

# Guaranteed and robust a posteriori error estimates for the reaction–diffusion and heat equations

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# Outline

## 1 Introduction

### 2 The reaction–diffusion equation

- Equivalence between error and dual norm of the residual
- Guaranteed upper bound
- Local efficiency and robustness

### 3 The heat equation

- Equivalence between error and dual norm of the residual
- High-order discretization & Radau reconstruction
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- Local space-time efficiency and robustness

### 4 Some numerical experiments (steady case)

### 5 Conclusions and future directions

# An optimal a posteriori estimate for singular problems

## Guaranteed upper bound

- $\|u - u_h\|_{?,\Omega}^2 \leq \sum_{K \in \mathcal{T}} \eta_K(u_h)^2$
- no undetermined constant: **error control**

## Local efficiency

- $\eta_K(u_h) \leq C_{\text{eff}} \|u - u_h\|_{?, \text{neighbors of } K}$
- **local** error lower bound (optimal space mesh refinement)

## Robustness

- $C_{\text{eff}}$  independent of domain  $\Omega$ , meshes, solution  $u$ , **data**

## Asymptotic exactness

- $\sum_{K \in \mathcal{T}} \eta_K(u_h)^2 / \|u - u_h\|_{?,\Omega}^2 \searrow 1$
- overestimation factor goes to one with increasing effort

## Small evaluation cost

- estimators  $\eta_K(u_h)$  can be evaluated cheaply (locally)

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# Previous results (reaction–diffusion equation)

- Verfürth (1998) / Ainsworth and Babuška (1999): **robustness** wrt. singular perturbation
- Cheddadi, Fučík, Prieto, Vohralík (2009): **guaranteed upper bound** & robustness,  $p = 1$
- Ainsworth and Vejchodský (2011, 2014): **guaranteed upper bound** & robustness but requires submesh (complicated), (2019) without submesh (simple);  $p = 1$
- Grosman (2006) / Kopteva (2017): **anisotropic meshes**

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# The reaction–diffusion equation

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Find  $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $d \geq 1$ , such that

$$-\varepsilon^2 \Delta u + \kappa^2 u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega$$

- $f \in L^2(\Omega)$ ,  $\varepsilon > 0$ ,  $\kappa \geq 0$  fixed real parameters

### Singular perturbation

- $\varepsilon \ll \kappa$

### Weak solution

Find  $u \in H_0^1(\Omega)$  such that

$$\varepsilon^2 (\nabla u, \nabla v) + \kappa^2 (u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

### Finite element approximation

Find  $u_h \in V_h := \mathcal{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$  such that

$$\varepsilon^2 (\nabla u_h, \nabla v_h) + \kappa^2 (u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h$$



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# Equivalence between error and residual

## Energy norm

$$\|\|v\|\|^2 := \varepsilon^2 \|\nabla v\|^2 + \kappa^2 \|v\|^2 \quad v \in H_0^1(\Omega)$$

Residual of  $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ , the misfit of  $u_h$  in the weak formulation:

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - \varepsilon^2 (\nabla u_h, \nabla v) - \kappa^2 (u_h, v) \quad v \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\| \mathcal{R}(u_h) \| \|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega); \|v\|=1} \langle \mathcal{R}(u_h), v \rangle$$

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$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega); \|v\|=1} \langle \mathcal{R}(u_h), v \rangle$$

## Energy error

$$\|(u - u_h)\| = \sup_{v \in H_0^1(\Omega); \|v\|=1} \{(f, v) - \varepsilon^2 (\nabla u_h, \nabla v) - \kappa^2 (u_h, v)\} = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

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**Residual of  $u_h \in H_0^1(\Omega)$**

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- **dual norm** of the residual

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**Energy error** is the dual norm of the residual

$$\|\varphi\| = \sup_{v \in H_0^1(\Omega); \|v\|=1} \{ \varepsilon^2 (\nabla \varphi, \nabla v) + \kappa^2 (\varphi, v) \} \quad \forall \varphi \in H_0^1(\Omega)$$

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$$\|\| (u - u_h) \|\| = \sup_{v \in H_0^1(\Omega); \|v\|=1} \{ \varepsilon^2 (\nabla(u - u_h), \nabla v) + \kappa^2 (u - u_h, v) \} = \|\| \mathcal{R}(u_h) \|\|_{H^{-1}(\Omega)}$$

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# Upper bound: motivation

## Bound on the residual

- let  $\sigma_h \in \mathbf{H}(\text{div}, \Omega)$  and  $\phi_h \in L^2(\Omega)$  be such that

$$\nabla \cdot \sigma_h + \kappa^2 \phi_h = f$$

- $\sigma_h$ : equilibrated flux reconstruction,  $\approx -\varepsilon^2 \nabla u$
- $\phi_h$ : potential reconstruction,  $\approx u$
- Green theorem  $(\nabla \cdot \sigma_h, v) + (\sigma_h, \nabla v) = 0$  for  $v \in H_0^1(\Omega)$ :

$$\begin{aligned} \langle \mathcal{R}(u_h), v \rangle &= (f, v) - \varepsilon^2 (\nabla u_h, \nabla v) - \kappa^2 (u_h, v) \\ &\equiv -(\varepsilon \nabla u_h + \varepsilon^{-1} \sigma_h, \varepsilon \nabla v) - (\kappa(u_h - \phi_h), \kappa v) \\ &\leq [\|\varepsilon \nabla u_h + \varepsilon^{-1} \sigma_h\|^2 + \|\kappa(u_h - \phi_h)\|^2]^{\frac{1}{2}} \|v\| \end{aligned}$$

- how to obtain suitable practical  $\sigma_h$  and  $\phi_h$ ?
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# Flux and potential reconstructions

Definition (Flux  $\sigma_h$  and potential  $\phi_h$ )

For each vertex  $\mathbf{a} \in \mathcal{V}$ , let

$$(\sigma_h^\mathbf{a}, \phi_h^\mathbf{a}) := \arg \min_{(\mathbf{v}_h, q_h) \in \mathcal{V}_h^\mathbf{a} \times Q_h^\mathbf{a}} J_h^{\mathbf{a}}(\mathbf{v}_h, q_h)$$

$$J_h^{\mathbf{a}}(\mathbf{v}_h, q_h) := \nu_h^2 \|\varepsilon \psi_\mathbf{a} \nabla \mathbf{v}_h + \varepsilon^{-1} \mathbf{v}_h\|_{\omega_\mathbf{a}}^2 + \|\kappa [\Pi_h(\psi_\mathbf{a} \mathbf{v}_h) - q_h]\|_{\omega_\mathbf{a}}^2$$

## Comments

- local discrete constrained minimization problems
-

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$$(\sigma_h^\mathbf{a}, \phi_h^\mathbf{a}) := \arg \quad (\mathbf{v}_h, q_h) \in \mathbf{V}_h^\mathbf{a} \times Q_h^\mathbf{a} \subset H_0(\text{div}, \omega_\mathbf{a}) \times L^2(\omega_\mathbf{a})$$

$$\nabla \cdot \mathbf{v}_h + \varepsilon^2 q_h - \Pi_h(v_h) = \varepsilon^{-1} \nabla \phi_h \cdot \nabla v_h$$

$$\mathcal{J}_{\varepsilon, \omega_\mathbf{a}}^0(\mathbf{v}_h, q_h) := w_\mathbf{a}^2 \|\varepsilon \mathbf{v}_h \nabla u_h + \varepsilon^{-1} v_h\|_{\omega_\mathbf{a}}^2 + \|\kappa [\Pi_h(u_h) - q_h]\|_{\omega_\mathbf{a}}^2$$

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# Guaranteed a posteriori error estimate

## Theorem (Guaranteed a posteriori error estimate)

Let  $\mathbf{u}$  be the weak solution and let  $\mathbf{u}_h \in V_h$  be its finite element approximation. Let  $\boldsymbol{\sigma}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\text{div}, \Omega)$  and  $\phi_h \in \mathcal{P}_p(\mathcal{T})$  be the flux and potential reconstructions. Then

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## 1 Introduction

## 2 The reaction–diffusion equation

- Equivalence between error and dual norm of the residual
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## 3 The heat equation

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$$w_K \|\varepsilon \nabla \mathbf{u}_h + \varepsilon^{-1} \boldsymbol{\sigma}_h\|_K + \|\kappa (\mathbf{u}_h - \phi_h)\|_K \leq C \| \mathbf{u} - \mathbf{u}_h \|_{\omega_K},$$

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# The heat equation

## The heat equation

$$\begin{aligned}\partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) &= u_0 \quad \text{in } \Omega\end{aligned}$$

$$f \in L^2(0, T; L^2(\Omega)), \quad u_0 \in L^2(\Omega)$$

## Spaces

$$X := L^2(0, T; H_0^1(\Omega)),$$

$$\|v\|_X^2 := \int_0^T \|\nabla v\|^2 dt,$$

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## Weak solution

Find  $u \in Y$  with  $u(0) = u_0$  such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) dt = \int_0^T (f, v) dt \quad \forall v \in X$$

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# The heat equation

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$$\partial_t u - \Delta u = f \quad \text{in } \Omega \times (0, T),$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

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$$f \in L^2(0, T; L^2(\Omega)), \quad u_0 \in L^2(\Omega)$$

## Spaces

$$\mathbf{X} := L^2(0, T; H_0^1(\Omega)),$$

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# An optimal a posteriori estimate for evolutive problems

## Guaranteed upper bound

- $\|u - u_{h\tau}\|_{?, \Omega \times (0, T)}^2 \leq \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

## Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{?, \text{neighbors of } K \times (t^{n-1}, t^n)}$
- optimal space-time mesh refinement
- **local** in **time** and in **space** error lower bound

## Robustness

- $C_{\text{eff}}$  independent of data, domain  $\Omega$ , **final time**  $T$ , meshes, solution  $u$ , **polynomial degrees** of  $u_{h\tau}$  in space and in time

## Asymptotic exactness

- $\sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|_{?, \Omega \times (0, T)}^2 \rightarrow 1$
- overestimation factor goes to one with increasing effort

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- estimators  $\eta_K^n(u_{h\tau})$  can be evaluated cheaply



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# Previous results (heat equation)

- Picasso / Verfürth (1998), work with the energy norm  $X$ :
  - ✓ upper bound  $\|u - u_{h\tau}\|_X^2 \leq C \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
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# Outline

## 1 Introduction

## 2 The reaction–diffusion equation

- Equivalence between error and dual norm of the residual
- Guaranteed upper bound
- Local efficiency and robustness

## 3 The heat equation

- Equivalence between error and dual norm of the residual
- High-order discretization & Radau reconstruction
- Guaranteed upper bound
- Local space-time efficiency and robustness

## 4 Some numerical experiments (steady case)

## 5 Conclusions and future directions

# Equivalence between error and residual

## The heat equation

$$\begin{aligned}\partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) &= u_0 \quad \text{in } \Omega\end{aligned}$$

## Spaces

$$\begin{aligned}X &:= L^2(0, T; H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 dt, \\ Y &:= L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 dt + \|v(T)\|^2\end{aligned}$$

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# Equivalence between error and residual

Theorem (Parabolic inf–sup identity)

For every  $\varphi \in Y$ , we have

$$\|\varphi\|_Y^2 = \left[ \sup_{v \in X, \|v\|_X=1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) dt \right]^2 + \|\varphi(0)\|^2.$$

Residual of  $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$ , the misfit of  $u_{h\tau}$  in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) dt \quad v \in X$$

- dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X, \|v\|_X=1} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

$Y$  norm error is the dual  $X$  norm of the residual + IC error

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Proof.

