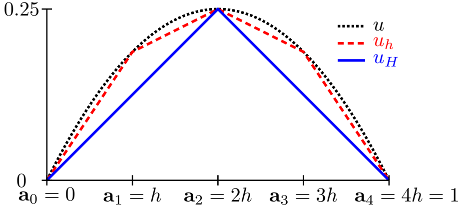

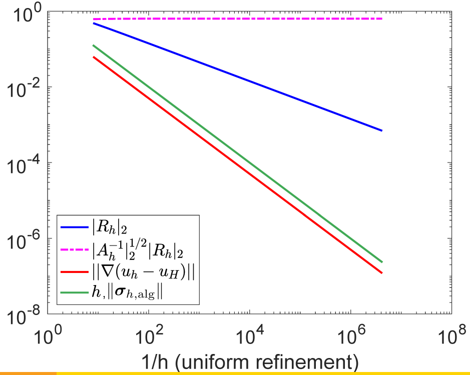

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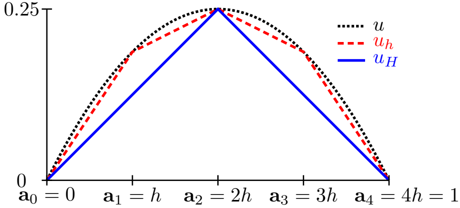

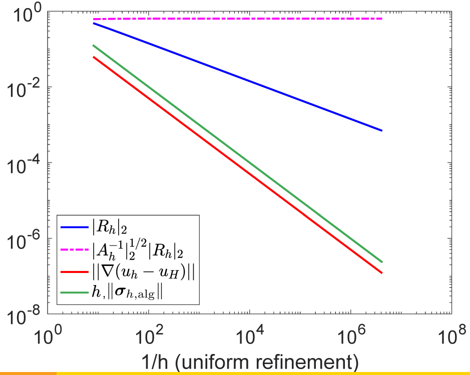

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Algebraic residual representer

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- $(r_h^i, \psi_l) = (R_h^i)_l$, for all basis functions $l = 1, \ldots, N$

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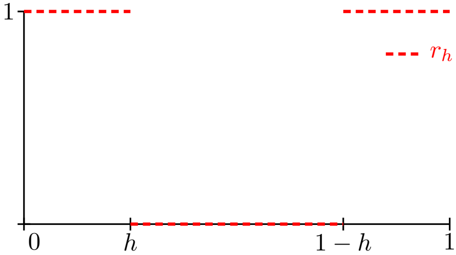
1D example:

$$R_h := F_h - \mathbb{A}_h U_H = \begin{pmatrix} 2h \\ -2h \\ 2h \\ -2h \\ \vdots \\ 2h \end{pmatrix}$$



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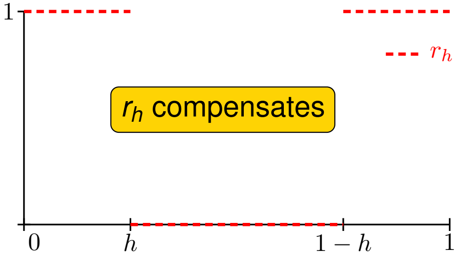
$\Rightarrow |R_h|_2$ explodes



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- gives **equivalent form of residual equation**: $u_h^i \in V_h$ s.t.

$$(\nabla u_h^i, \nabla \psi_l) = (f, \psi_l) - (r_h^i, \psi_l) \qquad \forall l = 1, \ldots, N$$

$$\Leftarrow \mathbb{A}_h U_h^i = F_h - R_h^i$$

- consequence of equations for u_h and u_h^i:

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Tools

- flux and potential reconstructions, $\boxed{\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}} = r_h^i}$
- local Neumann MFE & local Dirichlet FE problems
- separate components for algebraic & discretization errors
- multilevel hierarchy (algebraic components)

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Algebraic error upper bound

Theorem (Upper bound via algebraic error flux reconstruction)

Let $\boldsymbol{\sigma}^i_{h,\mathrm{alg}} \in \mathbf{H}(\mathrm{div},\Omega)$ *be such that* $\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$. *Then*

$$\underbrace{\|\nabla(u_h - u^i_h)\|}_{\text{algebraic error}} \leq \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{upper algebraic est.}}.$$

Proof.

$$\|\nabla(u_h - u^i_h)\| = \sup_{v_h\in V_h, \|\nabla v_h\|=1} (\nabla(u_h - u^i_h), \nabla v_h);$$