- let  $w_* \in X$  be defined by, a.e. in  $(0, T)$ ,

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$$(\nabla w_*, \nabla v) = \langle \partial_t \varphi, v \rangle \quad \forall v \in H_0^1(\Omega) \Rightarrow \|\nabla w_*\|^2 = \|\partial_t \varphi\|_{H^{-1}(\Omega)}^2$$

- using  $\int_0^T 2\langle \partial_t \varphi, \varphi \rangle dt = \|\varphi(T)\|^2 - \|\varphi(0)\|^2$  gives

$$\begin{aligned} & \left[ \sup_{v \in X, \|v\|_X=1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) dt \right]^2 \\ &= \|w_* + \varphi\|_X^2 = \int_0^T \|\nabla(w_* + \varphi)\|^2 dt \\ &= \int_0^T \|\nabla w_*\|^2 + 2(\nabla w_*, \nabla \varphi) + \|\nabla \varphi\|^2 dt \\ &= \int_0^T \|\partial_t \varphi\|_{H^{-1}(\Omega)}^2 + 2\langle \partial_t \varphi, \varphi \rangle + \|\nabla \varphi\|^2 dt = \|\varphi\|_Y^2 - \|\varphi(0)\|^2 \end{aligned}$$

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# Approximate solution and Radau reconstruction



## Approximate solution

- ✓  $u_{h\tau}(t)$ ,  $t \in I_n$ , is a piecewise continuous polynomial in space in  $V_h^n := \{v_h \in H_0^1(\Omega), v_h|_K \in \mathcal{P}_{pk}(K) \quad \forall K \in \mathcal{T}^n\}$
- ✗  $u_{h\tau}$  is a piecewise discontinuous polynomial in time
- ✗  $u_{h\tau} \notin Y \Rightarrow$  impossible to estimate  $\|u - u_{h\tau}\|_Y$

## Radau reconstruction

- ✓  $\mathcal{I}u_{h\tau} \in Y$ ,  $\mathcal{I}u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n+1}(I_n; \widetilde{V}_h^n)$  (Makridakis–Nochetto)

$$\int_{I_n} (\partial_t u_{h\tau}, v_m) + (\nabla u_{h\tau}, \nabla v_m) dt = \int_{I_n} (f, v_m) dt \quad \forall v_m \in \mathcal{Q}_{q_n}(I_n; V_h^n)$$

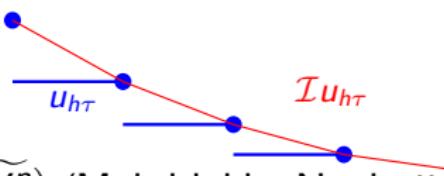
Final norm:  $\|u - \mathcal{I}u_{h\tau}\|_Y$

# Approximate solution and Radau reconstruction



## Approximate solution

- ✓  $u_{h\tau}(t)$ ,  $t \in I_n$ , is a piecewise **continuous** polynomial in space in  $V_h^n := \{v_h \in H_0^1(\Omega), v_h|_K \in \mathcal{P}_{pk}(K) \quad \forall K \in \mathcal{T}^n\}$
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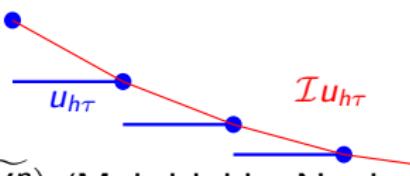
- ✓ final norm:  $\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 := \|u - \mathcal{I}u_{h\tau}\|_Y^2 + \|u_{h\tau} - \mathcal{I}u_{h\tau}\|_X^2$

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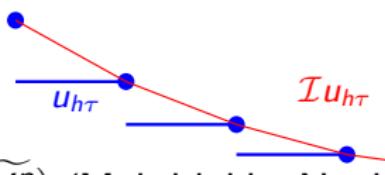
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# Approximate solution and Radau reconstruction



## Approximate solution

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# Results in the $Y$ norm

## Theorem (Reliability in the $Y$ norm)

Suppose no data oscillation for simplicity. Then, for any  $\sigma_{h\tau} \in L^2(0, T; \mathbf{H}(\text{div}, \Omega))$  with  $\nabla \cdot \sigma_{h\tau} = f - \partial_t \mathcal{I}u_{h\tau}$ , there holds

$$\|u - \mathcal{I}u_{h\tau}\|_Y^2 \leq \int_0^T \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2 dt.$$

# Proof of the upper bound

## Proof.

- equivalence error-residual (no error in the initial condition):

$$\|u - \mathcal{I}u_{h\tau}\|_Y = \sup_{v \in X, \|v\|_X=1} \langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle$$

- Green theorem

$$\int_0^T (\sigma_{h\tau}, \nabla \mathcal{I}u_{h\tau}) + (\nabla \cdot \sigma_{h\tau}, \mathcal{I}u_{h\tau}) dt = 0$$

- residual definition, Cauchy–Schwarz inequality:

$$\begin{aligned} \langle \mathcal{R}(\mathcal{I}u_{h\tau}), v \rangle &= \int_0^T (f, v) - (\partial_t \mathcal{I}u_{h\tau}, v) - (\nabla \mathcal{I}u_{h\tau}, \nabla v) dt \\ &= \int_0^T \underbrace{(f - \partial_t \mathcal{I}u_{h\tau} - \nabla \cdot \sigma_{h\tau}, v)}_{=0} - (\nabla \mathcal{I}u_{h\tau} + \sigma_{h\tau}, \nabla v) dt \\ &\leq \left\{ \int_0^T \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2 dt \right\}^{\frac{1}{2}} \|v\|_X \end{aligned}$$

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# Equilibrated flux reconstruction

## Definition (Equilibrated flux reconstruction)

For each time-step interval  $I_h$  and for each vertex  $\mathbf{a} \in \mathcal{V}^n$ , let

$$\sigma_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_{h\tau}^{\mathbf{a},n} \\ \nabla \cdot \mathbf{v}_h = \psi_{\mathbf{a}}(f - \partial_t \mathcal{I} \mathbf{u}_{h\tau}) - \nabla \psi_{\mathbf{a}} \cdot \nabla \mathbf{u}_{h\tau}}} \int_{I_h} \|\mathbf{v}_h + \psi_{\mathbf{a}} \nabla \mathbf{u}_{h\tau}\|_{\omega_{\mathbf{a}}}^2 dt.$$

Then set

$$\sigma_{h\tau} := \sum_{n=1}^N \sum_{\mathbf{a} \in \mathcal{V}^n} \sigma_{h\tau}^{\mathbf{a},n}.$$

## Comments

- ✓ satisfies  $\sigma_{h\tau} \in L^2(0, T; \mathbf{H}(\text{div}, \Omega))$  with  $\nabla \cdot \sigma_{h\tau} = f - \partial_t \mathcal{I} \mathbf{u}_{h\tau}$
- works on the common refinement  $\widetilde{\mathcal{T}}^{\mathbf{a},n}$  of the patch  $\omega_{\mathbf{a}}$
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# Guaranteed upper bound

## Theorem (Guaranteed upper bound)

*In the absence of data oscillation ( $f$  and  $u_0$  piecewise polynomial), there holds*

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 \leq \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 dt.$$

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# Local space-time efficiency and robustness

## Local error contributions

$$\begin{aligned} |u - u_{h\tau}|_{\mathcal{E}_Y^{\mathbf{a},n}}^2 = & \int_{I_n} \|\partial_t(u - \mathcal{I}u_{h\tau})\|_{H^{-1}(\omega_{\mathbf{a}})}^2 + \|\nabla(u - \mathcal{I}u_{h\tau})\|_{\omega_{\mathbf{a}}}^2 dt \\ & + \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_{\omega_{\mathbf{a}}}^2 dt \end{aligned}$$

Theorem (Local space-time efficiency and robustness)

For each time-step interval  $I_n$  and for each element  $K \in \mathcal{T}^n$ , there holds, in the absence of data oscillation,

$$\int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|_K^2 + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|_K^2 dt \leq C_{\text{eff}}^2 \sum_{\mathbf{a} \in \mathcal{V}_K} |u - u_{h\tau}|_{\mathcal{E}_Y^{\mathbf{a},n}}^2.$$

## Comments

- ✓ local in space and time
- ✓  $C_{\text{eff}}$  only depends on shape regularity  $\Rightarrow$  robustness w.r.t. the final time  $T$  and the polynomial degrees  $p$  and  $q$
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recall

$$\begin{aligned} \|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 = & \int_0^T \|\partial_t(u - \mathcal{I}u_{h\tau})\|_{H^{-1}(\Omega)}^2 dt + \int_0^T \|\nabla(u - \mathcal{I}u_{h\tau})\|^2 dt \\ & + \int_0^T \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 dt + \|(u - \mathcal{I}u_{h\tau})(T)\|^2 \end{aligned}$$

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# Numerics: smooth case

## Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (0, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

## Exact solution

$$u(x, y) = \sin(2\pi x) \sin(2\pi y)$$

## Discretization

- symmetric interior penalty discontinuous Galerkin method:  
 $u_h \notin H_0^1(\Omega)$
- unstructured triangular grids
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# How large is the overall error?

(model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u\ }$	$\beta = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^{-1}\%$	1.7
$\approx h_0/2$						
$\approx h_0/4$						
$\approx h_0/8$						
$\approx h_0/16$						
$\approx h_0/32$						
$\approx h_0/64$						
$\approx h_0/128$						

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$h$	$p$	$\eta(\mathbf{u}_h)$	rel. error estimate $\frac{\eta(\mathbf{u}_h)}{\ \nabla \mathbf{u}_h\ }$	$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $	rel. error $\frac{\ \nabla(\mathbf{u} - \mathbf{u}_h)\ }{\ \nabla \mathbf{u}_h\ }$	$\ \mathbf{u} - \mathbf{u}_h\ $	rel. error $\frac{\ \mathbf{u} - \mathbf{u}_h\ }{\ \mathbf{u}\ }$
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$\approx h_0/32$							
$\approx h_0/64$							
$\approx h_0/128$							
$\approx h_0/256$							

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M. Ern, I. Smears, M. Vohralík, A posteriori error estimates for reaction-diffusion problems

How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$P^H = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^1\%$	1.1	$2.4 \times 10^1\%$	1.37
$\approx h_0/8$	4	$2.6 \times 10^{-2}$	$8.2 \times 10^{-3}\%$	$2.6 \times 10^{-2}$	$1.0 \times 10^{-2}\%$	1.37
$\approx h_0/16$	8	$3.3 \times 10^{-3}$	$1.0 \times 10^{-3}\%$	$3.3 \times 10^{-3}$	$1.0 \times 10^{-3}\%$	1.37
$\approx h_0/32$	16	$4.1 \times 10^{-4}$	$1.3 \times 10^{-4}\%$	$4.1 \times 10^{-4}$	$1.3 \times 10^{-4}\%$	1.37
$\approx h_0/64$	32	$5.1 \times 10^{-5}$	$1.7 \times 10^{-5}\%$	$5.1 \times 10^{-5}$	$1.7 \times 10^{-5}\%$	1.37
$\approx h_0/128$	64	$6.3 \times 10^{-6}$	$2.2 \times 10^{-6}\%$	$6.3 \times 10^{-6}$	$2.2 \times 10^{-6}\%$	1.37
$\approx h_0/256$	128	$7.7 \times 10^{-7}$	$2.8 \times 10^{-7}\%$	$7.7 \times 10^{-7}$	$2.8 \times 10^{-7}\%$	1.37
$\approx h_0/512$	256	$9.4 \times 10^{-8}$	$3.6 \times 10^{-8}\%$	$9.4 \times 10^{-8}$	$3.6 \times 10^{-8}\%$	1.37
$\approx h_0/1024$	512	$1.14 \times 10^{-8}$	$4.6 \times 10^{-9}\%$	$1.14 \times 10^{-8}$	$4.6 \times 10^{-9}\%$	1.37
$\approx h_0/2048$	1024	$1.37 \times 10^{-9}$	$5.8 \times 10^{-10}\%$	$1.37 \times 10^{-9}$	$5.8 \times 10^{-10}\%$	1.37
$\approx h_0/4096$	2048	$1.64 \times 10^{-10}$	$7.2 \times 10^{-11}\%$	$1.64 \times 10^{-10}$	$7.2 \times 10^{-11}\%$	1.37
$\approx h_0/8192$	4096	$2.0 \times 10^{-11}$	$9.0 \times 10^{-12}\%$	$2.0 \times 10^{-11}$	$9.0 \times 10^{-12}\%$	1.37
$\approx h_0/16384$	8192	$2.4 \times 10^{-12}$	$1.1 \times 10^{-12}\%$	$2.4 \times 10^{-12}$	$1.1 \times 10^{-12}\%$	1.37
$\approx h_0/32768$	16384	$3.0 \times 10^{-13}$	$1.3 \times 10^{-13}\%$	$3.0 \times 10^{-13}$	$1.3 \times 10^{-13}\%$	1.37
$\approx h_0/65536$	32768	$3.8 \times 10^{-14}$	$1.6 \times 10^{-14}\%$	$3.8 \times 10^{-14}$	$1.6 \times 10^{-14}\%$	1.37
$\approx h_0/131072$	65536	$4.8 \times 10^{-15}$	$2.0 \times 10^{-15}\%$	$4.8 \times 10^{-15}$	$2.0 \times 10^{-15}\%$	1.37
$\approx h_0/262144$	131072	$6.1 \times 10^{-16}$	$2.5 \times 10^{-16}\%$	$6.1 \times 10^{-16}$	$2.5 \times 10^{-16}\%$	1.37
$\approx h_0/524288$	262144	$7.7 \times 10^{-17}$	$3.2 \times 10^{-17}\%$	$7.7 \times 10^{-17}$	$3.2 \times 10^{-17}\%$	1.37
$\approx h_0/1048576$	524288	$9.7 \times 10^{-18}$	$4.1 \times 10^{-18}\%$	$9.7 \times 10^{-18}$	$4.1 \times 10^{-18}\%$	1.37
$\approx h_0/2097152$	1097152	$1.21 \times 10^{-18}$	$5.2 \times 10^{-19}\%$	$1.21 \times 10^{-18}$	$5.2 \times 10^{-19}\%$	1.37
$\approx h_0/4194304$	4194304	$1.51 \times 10^{-19}$	$6.6 \times 10^{-20}\%$	$1.51 \times 10^{-19}$	$6.6 \times 10^{-20}\%$	1.37
$\approx h_0/8388608$	8388608	$1.87 \times 10^{-20}$	$8.3 \times 10^{-21}\%$	$1.87 \times 10^{-20}$	$8.3 \times 10^{-21}\%$	1.37
$\approx h_0/16777216$	16777216	$2.32 \times 10^{-21}$	$1.04 \times 10^{-21}\%$	$2.32 \times 10^{-21}$	$1.04 \times 10^{-21}\%$	1.37
$\approx h_0/33554432$	33554432	$2.84 \times 10^{-22}$	$1.28 \times 10^{-22}\%$	$2.84 \times 10^{-22}$	$1.28 \times 10^{-22}\%$	1.37
$\approx h_0/67108864$	67108864	$3.49 \times 10^{-23}$	$1.56 \times 10^{-23}\%$	$3.49 \times 10^{-23}$	$1.56 \times 10^{-23}\%$	1.37
$\approx h_0/134217728$	134217728	$4.26 \times 10^{-24}$	$1.88 \times 10^{-24}\%$	$4.26 \times 10^{-24}$	$1.88 \times 10^{-24}\%$	1.37
$\approx h_0/268435456$	268435456	$5.16 \times 10^{-25}$	$2.24 \times 10^{-25}\%$	$5.16 \times 10^{-25}$	$2.24 \times 10^{-25}\%$	1.37
$\approx h_0/536870912$	536870912	$6.22 \times 10^{-26}$	$2.64 \times 10^{-26}\%$	$6.22 \times 10^{-26}$	$2.64 \times 10^{-26}\%$	1.37
$\approx h_0/1073741824$	1073741824	$7.48 \times 10^{-27}$	$3.1 \times 10^{-27}\%$	$7.48 \times 10^{-27}$	$3.1 \times 10^{-27}\%$	1.37
$\approx h_0/2147483648$	2147483648	$9.0 \times 10^{-28}$	$3.6 \times 10^{-28}\%$	$9.0 \times 10^{-28}$	$3.6 \times 10^{-28}\%$	1.37
$\approx h_0/4294967296$	4294967296	$1.08 \times 10^{-28}$	$4.2 \times 10^{-29}\%$	$1.08 \times 10^{-28}$	$4.2 \times 10^{-29}\%$	1.37
$\approx h_0/8589934592$	8589934592	$1.29 \times 10^{-29}$	$4.9 \times 10^{-30}\%$	$1.29 \times 10^{-29}$	$4.9 \times 10^{-30}\%$	1.37
$\approx h_0/17179869184$	17179869184	$1.54 \times 10^{-29}$	$5.7 \times 10^{-31}\%$	$1.54 \times 10^{-29}$	$5.7 \times 10^{-31}\%$	1.37
$\approx h_0/34359738368$	34359738368	$1.84 \times 10^{-29}$	$6.6 \times 10^{-32}\%$	$1.84 \times 10^{-29}$	$6.6 \times 10^{-32}\%$	1.37
$\approx h_0/68719476736$	68719476736	$2.2 \times 10^{-29}$	$7.6 \times 10^{-33}\%$	$2.2 \times 10^{-29}$	$7.6 \times 10^{-33}\%$	1.37
$\approx h_0/137438953472$	137438953472	$2.64 \times 10^{-29}$	$8.7 \times 10^{-34}\%$	$2.64 \times 10^{-29}$	$8.7 \times 10^{-34}\%$	1.37
$\approx h_0/274877906944$	274877906944	$3.16 \times 10^{-29}$	$1.0 \times 10^{-34}\%$	$3.16 \times 10^{-29}$	$1.0 \times 10^{-34}\%$	1.37
$\approx h_0/549755813888$	549755813888	$3.77 \times 10^{-29}$	$1.14 \times 10^{-35}\%$	$3.77 \times 10^{-29}$	$1.14 \times 10^{-35}\%$	1.37
$\approx h_0/1099511627776$	1099511627776	$4.49 \times 10^{-29}$	$1.3 \times 10^{-36}\%$	$4.49 \times 10^{-29}$	$1.3 \times 10^{-36}\%$	1.37
$\approx h_0/2199023255552$	2199023255552	$5.31 \times 10^{-29}$	$1.5 \times 10^{-37}\%$	$5.31 \times 10^{-29}$	$1.5 \times 10^{-37}\%$	1.37
$\approx h_0/4398046511104$	4398046511104	$6.25 \times 10^{-29}$	$1.72 \times 10^{-38}\%$	$6.25 \times 10^{-29}$	$1.72 \times 10^{-38}\%$	1.37
$\approx h_0/8796093022208$	8796093022208	$7.31 \times 10^{-29}$	$1.96 \times 10^{-39}\%$	$7.31 \times 10^{-29}$	$1.96 \times 10^{-39}\%$	1.37