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Previous cheap constructions of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$

1. sequential sweep trough $\mathcal{T}_h$, local min. (JSV (2010))
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Algebraic error flux reconstruction, two-level setting

Definition (Coarse grid solve)

Find $\rho^i_{H,\mathrm{alg}} \in V_H := \mathbb{P}_1(\mathcal{T}_H) \cap H^1_0(\Omega)$ such that

$$(\nabla \rho^i_{H,\mathrm{alg}}, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (r^i_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_H$$

- $\mathbb{P}_1$ finite element solve on coarse mesh $\mathcal{T}_H$
- no need for multigrid without post-smoothing

Definition (Algebraic error flux reconstruction)

$$\sigma^{\mathbf{a},i}_{h,\mathrm{alg}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}^{\mathbf{a}}_h,\ \nabla \cdot \mathbf{v}_h = \Pi_{Q^{\mathbf{a}}_h}(\psi_{\mathbf{a}} r^i_h - \nabla \psi_{\mathbf{a}} \cdot \nabla \rho^i_{H,\mathrm{alg}})} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}},$$

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Algebraic error flux reconstruction, two-level setting

Definition (Coarse grid solve)

Find $\rho^i_{H,\mathrm{alg}} \in V_H := \mathbb{P}_1(\mathcal{T}_H) \cap H^1_0(\Omega)$ such that

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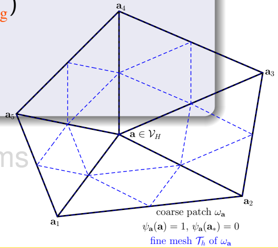

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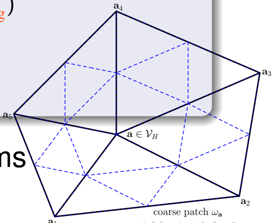

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Divergence of the algebraic error flux reconstruction

Lemma (Divergence of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$)

There holds $\nabla \cdot \boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$.

Proof.

- partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_H} \psi^{\mathbf{a}} = 1$
-

$$\nabla \cdot \boldsymbol{\sigma}^i_{h,\mathrm{alg}} = \sum_{\mathbf{a} \in \mathcal{V}_H} \nabla \cdot \boldsymbol{\sigma}^{\mathbf{a},i}_{h,\mathrm{alg}}$$

$$= \sum_{\mathbf{a} \in \mathcal{V}_H} \Pi_{Q_h} (\psi_{\mathbf{a}} r^i_h - \nabla \psi_{\mathbf{a}} \cdot \nabla \rho^i_{H,\mathrm{alg}}) = r^i_h$$

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\begin{aligned}
\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} &= \sum_{\mathbf{a}\in\mathcal{V}_H} \nabla\cdot\boldsymbol{\sigma}^{\mathbf{a},i}_{h,\mathrm{alg}} \\
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\end{aligned}
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Algebraic error lower bound

Theorem (Lower bound via algebraic residual liftings)

Let $\rho_{h,\mathrm{alg}}^i \in V_h$ be arbitrary. Then

$$\underbrace{\|\nabla(u_h - u_h^i)\|}_{\text{algebraic error}} \geq \underbrace{\frac{(r_h^i, \rho_{h,\mathrm{alg}}^i)}{\|\nabla \rho_{h,\mathrm{alg}}^i\|}}_{\text{lower algebraic est.}}.$$

Proof.

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\|=1} (r_h^i, v_h) \geq \frac{(r_h^i, \rho_{h,\mathrm{alg}}^i)}{\|\nabla \rho_{h,\mathrm{alg}}^i\|}.$$

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Algebraic residual lifting, two-level setting

Definition (Algebraic residual lifting, ≈ Bank & Smith (1993), Oswald (1993), Rüde (1993), EV (2015))

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Set

$$\rho_{h,\mathrm{alg}}^i := \rho_{H,\mathrm{alg}}^i + \sum_{\mathbf{a} \in \mathcal{V}_H} \rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in V_h.$$

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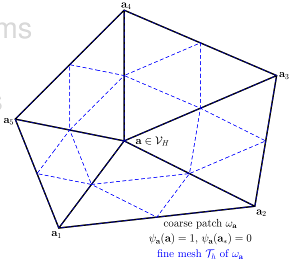