$\approx h_0/17592186044416$	17592186044416	$8.5 \times 10^{-29}$	$2.22 \times 10^{-40}\%$	$8.5 \times 10^{-29}$	$2.22 \times 10^{-40}\%$	1.37
$\approx h_0/35184372088832$	35184372088832	$9.8 \times 10^{-29}$	$2.5 \times 10^{-41}\%$	$9.8 \times 10^{-29}$	$2.5 \times 10^{-41}\%$	1.37
$\approx h_0/70368744177664$	70368744177664	$11.2 \times 10^{-29}$	$2.8 \times 10^{-42}\%$	$11.2 \times 10^{-29}$	$2.8 \times 10^{-42}\%$	1.37
$\approx h_0/140737488355328$	140737488355328	$12.7 \times 10^{-29}$	$3.1 \times 10^{-43}\%$	$12.7 \times 10^{-29}$	$3.1 \times 10^{-43}\%$	1.37
$\approx h_0/281474976710656$	281474976710656	$14.3 \times 10^{-29}$	$3.44 \times 10^{-44}\%$	$14.3 \times 10^{-29}$	$3.44 \times 10^{-44}\%$	1.37
$\approx h_0/562949953421312$	562949953421312	$16.0 \times 10^{-29}$	$3.8 \times 10^{-45}\%$	$16.0 \times 10^{-29}$	$3.8 \times 10^{-45}\%$	1.37
$\approx h_0/1125899906842624$	1125899906842624	$17.8 \times 10^{-29}$	$4.2 \times 10^{-46}\%$	$17.8 \times 10^{-29}$	$4.2 \times 10^{-46}\%$	1.37
$\approx h_0/2251799813685248$	2251799813685248	$19.7 \times 10^{-29}$	$4.64 \times 10^{-47}\%$	$19.7 \times 10^{-29}$	$4.64 \times 10^{-47}\%$	1.37
$\approx h_0/4503599627370496$	4503599627370496	$21.7 \times 10^{-29}$	$5.1 \times 10^{-48}\%$	$21.7 \times 10^{-29}$	$5.1 \times 10^{-48}\%$	1.37
$\approx h_0/9007199254740992$	9007199254740992	$23.8 \times 10^{-29}$	$5.6 \times 10^{-49}\%$	$23.8 \times 10^{-29}$	$5.6 \times 10^{-49}\%$	1.37
$\approx h_0/18014398509481984$	18014398509481984	$26.0 \times 10^{-29}$	$6.16 \times 10^{-50}\%$	$26.0 \times 10^{-29}$	$6.16 \times 10^{-50}\%$	1.37
$\approx h_0/36028797018963968$	36028797018963968	$28.3 \times 10^{-29}$	$6.76 \times 10^{-51}\%$	$28.3 \times 10^{-29}$	$6.76 \times 10^{-51}\%$	1.37
$\approx h_0/72057594037927936$	72057594037927936	$30.7 \times 10^{-29}$	$7.4 \times 10^{-52}\%$	$30.7 \times 10^{-29}$	$7.4 \times 10^{-52}\%$	1.37
$\approx h_0/144115188075855872$	144115188075855872	$33.2 \times 10^{-29}$	$8.1 \times 10^{-53}\%$	$33.2 \times 10^{-29}$	$8.1 \times 10^{-53}\%$	1.37
$\approx h_0/288230376151711744$	288230376151711744	$35.8 \times 10^{-29}$	$8.84 \times 10^{-54}\%$	$35.8 \times 10^{-29}$	$8.84 \times 10^{-54}\%$	1.37
$\approx h_0/576460752303423488$	576460752303423488	$38.5 \times 10^{-29}$	$9.6 \times 10^{-55}\%$	$38.5 \times 10^{-29}$	$9.6 \times 10^{-55}\%$	1.37
$\approx h_0/1152921504606846976$	1152921504606846976	$41.3 \times 10^{-29}$	$1.04 \times 10^{-55}\%$	$41.3 \times 10^{-29}$	$1.04 \times 10^{-55}\%$	1.37
$\approx h_0/2305843009213693952$	2305843009213693952	$44.2 \times 10^{-29}$	$1.13 \times 10^{-56}\%$	$44.2 \times 10^{-29}$	$1.13 \times 10^{-56}\%$	1.37
$\approx h_0/4611686018427387904$	4611686018427387904	$47.2 \times 10^{-29}$	$1.22 \times 10^{-57}\%$	$47.2 \times 10^{-29}$	$1.22 \times 10^{-57}\%$	1.37
$\approx h_0/9223372036854775808$	9223372036854775808	$50.3 \times 10^{-29}$	$1.32 \times 10^{-58}\%$	$50.3 \times 10^{-29}$	$1.32 \times 10^{-58}\%$	1.37
$\approx h_0/18446744073709551616$	18446744073709551616	$53.5 \times 10^{-29}$	$1.42 \times 10^{-59}\%$	$53.5 \times 10^{-29}$	$1.42 \times 10^{-59}\%$	1.37
$\approx h_0/36893488147419103232$	36893488147419103232	$56.8 \times 10^{-29}$	$1.53 \times 10^{-60}\%$	$56.8 \times 10^{-29}$	$1.53 \times 10^{-60}\%$	1.37
$\approx h_0/73786976294838206464$	73786976294838206464	$60.2 \times 10^{-29}$	$1.64 \times 10^{-61}\%$	$60.2 \times 10^{-29}$	$1.64 \times 10^{-61}\%$	1.37
$\approx h_0/147573952589676412928$	147573952589676412928	$63.7 \times 10^{-29}$	$1.76 \times 10^{-62}\%$	$63.7 \times 10^{-29}$	$1.76 \times 10^{-62}\%$	1.37
$\approx h_0/295147905179352825856$	295147905179352825856	$67.3 \times 10^{-29}$	$1.88 \times 10^{-63}\%$	$67.3 \times 10^{-29}$	$1.88 \times 10^{-63}\%$	1.37
$\approx h_0/590295810358705651712$	590295810358705651712	$71.0 \times 10^{-29}$	$2.01 \times 10^{-64}\%$	$71.0 \times 10^{-29}$	$2.01 \times 10^{-64}\%$	1.37
$\approx h_0/1180591620717411303424$	1180591620717411303424	$74.7 \times 10^{-29}$	$2.14 \times 10^{-65}\%$	$74.7 \times 10^{-29}$	$2.14 \times 10^{-65}\%$	1.37
$\approx h_0/2361183241434822606848$	2361183241434822606848	$78.5 \times 10^{-29}$	$2.28 \times 10^{-66}\%$	$78.5 \times 10^{-29}$	$2.28 \times 10^{-66}\%$	1.37
$\approx h_0/4722366482869645213696$	4722366482869645213696	$82.3 \times 10^{-29}$	$2.42 \times 10^{-67}\%$	$82.3 \times 10^{-29}$	$2.42 \times 10^{-67}\%$	1.37
$\approx h_0/9444732965739290427392$	9444732965739290427392	$86.2 \times 10^{-29}$	$2.57 \times 10^{-68}\%$	$86.2 \times 10^{-29}$	$2.57 \times 10^{-68}\%$	1.37
$\approx h_0/18889465931478580854784$	18889465931478580854784	$90.1 \times 10^{-29}$	$2.72 \times 10^{-69}\%$	$90.1 \times 10^{-29}$	$2.72 \times 10^{-69}\%$	1.37
$\approx h_0/37778931862957161709568$	37778931862957161709568	$94.1 \times 10^{-29}$	$2.88 \times 10^{-70}\%$	$94.1 \times 10^{-29}$	$2.88 \times 10^{-70}\%$	1.37
$\approx h_0/75557863725914323419136$	75557863725914323419136	$98.1 \times 10^{-29}$	$3.04 \times 10^{-71}\%$	$98.1 \times 10^{-29}$	$3.04 \times 10^{-71}\%$	1.37
$\approx h_0/151115727451826646838272$	151115727451826646838272	$102.1 \times 10^{-29}$	$3.21 \times 10^{-72}\%$	$102.1 \times 10^{-29}$	$3.21 \times 10^{-72}\%$	1.37
$\approx h_0/302231454903653293676544$	302231454903653293676544	$106.1 \times 10^{-29}$	$3.39 \times 10^{-73}\%$	$106.1 \times 10^{-29}$	$3.39 \times 10^{-73}\%$	1.37
$\approx h_0/604462909807306587353088$	604462909807306587353088	$110.1 \times 10^{-29}$	$3.57 \times 10^{-74}\%$	$110.1 \times 10^{-29}$	$3.57 \times 10^{-74}\%$	1.37
$\approx h_0/1208925819614613174706176$	1208925819614613174706176	$114.1 \times 10^{-29}$	$3.76 \times 10^{-75}\%$	$114.1 \times 10^{-29}$	$3.76 \times 10^{-75}\%$	1.37
$\approx h_0/241785163922922634941232$	241785163922922634941232	$118.1 \times 10^{-29}$	$3.95 \times 10^{-76}\%$	$118.1 \times 10^{-29}$	$3.95 \times 10^{-76}\%$	1.37
$\approx h_0/483570327845845269882464$	483570327845845269882464	$122.1 \times 10^{-29}$	$4.15 \times 10^{-77}\%$	$122.1 \times 10^{-29}$	$4.15 \times 10^{-77}\%$	1.37
$\approx h_0/967140655691690539764928$	967140655691690539764928	$126.1 \times 10^{-29}$	$4.35 \times 10^{-78}\%$	$126.1 \times 10^{-29}$	$4.35 \times 10^{-78}\%$	1.37
$\approx h_0/1934281311383381079529856$	1934281311383381079529856	$130.1 \times 10^{-29}$	$4.56 \times 10^{-79}\%$	$130.1 \times 10^{-29}$	$4.56 \times 10^{-79}\%$	1.37
$\approx h_0/3868562622766762159$						

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.0 \times 10^{1\%}$	
$\approx h_0/4$		$3.1 \times 10^{-1}$	7%	$2.9 \times 10^{-1}$	5%	
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.5%	$1.4 \times 10^{-1}$	3.5%	
$\approx h_0/16$		$7.8 \times 10^{-2}$	1.8%	$7.6 \times 10^{-2}$	1.8%	
$\approx h_0/32$		$3.9 \times 10^{-2}$	0.95%	$4.1 \times 10^{-2}$	0.95%	
$\approx h_0/64$		$1.9 \times 10^{-2}$	0.48%	$1.9 \times 10^{-2}$	0.48%	
$\approx h_0/128$		$9.5 \times 10^{-3}$	0.24%	$9.5 \times 10^{-3}$	0.24%	
$\approx h_0/256$		$4.7 \times 10^{-3}$	0.12%	$4.7 \times 10^{-3}$	0.12%	
$\approx h_0/512$		$2.3 \times 10^{-3}$	0.06%	$2.3 \times 10^{-3}$	0.06%	
$\approx h_0/1024$		$1.15 \times 10^{-3}$	0.03%	$1.15 \times 10^{-3}$	0.03%	
$\approx h_0/2048$		$5.7 \times 10^{-4}$	0.015%	$5.7 \times 10^{-4}$	0.015%	
$\approx h_0/4096$		$2.8 \times 10^{-4}$	0.0075%	$2.8 \times 10^{-4}$	0.0075%	
$\approx h_0/8192$		$1.4 \times 10^{-4}$	0.00375%	$1.4 \times 10^{-4}$	0.00375%	
$\approx h_0/16384$		$7.1 \times 10^{-5}$	0.001875%	$7.1 \times 10^{-5}$	0.001875%	
$\approx h_0/32768$		$3.5 \times 10^{-5}$	0.0009375%	$3.5 \times 10^{-5}$	0.0009375%	
$\approx h_0/65536$		$1.75 \times 10^{-5}$	0.00046875%	$1.75 \times 10^{-5}$	0.00046875%	
$\approx h_0/131072$		$8.75 \times 10^{-6}$	0.000234375%	$8.75 \times 10^{-6}$	0.000234375%	
$\approx h_0/262144$		$4.375 \times 10^{-6}$	0.0001171875%	$4.375 \times 10^{-6}$	0.0001171875%	
$\approx h_0/524288$		$2.1875 \times 10^{-6}$	0.00005859375%	$2.1875 \times 10^{-6}$	0.00005859375%	
$\approx h_0/1048576$		$1.09375 \times 10^{-6}$	0.000029296875%	$1.09375 \times 10^{-6}$	0.000029296875%	
$\approx h_0/2097152$		$5.46875 \times 10^{-7}$	0.0000146484375%	$5.46875 \times 10^{-7}$	0.0000146484375%	
$\approx h_0/4194304$		$2.734375 \times 10^{-7}$	0.00000732421875%	$2.734375 \times 10^{-7}$	0.00000732421875%	
$\approx h_0/8388608$		$1.3671875 \times 10^{-7}$	0.000003662109375%	$1.3671875 \times 10^{-7}$	0.000003662109375%	
$\approx h_0/16777216$		$6.8359375 \times 10^{-8}$	0.0000018310546875%	$6.8359375 \times 10^{-8}$	0.0000018310546875%	
$\approx h_0/33554432$		$3.41796875 \times 10^{-8}$	0.00000091552734375%	$3.41796875 \times 10^{-8}$	0.00000091552734375%	
$\approx h_0/67108864$		$1.708984375 \times 10^{-8}$	0.000000457763671875%	$1.708984375 \times 10^{-8}$	0.000000457763671875%	
$\approx h_0/134217728$		$8.544921875 \times 10^{-9}$	0.0000002288818359375%	$8.544921875 \times 10^{-9}$	0.0000002288818359375%	
$\approx h_0/268435456$		$4.2724609375 \times 10^{-9}$	0.00000011444091796875%	$4.2724609375 \times 10^{-9}$	0.00000011444091796875%	
$\approx h_0/536870912$		$2.13623046875 \times 10^{-9}$	0.000000057220458984375%	$2.13623046875 \times 10^{-9}$	0.000000057220458984375%	
$\approx h_0/1073741824$		$1.068115234375 \times 10^{-9}$	0.0000000286102294921875%	$1.068115234375 \times 10^{-9}$	0.0000000286102294921875%	
$\approx h_0/2147483648$		$5.340576171875 \times 10^{-10}$	0.00000001430511474609375%	$5.340576171875 \times 10^{-10}$	0.00000001430511474609375%	
$\approx h_0/4294967296$		$2.6702880859375 \times 10^{-10}$	0.000000007152557373046875%	$2.6702880859375 \times 10^{-10}$	0.000000007152557373046875%	
$\approx h_0/8589934592$		$1.33514404296875 \times 10^{-10}$	0.0000000035762786865234375%	$1.33514404296875 \times 10^{-10}$	0.0000000035762786865234375%	
$\approx h_0/17179869184$		$6.67572021484375 \times 10^{-11}$	0.00000000178813934326171875%	$6.67572021484375 \times 10^{-11}$	0.00000000178813934326171875%	
$\approx h_0/34359738368$		$3.337860107421875 \times 10^{-11}$	0.000000000894069671630859375%	$3.337860107421875 \times 10^{-11}$	0.000000000894069671630859375%	
$\approx h_0/68719476736$		$1.6689300537109375 \times 10^{-11}$	0.0000000004470348358154296875%	$1.6689300537109375 \times 10^{-11}$	0.0000000004470348358154296875%	
$\approx h_0/137438953472$		$8.3446502685546875 \times 10^{-12}$	0.00000000022351741790771484375%	$8.3446502685546875 \times 10^{-12}$	0.00000000022351741790771484375%	
$\approx h_0/274877906944$		$4.17232513427734375 \times 10^{-12}$	0.000000000111758708903857421875%	$4.17232513427734375 \times 10^{-12}$	0.000000000111758708903857421875%	
$\approx h_0/549755813888$		$2.086162567138671875 \times 10^{-12}$	0.000000000558793544511523609375%	$2.086162567138671875 \times 10^{-12}$	0.000000000558793544511523609375%	
$\approx h_0/1099511627776$		$1.0430812835693359375 \times 10^{-12}$	0.0000000002793967722557618046875%	$1.0430812835693359375 \times 10^{-12}$	0.0000000002793967722557618046875%	
$\approx h_0/2199023255552$		$5.2154064178346796875 \times 10^{-13}$	0.00000000013969838612788090234375%	$5.2154064178346796875 \times 10^{-13}$	0.00000000013969838612788090234375%	
$\approx h_0/4398046511104$		$2.60770320891733984375 \times 10^{-13}$	0.0000000006984919306394018016875%	$2.60770320891733984375 \times 10^{-13}$	0.0000000006984919306394018016875%	
$\approx h_0/8796093022208$		$1.303851604458669921875 \times 10^{-13}$	0.00000000034924596531950360084375%	$1.303851604458669921875 \times 10^{-13}$	0.00000000034924596531950360084375%	
$\approx h_0/17592186044416$		$6.519258022293349609375 \times 10^{-14}$	0.000000000174622982659751800421875%	$6.519258022293349609375 \times 10^{-14}$	0.000000000174622982659751800421875%	
$\approx h_0/35184372088832$		$3.2596290111466748046875 \times 10^{-14}$	0.0000000008731149131493790002109375%	$3.2596290111466748046875 \times 10^{-14}$	0.0000000008731149131493790002109375%	
$\approx h_0/70368744177664$		$1.62981450557333740234375 \times 10^{-14}$	0.00000000043655745657469700010546875%	$1.62981450557333740234375 \times 10^{-14}$	0.00000000043655745657469700010546875%	
$\approx h_0/140737488355328$		$8.149072527866687011709375 \times 10^{-15}$	0.000000000218278728287348500052734375%	$8.149072527866687011709375 \times 10^{-15}$	0.000000000218278728287348500052734375%	
$\approx h_0/281474976710656$		$4.0745362639333435058546875 \times 10^{-15}$	0.00000000010913936414362425002636875%	$4.0745362639333435058546875 \times 10^{-15}$	0.00000000010913936414362425002636875%	
$\approx h_0/562949953421312$		$2.03726813196667175292734375 \times 10^{-15}$	0.000000000545696830358812125013184375%	$2.03726813196667175292734375 \times 10^{-15}$	0.000000000545696830358812125013184375%	
$\approx h_0/1125899906842624$		$1.018634065983335876463678125 \times 10^{-15}$	0.000000000272848415189406062506592125%	$1.018634065983335876463678125 \times 10^{-15}$	0.000000000272848415189406062506592125%	
$\approx h_0/2251799813685248$		$5.093170329916679382231390625 \times 10^{-16}$	0.0000000001364242075947030312532960625%	$5.093170329916679382231390625 \times 10^{-16}$	0.0000000001364242075947030312532960625%	
$\approx h_0/4503599627370496$		$2.5465851649583396911156953125 \times 10^{-16}$	0.0000000006821210379735060156264980625%	$2.5465851649583396911156953125 \times 10^{-16}$	0.0000000006821210379735060156264980625%	
$\approx h_0/9007199254740992$		$1.27329258247916984555784765625 \times 10^{-16}$	0.0000000003410605189867530078132493125%	$1.27329258247916984555784765625 \times 10^{-16}$	0.0000000003410605189867530078132493125%	
$\approx h_0/18014398509481984$		$6.366462912395849227789238125 \times 10^{-17}$	0.00000000017053025949218850390662465625%	$6.366462912395849227789238125 \times 10^{-17}$	0.00000000017053025949218850390662465625%	
$\approx h_0/36028797018963968$		$3.1832314561979246138946190625 \times 10^{-17}$	0.0000000008526512974754715195331223125%	$3.1832314561979246138946190625 \times 10^{-17}$	0.0000000008526512974754715195331223125%	
$\approx h_0/72057594037927936$		$1.59161572809896230694730953125 \times 10^{-17}$	0.00000000042632564873771075976656115625%	$1.59161572809896230694730953125 \times 10^{-17}$	0.00000000042632564873771075976656115625%	
$\approx h_0/144115188075855872$		$7.95807864049481153473654765625 \times 10^{-18}$	0.000000000213162824344427879883280578125%	$7.95807864049481153473654765625 \times 10^{-18}$	0.000000000213162824344427879883280578125%	
$\approx h_0/288230376151711744$		$3.9790393202474057673682738125 \times 10^{-18}$	0.00000000010658041211110721994144115625%	$3.9790393202474057673682738125 \times 10^{-18}$	0.00000000010658041211110721994144115625%	
$\approx h_0/576460752303423488$		$1.98951966012370288368413690625 \times 10^{-18}$	0.0000000004164510552777144398828823125%	$1.98951966012370288368413690625 \times 10^{-18}$	0.0000000004164510552777144398828823125%	
$\approx h_0/1152921504606846976$		$9.94759830061851441834068453125 \times 10^{-19}$	0.00000000020411552769442881994144115625%	$9.94759830061851441834068453125 \times 10^{-19}$	0.00000000020411552769442881994144115625%	
$\approx h_0/2305843009213693952$		$4.973799150309257209170342265625 \times 10^{-19}$	0.0000000001018537638488576099707281125%	$4.973799150309257209170342265625 \times 10^{-19}$	0.0000000001018537638488576099707281125%	
$\approx h_0/4611686018427387904$		$2.48689957515462880366813813125 \times 10^{-19}$	0.00000000050918488412211521994144115625%	$2.48689957515462880366813813125 \times 10^{-19}$	0.00000000050918488412211521994144115625%	
$\approx h_0/9223372036854775808$		$1.243439830061851441834068453125 \times 10^{-19}$	0.0000000002545474420550304099707281125%	$1.243439830061851441834068453125 \times 10^{-19}$	0.0000000002545474420550304099707281125%	
$\approx h_0/18446744073709551616$		$6.217199150309257209170342265625 \times 10^{-20}$	0.000000000127273721027507204994144115625%	$6.217199150309257209170342265625 \times 10^{-20}$	0.000000000127273721027507204994144115625%	
$\approx h_0/36893488147419103232$		$3.10859957515462880366813813125 \times 10^{-20}$	0.0000000006363686051225184099707281125%	$3.10859957515462880366813813125 \times 10^{-20}$	0.0000000006363686051225184099707281125%	
$\approx h_0/73786976294838206464$		$1.554299787577314401834068453125 \times 10^{-20}$	0.000000000318184302561259204994144115625%	$1.554299787577314401834068453125 \times 10^{-20}$	0.000000000318184302561259204994144115625%	
$\approx h_0/147573952589676412928$		$7.771498937886572009170342265625 \times 10^{-21}$	0.0000000001590410512806296099707281125%	$7.771498937886572009170342265625 \times 10^{-21}$	0.0000000001590410512806296099707281125%	
$\approx h_0/295147905179352825856$		$3.88574946894328600458513813125 \times 10^{-21}$	0.000000000795105253161259204994144115625%	$3.88574946894328600458513813125 \times 10^{-21}$	0.000000000795105253161259204994144115625%	
$\approx h_0/590295810358705651712$		$1.9428747344716430022925690625 \times 10^{-21}$	0.0000000003975526265806296099707281125%	$1.9428747344716430022925690625 \times 10^{-21}$	0.0000000003975526265806296099707281125%	
$\approx h_0/1180591620717411303424$		$9.7143736723582150011878453125 \times 10^{-22}$	0.000000000198776313140314804994144115625%	$9.7143736723582150011878453125 \times 10^{-22}$	0.000000000198776313140314804994144115625%	
$\approx h_0/2361183241434822606848$		$4.857186836179$				