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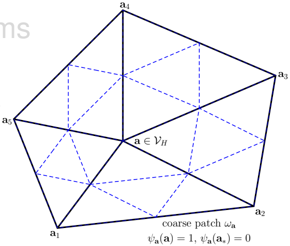

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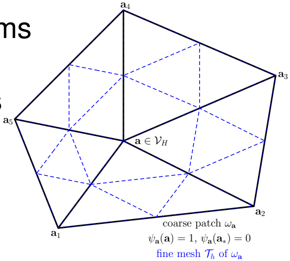

Outline

Discretization flux reconstruction

Definition (Discretization flux reconstruction, Braes & Schöberl (2008), EV (2013))

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^{\mathbf{a},i} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(f\psi^{\mathbf{a}} - \nabla u_h^i \cdot \nabla \psi_{\mathbf{a}} - r_h^i \psi^{\mathbf{a}})} \|\psi^{\mathbf{a}} \nabla u_h^i + \mathbf{v}_h\|_{\omega_{\mathbf{a}}},$$

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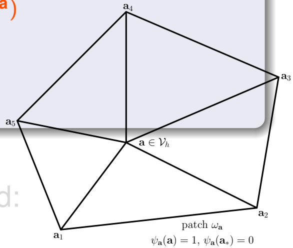

Neumann compatibility condition satisfied:

$$(\nabla u_h^i, \nabla \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} - (r_h^i, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Lemma (Divergence of $\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i}$)

There holds

$$\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i} = \Pi_{Q_h} f - r_h^i.$$

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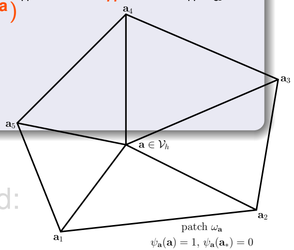

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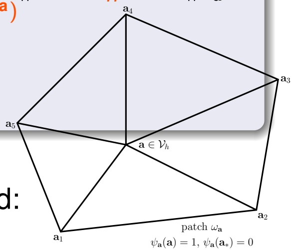

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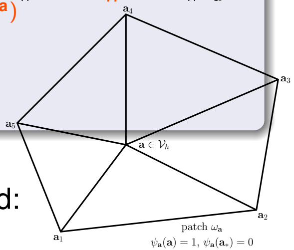

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Upper bound on the total error

Theorem (Total error upper bound)

On each iteration $i \geq 1$, there holds

$$\underbrace{\|\nabla(u-u_h^i)\|}_{\text{total error}} \leq \underbrace{\|\nabla u_h^i+\boldsymbol{\sigma}_{h,\text{dis}}^i\|}_{\text{discretization est.}} + \underbrace{\|\boldsymbol{\sigma}_{h,\text{alg}}^i\|}_{\text{algebraic est.}} + \underbrace{\left\{\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\pi^2}\|f-\Pi_{Q_h} f\|_K^2\right\}^{1/2}}_{\text{data osc. est.}}.$$

Proof.

$$\|\nabla(u-u_h^i)\| = \sup_{v\in H_0^1(\Omega),\, \|\nabla v\|=1} (\nabla(u-u_h^i), \nabla v)$$

$$(\nabla(u-u_h^i), \nabla v) = (f, v) - (\nabla u_h^i, \nabla v) = (f - \overbrace{\nabla\cdot(\boldsymbol{\sigma}_{h,\text{alg}}^i+\boldsymbol{\sigma}_{h,\text{dis}}^i)}, v)$$
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Definition (Total residual lifting, $\approx$ Babuška and Strouboulis (2001), Repin (2008))

Find $\rho_{h,\mathrm{tot}}^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H^1(\omega_{\mathbf{a}})$ (together with mean value or value on $\partial\Omega$ zero) such that

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- local homogeneous Neumann FE problems

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Outline

Stopping criteria

Galerkin orthogonality

$$\underbrace{\|\nabla(u - u_h^i)\|^2}_{\text{total error}} = \underbrace{\|\nabla(u - u_h)\|^2}_{\text{discretization error}} + \underbrace{\|\nabla(u_h - u_h^i)\|^2}_{\text{algebraic error}}$$

Discretization error upper and lower bounds

- upper bound on total error & lower bound on algebraic error $\Rightarrow$ upper bound on the discretization error
- lower bound on total error & upper bound on algebraic error $\Rightarrow$ lower bound on the discretization error