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h\ )}$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.5%	$1.4 \times 10^{-1}$	3.1%	
$\approx h_0/16$		$7.8 \times 10^{-2}$	1.8%	$7.6 \times 10^{-2}$	1.5%	
$\approx h_0/32$		$4.2 \times 10^{-2}$	0.95%	$4.1 \times 10^{-2}$	0.92%	
$\approx h_0/64$		$2.1 \times 10^{-2}$	0.48%	$2.0 \times 10^{-2}$	0.47%	
$\approx h_0/128$		$1.05 \times 10^{-2}$	0.24%	$1.04 \times 10^{-2}$	0.23%	
$\approx h_0/256$		$5.25 \times 10^{-3}$	0.12%	$5.24 \times 10^{-3}$	0.11%	
$\approx h_0/512$		$2.625 \times 10^{-3}$	0.06%	$2.62 \times 10^{-3}$	0.05%	
$\approx h_0/1024$		$1.3125 \times 10^{-3}$	0.03%	$1.312 \times 10^{-3}$	0.02%	

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# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$\approx h_0/16$		$7.8 \times 10^{-2}$	1.7%	$7.0 \times 10^{-2}$	1.6%	1.02
$\approx h_0/32$		$4.2 \times 10^{-2}$	0.95%	$4.1 \times 10^{-2}$	0.92%	1.01
$\approx h_0/64$		$2.1 \times 10^{-2}$	0.48%	$2.1 \times 10^{-2}$	0.47%	1.00
$\approx h_0/128$		$1.05 \times 10^{-2}$	0.24%	$1.05 \times 10^{-2}$	0.24%	1.00
$\approx h_0/256$		$5.2 \times 10^{-3}$	0.12%	$5.2 \times 10^{-3}$	0.12%	1.00
$\approx h_0/512$		$2.6 \times 10^{-3}$	0.06%	$2.6 \times 10^{-3}$	0.06%	1.00
$\approx h_0/1024$		$1.3 \times 10^{-3}$	0.03%	$1.3 \times 10^{-3}$	0.03%	1.00
$\approx h_0/2048$		$6.5 \times 10^{-4}$	0.015%	$6.5 \times 10^{-4}$	0.015%	1.00
$\approx h_0/4096$		$3.25 \times 10^{-4}$	0.0075%	$3.25 \times 10^{-4}$	0.0075%	1.00

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# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$\approx h_0/4$	2	$1.4 \times 10^{-2}$	3.4%	$1.4 \times 10^{-2}$	$3.1 \times 10^{-2\%}$	1.03
$\approx h_0/8$	2	$2.6 \times 10^{-3}$	$5.9 \times 10^{-2\%}$	$2.6 \times 10^{-3}$	$5.9 \times 10^{-2\%}$	1.01
$\approx h_0/16$	2	$1.0 \times 10^{-3}$	1.1%	$9.9 \times 10^{-4}$	$2.2 \times 10^{-3\%}$	1.01
$\approx h_0/32$	2	$2.6 \times 10^{-4}$	2.8%	$2.6 \times 10^{-4}$	$5.8 \times 10^{-4\%}$	1.01

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W. Dörfler, J. Schöberl, A posteriori error estimates for reaction-diffusion problems

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$h_0$	3	$1.4 \times 10^{-2}$	$3.2 \times 10^{-1\%}$	$1.4 \times 10^{-2}$	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	1.01
$h_0$	4	$1.0 \times 10^{-4}$	0.0%	$9.9 \times 10^{-5}$	$2.2 \times 10^{-5\%}$	1.00
$\approx h_0/8$	4	$2.6 \times 10^{-7}$	0.0%	$2.6 \times 10^{-7}$	$5.8 \times 10^{-7\%}$	1.00

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# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$h_0$	3	$1.4 \times 10^{-2}$	$3.2 \times 10^{-1\%}$	$1.4 \times 10^{-2}$	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	1.01
$h_0$	4	$1.0 \times 10^{-3}$	$2.3 \times 10^{-2\%}$	$9.9 \times 10^{-4}$	$2.2 \times 10^{-2\%}$	1.02
$\approx h_0/8$	4	$2.6 \times 10^{-7}$	$5.9 \times 10^{-6\%}$	$2.6 \times 10^{-7}$	$5.8 \times 10^{-6\%}$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)  
 V. Dolejš, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$h_0$	3	$1.4 \times 10^{-2}$	$3.2 \times 10^{-1\%}$	$1.4 \times 10^{-2}$	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	1.01
$h_0$	4	$1.0 \times 10^{-3}$	$2.3 \times 10^{-2\%}$	$9.9 \times 10^{-4}$	$2.2 \times 10^{-2\%}$	1.02
$\approx h_0/8$	4	$2.6 \times 10^{-7}$	$5.9 \times 10^{-6\%}$	$2.6 \times 10^{-7}$	$5.8 \times 10^{-6\%}$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)  
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$h_0$	3	$1.4 \times 10^{-2}$	$3.2 \times 10^{-1\%}$	$1.4 \times 10^{-2}$	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	1.01
$h_0$	4	$1.0 \times 10^{-3}$	$2.3 \times 10^{-2\%}$	$9.9 \times 10^{-4}$	$2.2 \times 10^{-2\%}$	1.02
$\approx h_0/8$	4	$2.6 \times 10^{-7}$	$5.9 \times 10^{-6\%}$	$2.6 \times 10^{-7}$	$5.8 \times 10^{-6\%}$	1.01

A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)  
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

# How large is the overall error? (model pb, known sol.)

$h$	$p$	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\ \nabla u_h\ }$	$\ \nabla(u - u_h)\ $	rel. error $\frac{\ \nabla(u - u_h)\ }{\ \nabla u_h\ }$	$J^{\text{eff}} = \frac{\eta(u_h)}{\ \nabla(u - u_h)\ }$
$h_0$	1	1.3	$2.8 \times 10^{1\%}$	1.1	$2.4 \times 10^{1\%}$	1.17
$\approx h_0/2$		$6.1 \times 10^{-1}$	$1.4 \times 10^{1\%}$	$5.6 \times 10^{-1}$	$1.3 \times 10^{1\%}$	1.09
$\approx h_0/4$		$3.1 \times 10^{-1}$	7.0%	$2.9 \times 10^{-1}$	6.6%	1.06
$\approx h_0/8$		$1.5 \times 10^{-1}$	3.3%	$1.4 \times 10^{-1}$	3.1%	1.04
$h_0$	2	$1.6 \times 10^{-1}$	3.7%	$1.5 \times 10^{-1}$	3.5%	1.06
$\approx h_0/2$	2	$4.2 \times 10^{-2}$	$9.5 \times 10^{-1\%}$	$4.1 \times 10^{-2}$	$9.2 \times 10^{-1\%}$	1.04
$h_0$	3	$1.4 \times 10^{-2}$	$3.2 \times 10^{-1\%}$	$1.4 \times 10^{-2}$	$3.1 \times 10^{-1\%}$	1.03
$\approx h_0/4$	3	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	$2.6 \times 10^{-4}$	$5.9 \times 10^{-3\%}$	1.01
$h_0$	4	$1.0 \times 10^{-3}$	$2.3 \times 10^{-2\%}$	$9.9 \times 10^{-4}$	$2.2 \times 10^{-2\%}$	1.02
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)  
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

# Numerics: smooth case with localized features

## Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (-1, 1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

## Exact solution

$$u(x, y) = (x^2 - 1)(y^2 - 1) \exp(-100(x^2 + y^2))$$

## Discretization

- conforming finite elements:  $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- $hp$ -adaptive refinement

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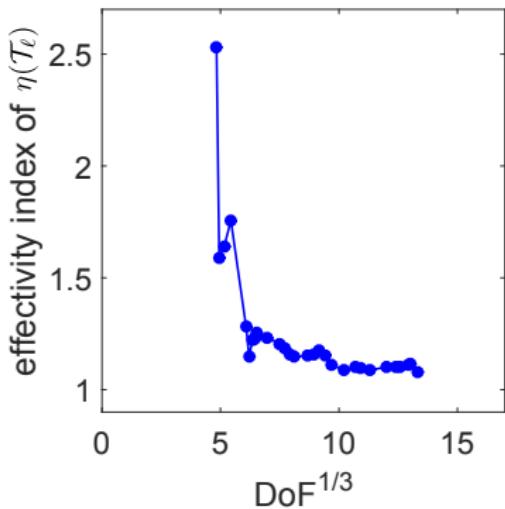
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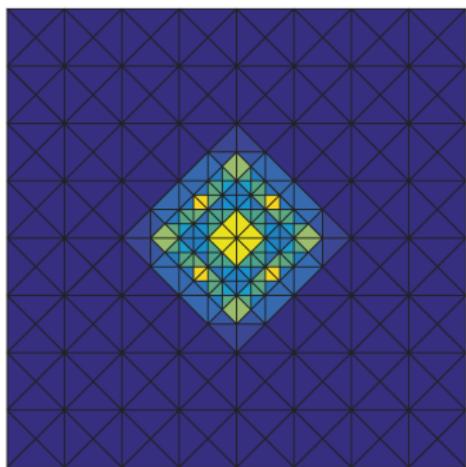
# How precise are the estimates?



Effectivity indices on  $hp$  meshes

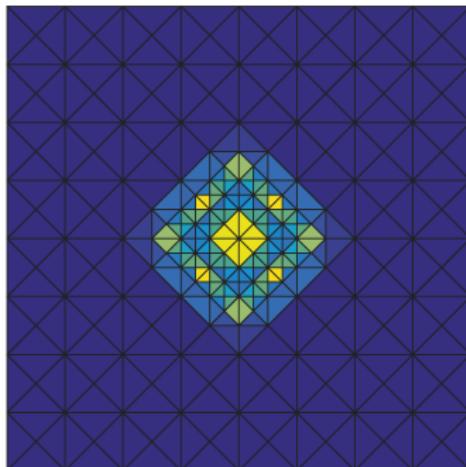
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

# Where (in space) is the error localized?



Estimated error distribution

$$\eta_K(u_h)$$

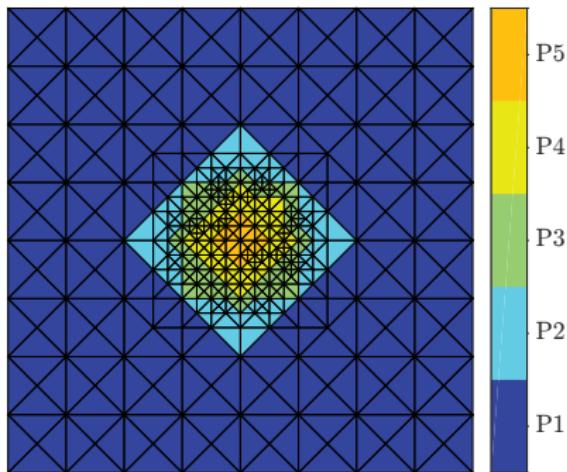


Exact error distribution

$$\|\nabla(u - u_h)\|_K$$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

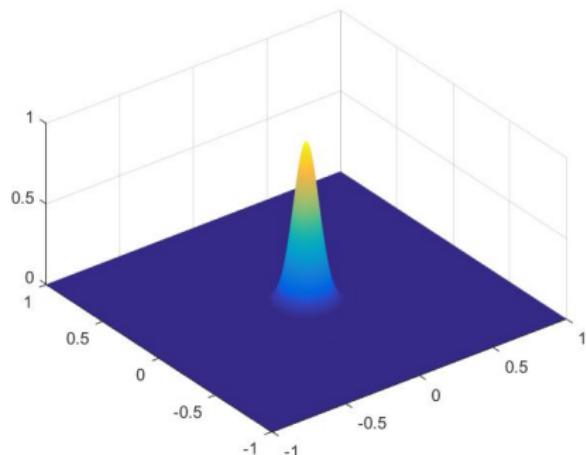
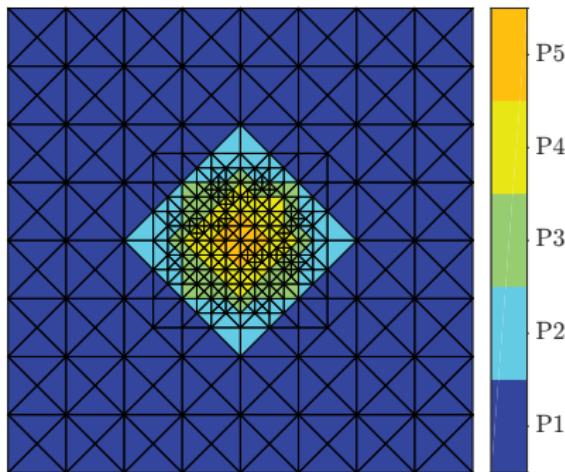
# Can we decrease the error efficiently?