Safe stopping criterion ($\gamma_{\mathrm{alg}} \approx 0.1$)

$$\text{algebraic error} \leq \gamma_{\mathrm{alg}}\ \text{discretization error}$$

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Efficiency and polynomial-degree-robustness

Theorem (Efficiency & p-robustness)

Let the algebraic estimate be below the discretization estimate. Let $f \in \mathbb{P}_p(\mathcal{T}_h)$. Then

$$\underbrace{\|\nabla u_h^i + \boldsymbol{\sigma}_{h,\mathrm{dis}}^i\| + \|\boldsymbol{\sigma}_{h,\mathrm{alg}}^i\|}_{\text{total estimate}} \lesssim \underbrace{\|\nabla(u - u_h^i)\|}_{\text{total error}}.$$

Theorem (Local efficiency & p-robustness)

Let patchwise the algebraic estimate be below the discretization estimate. Let $f \in \mathbb{P}_p(\mathcal{T}_h)$. Then

$$\underbrace{\|\nabla u_h^i + \boldsymbol{\sigma}_{h,\mathrm{dis}}^i\|_K + \|\boldsymbol{\sigma}_{h,\mathrm{alg}}^i\|_K}_{\text{element total estimate}} \lesssim \underbrace{\sum_{\mathbf{a} \in \mathcal{V}_h,\, \mathbf{a} \subset \partial K} \|\nabla(u - u_h^i)\|_{\omega_{\mathbf{a}}}}_{\text{patch total error}} \quad \forall K \in \mathcal{T}_h.$$

stopping criterion $\Rightarrow$ efficiency & p-robustness

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Outline

Numerical illustration

Peak

$$\Omega = (0,1) \times (0,1),$$
$$u(x,y) = x(x-1)y(y-1)e^{-100(x-0.5)^2-100(y-117/1000)^2}$$

L-shape

$$\Omega = (-1,1) \times (-1,1) \setminus \; [0,1] \times [-1,0],$$
$$u(r,\theta) = r^{2/3}\sin(2\theta/3)$$

Discretization

- conforming finite elements, $p = 1, \ldots, 4$
- unstructured triangular meshes
- 4 uniform refinements

Multigrid

- geometric multigrid V-cycle
- 5 pre-smoothing steps of Gauss–Seidel

PCG

- incomplete Cholesky with drop-off tolerance 1e-4 prec.

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Peak problem, multigrid

p (unknowns)	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (9.31×10^3)	1	6.09×10^{-3}	1.13	1.02^{-1}	6.93×10^{-3}	1.61	1.21^{-1}	3.32×10^{-3}	2.84	—
	2	1.90×10^{-4}	1.13	1.03^{-1}	3.32×10^{-3}	1.10	1.03^{-1}		1.10	1.03^{-1}
2 (3.76×10^4)	1	7.49×10^{-3}	1.13	1.00^{-1}	7.49×10^{-3}	1.61	1.23^{-1}	1.11×10^{-4}	8.53×10^1	—
	3	8.11×10^{-6}	1.17	1.01^{-1}	1.12×10^{-4}	1.10	1.03^{-1}		1.10	1.03^{-1}
3 (8.48×10^4)	1	4.94×10^{-3}	1.10	1.00^{-1}	4.94×10^{-3}	1.40	1.44^{-1}	2.87×10^{-6}	1.68×10^3	—
	5	7.79×10^{-9}	1.17	1.00^{-1}	2.87×10^{-6}	1.01	1.11^{-1}		1.01	1.11^{-1}
4 (1.51×10^5)	1	4.45×10^{-3}	1.09	1.00^{-1}	4.45×10^{-3}	1.44	1.37^{-1}	6.33×10^{-8}	7.28×10^4	—
	6	1.06×10^{-9}	1.11	1.00^{-1}	6.33×10^{-8}	1.02	1.15^{-1}		1.02	1.15^{-1}