Mesh  $\mathcal{T}$  and pol. degrees  $p_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

# Can we decrease the error efficiently?



P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

# Numerics: singular case

## Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1, 1)^2 \setminus [0, 1]^2, \\ u &= u_D \quad \text{on } \partial\Omega \end{aligned}$$

## Exact solution

$$u(r, \phi) = r^{2/3} \sin(2\phi/3)$$

## Discretization

- conforming finite elements:  $u_h \in H^1(\Omega)$
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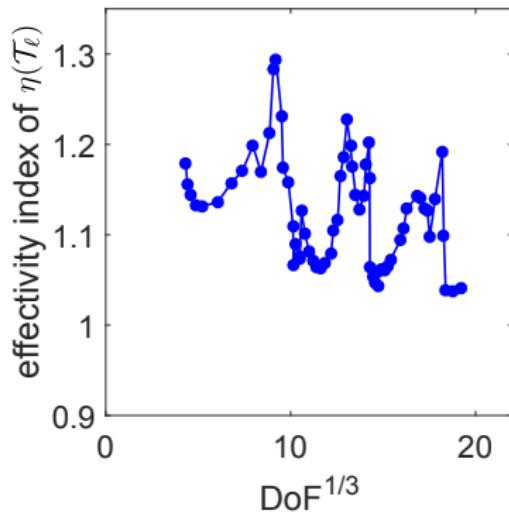
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$$u(r, \phi) = r^{2/3} \sin(2\phi/3)$$

## Discretization

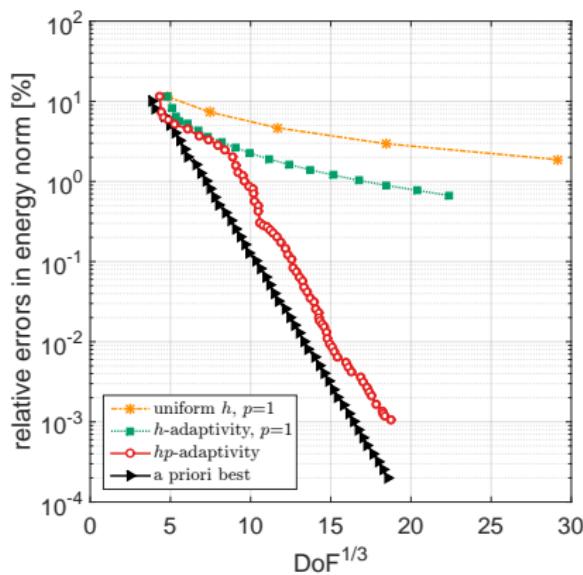
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- unstructured nested triangular grids
- $hp$ -adaptive refinement

# How precise are the estimates?



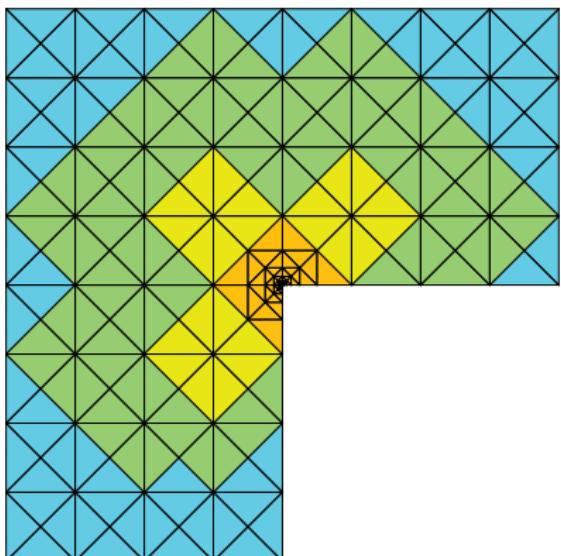
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

# Can we decrease the error efficiently?

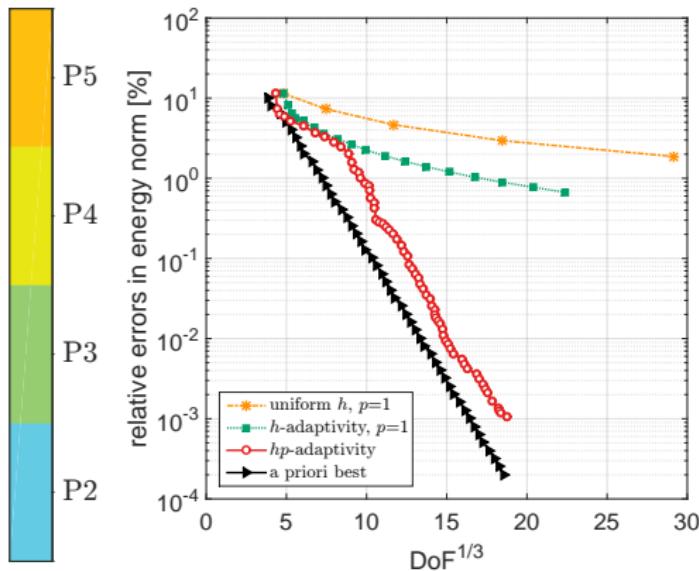


Relative error as a function of  
no. of unknowns

# Can we decrease the error efficiently?



Mesh  $\mathcal{T}$  and polynomial degrees  $p_k$



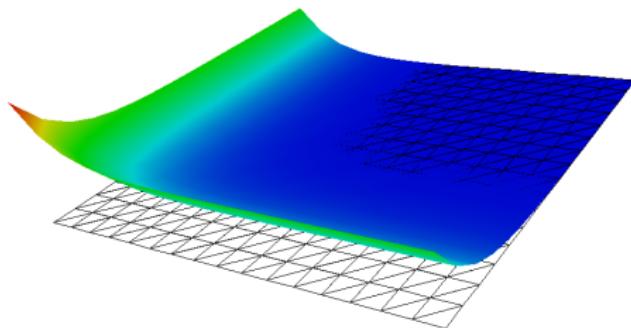
Relative error as a function of no. of unknowns

# Problem and exact solution



## Problem

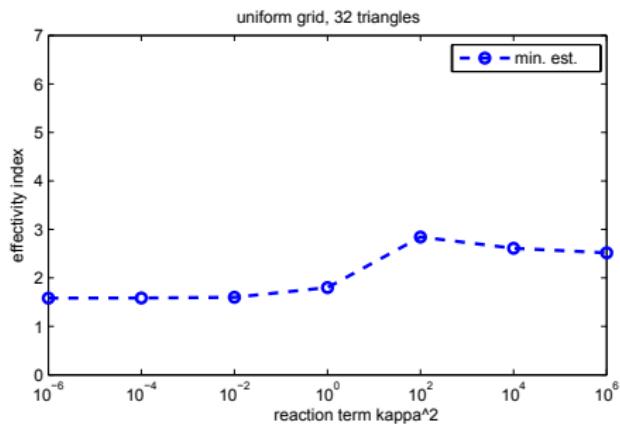
$$\begin{aligned} -\Delta u + \kappa^2 u &= 0 && \text{in } \Omega, \\ u &= u_D && \text{on } \partial\Omega \end{aligned}$$



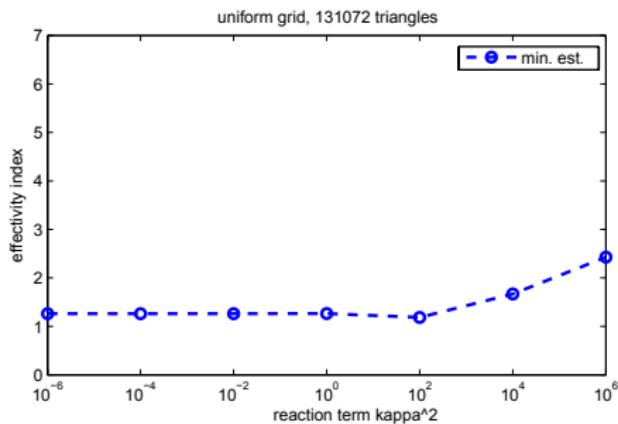
## Solution

$$u(x, y) = e^{-\kappa x} + e^{-\kappa y}$$

# Effectivity indices in dependence on $\kappa$ : robustness



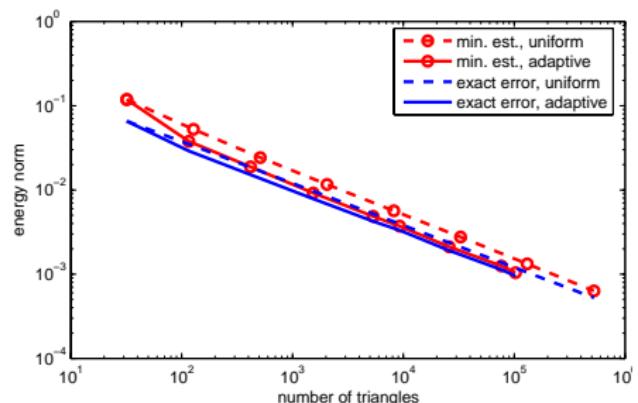
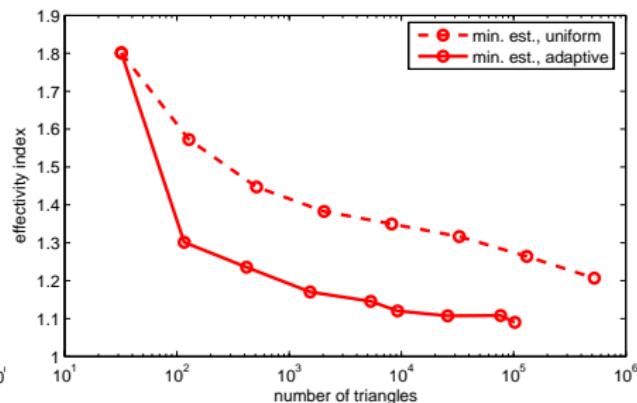
Mesh with 32 triangles



Mesh with 131072 triangles

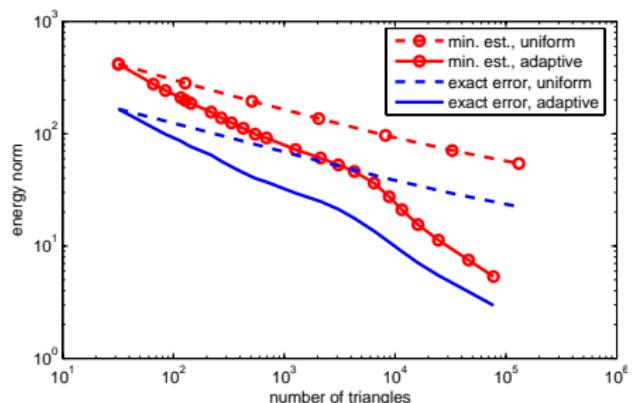
I. Cheddadi, R. Fučík, M. Prieto, M. Vohralík, M2AN Math. Model. Numer. Anal. (2009)