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L-shape problem, PCG

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2 (1.01×10^5)	4	6.24×10^{-1}	1.01	1.00^{-1}	6.24×10^{-1}	1.07	9.06^{-1}	8.93×10^{-3}	2.61×10^1	—
	12	1.87×10^{-4}	1.01	1.00^{-1}	8.93×10^{-3}	1.33	1.28^{-1}		1.33	1.28^{-1}
3 (2.27×10^5)	7	1.02	1.00	1.00^{-1}	1.02	1.05	10.0^{-1}	5.29×10^{-3}	6.29×10^1	—
	28	9.58×10^{-5}	1.00	1.00^{-1}	5.29×10^{-3}	1.46	1.41^{-1}		1.46	1.41^{-1}
4 (4.04×10^5)	7	1.17	1.01	1.00^{-1}	1.17	1.08	7.56^{-1}	3.77×10^{-3}	1.30×10^2	—
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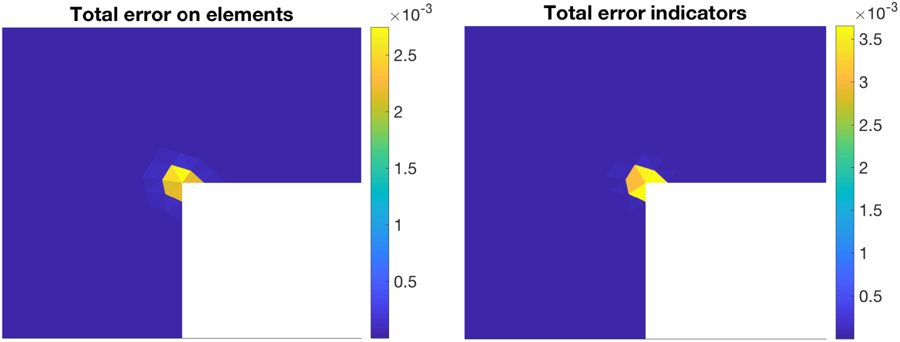
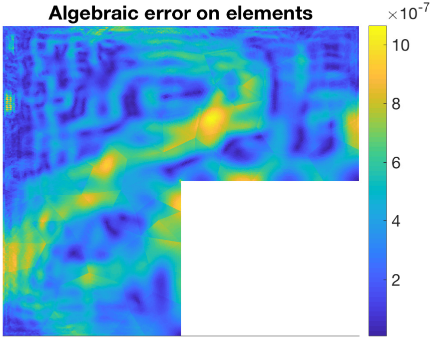
L-shape problem, PCG

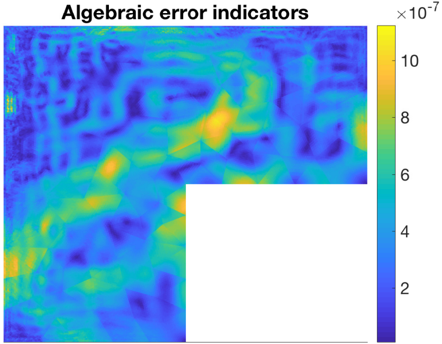
p (unknowns)	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.50×10^4)	4	8.86×10^{-2}	1.02	1.00^{-1}	9.13×10^{-2}	1.26	4.33^{-1}	2.22×10^{-2}	3.35	—
	8	3.82×10^{-4}	1.01	1.00^{-1}	2.22×10^{-2}	1.22	1.12^{-1}		1.22	1.12^{-1}
2 (1.01×10^5)	4	6.24×10^{-1}	1.01	1.00^{-1}	6.24×10^{-1}	1.07	9.06^{-1}	8.93×10^{-3}	2.61×10^1	—
	12	1.87×10^{-4}	1.01	1.00^{-1}	8.93×10^{-3}	1.33	1.28^{-1}		1.33	1.28^{-1}
3 (2.27×10^5)	7	1.02	1.00	1.00^{-1}	1.02	1.05	10.0^{-1}	5.29×10^{-3}	6.29×10^1	—
	28	9.58×10^{-5}	1.00	1.00^{-1}	5.29×10^{-3}	1.46	1.41^{-1}		1.46	1.41^{-1}
4 (4.04×10^5)	7	1.17	1.01	1.00^{-1}	1.17	1.08	7.56^{-1}	3.77×10^{-3}	1.30×10^2	—
	28	1.84×10^{-4}	1.01	1.00^{-1}	3.77×10^{-3}	1.52	1.60^{-1}		1.52	1.60^{-1}

L-shape problem, $p = 3$, total error, 28th PCG iteration



L-shape problem, $p = 3$, alg. error, 28th PCG iteration





Outline

Conclusions and outlook

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- **guaranteed** estimates on the **algebraic** and total **errors**
- **hierarchical construction**
- **local efficiency** and **robustness** wrt polynomial degree

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BLECHTA J., MÁLEK J., VOHRALÍK M., Localization of the $W^{-1,q}$ norm for local a posteriori efficiency, HAL Preprint 01332481.

Thank you for your attention!

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