# Estimated and actual errors in uniformly/adaptively refined meshes and effectivity indices

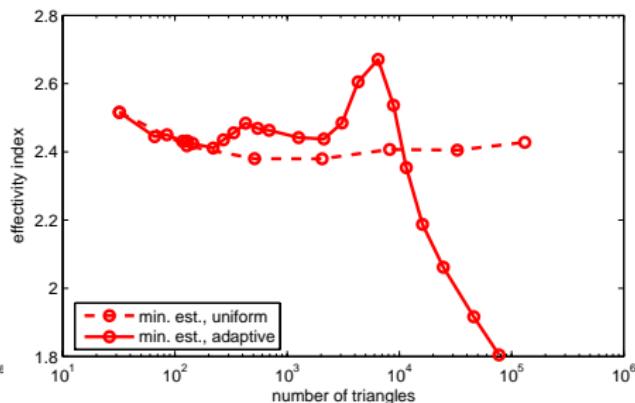
Est. and act. errors,  $\kappa = 1$ Effectivity indices,  $\kappa = 1$ 

I. Cheddadi, R. Fučík, M. Prieto, M. Vohralík, M2AN Math. Model. Numer. Anal. (2009)

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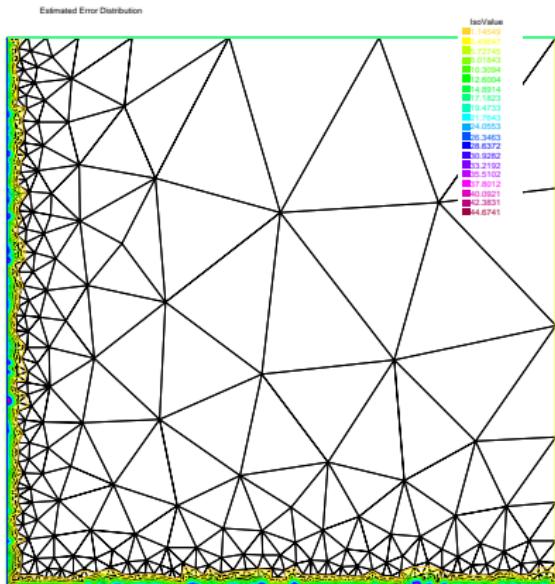
Est. and act. errors,  $\kappa = 10^3$



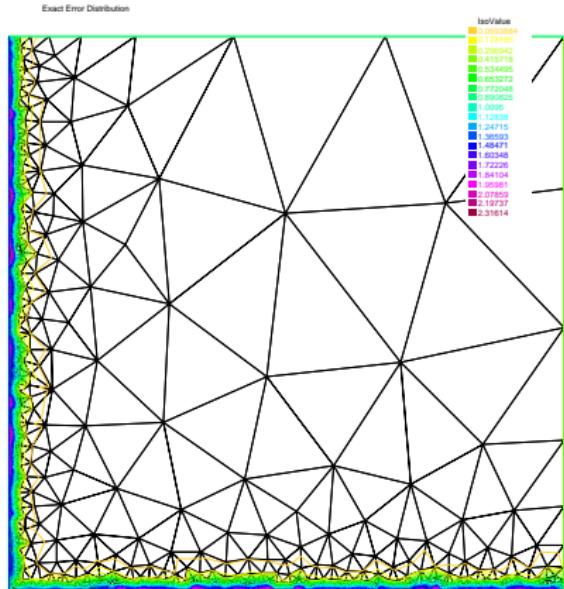
Effectivity indices,  $\kappa = 10^3$

I. Cheddadi, R. Fučík, M. Prieto, M. Vohralík, M2AN Math. Model. Numer. Anal. (2009)

# Error distribution, adaptively refined mesh, $\kappa = 10^3$



Estimated error distribution



Exact error distribution

I. Cheddadi, R. Fučík, M. Prieto, M. Vohralík, M2AN Math. Model. Numer. Anal. (2009)

# Outline

## 1 Introduction

## 2 The reaction–diffusion equation

- Equivalence between error and dual norm of the residual
- Guaranteed upper bound
- Local efficiency and robustness

## 3 The heat equation

- Equivalence between error and dual norm of the residual
- High-order discretization & Radau reconstruction
- Guaranteed upper bound
- Local space-time efficiency and robustness

## 4 Some numerical experiments (steady case)

## 5 Conclusions and future directions

# Conclusions and future directions

## Conclusions (reaction-diffusion)

- ✓ **guaranteed** upper bound
- ✓ local **efficiency** and **robustness** with respect to reaction and diffusion parameters
- ✓ **simple** form, any polynomial degree

## Conclusions (heat)

- ✓ **guaranteed** upper bound
- ✓ local **space-time** efficiency
- ✓ **robustness** with respect to both spatial and temporal polynomial degree
- ✓ arbitrarily large **coarsening** allowed

## Future directions

- nonlinear and coupled problems

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# Bibliography

- SMEARS, I., VOHRALÍK M., Simple and robust equilibrated flux a posteriori estimates for singularly perturbed reaction–diffusion problems, HAL Preprint 01956180, submitted for publication, 2018.
- ERN A., SMEARS, I., VOHRALÍK M., Guaranteed, locally space-time efficient, and polynomial-degree robust a posteriori error estimates for high-order discretizations of parabolic problems, *SIAM J. Numer. Anal.* **55** (2017), 2811–2834.
- ERN A., SMEARS, I., VOHRALÍK M., Discrete  $p$ -robust  $\mathbf{H}(\text{div})$ -liftings and a posteriori estimates for elliptic problems with  $H^{-1}$  source terms, *Calcolo* **54** (2017), 1009–1025.

Thank you for your attention!

# Bibliography

- SMEARS, I., VOHRALÍK M., Simple and robust equilibrated flux a posteriori estimates for singularly perturbed reaction–diffusion problems, HAL Preprint 01956180, submitted for publication, 2018.
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**Thank you for your attention!**

# Fundamental results on a reference tetrahedron

## Bounded right inverse of the divergence operator

- polynomial volume data
- Costabel & McIntosh (2010):

Let  $K \in \mathcal{T}$  and  $r \in \mathcal{P}_p(K)$ . Then there exists  $\xi_h \in \mathbf{RTN}_p(K)$  s.t.  
 $\nabla \cdot \xi_h = r$  and

$$\|\xi_h\|_K \leq C \|r\|_{H^{-1}(K)} = \sup_{v \in H_0^1(K), \|\nabla v\|_K=1} (r, v)_K.$$

## Polynomial extensions in $H(\text{div})$

- polynomial boundary data
- Demkowicz, Gopalakrishnan, Schöberl (2012):

Let  $K \in \mathcal{T}$  and  $r \in \mathcal{P}_p(\mathcal{F}_K)$  satisfying  $(r, 1)_{\partial K} = 0$ . Then there exists  $\xi_h \in \mathbf{RTN}_p(K)$  s.t.  $\xi_h \cdot \mathbf{n}_K = r$  on  $\partial K$ ,  $\nabla \cdot \xi_h = 0$  in  $K$ , and

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# General result on a physical tetrahedron

Lemma ( $\mathbf{H}(\text{div})$  polynomial extension on a tetrahedron)

Let  $K \in \mathcal{T}$ ,  $\mathcal{F}_K^N \subset \mathcal{F}_K$ . Let  $r \in \mathcal{P}_p(\mathcal{F}_K^N) \times \mathcal{P}_p(K)$ , satisfying

$\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$  if  $\mathcal{F}_K^N = \mathcal{F}_K$ . Then for  $C = C(\kappa_K) > 0$ ,

$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K .$$

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## Context

- $-\Delta \zeta_K = r_K \quad \text{in } K,$
- $-\nabla \zeta_K \cdot \mathbf{n}_K = r_F \quad \text{on all } F \in \mathcal{F}_K^N,$
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Set  $\xi_K := -\nabla \zeta_K$ .

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$$\|\boldsymbol{\xi}_{h,K}\|_K \stackrel{MFEs}{=} \min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_F \quad \forall F \in \mathcal{F}_K^N \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K = C \|\boldsymbol{\xi}_K\|_K.$$

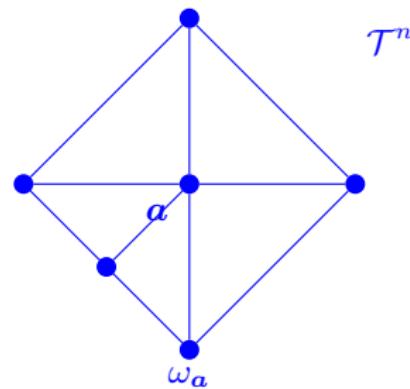
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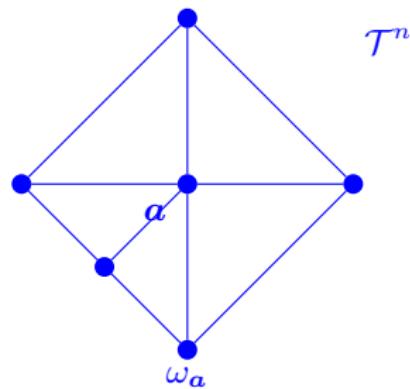
# Stable broken $\mathbf{H}(\text{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
- Ern & V. (2016), 3D
- Ern, Smeers, & V. (2017), 2-3D, patches with subrefinement



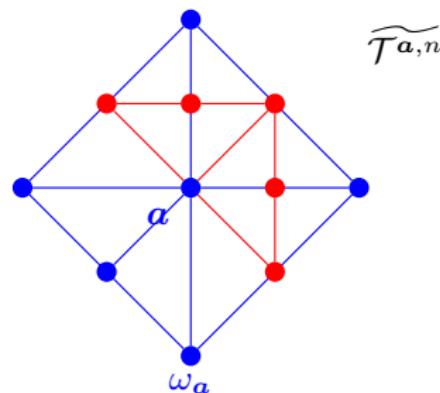
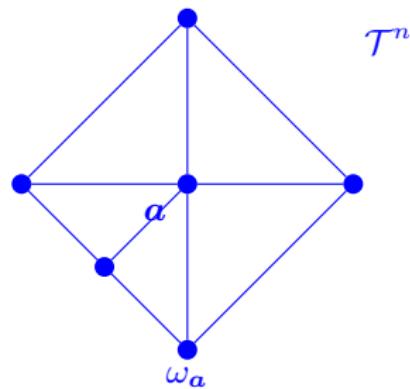
# Stable broken $\mathbf{H}(\text{div})$ polynomial extension on a patch

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# High-order space-time discretization

## CG in space & DG in time

- $p$ -degree **continuous** piecewise polynomials in **space**

$$V_h^n := \{v_h \in H_0^1(\Omega), v_h|_K \in \mathcal{P}_{pk}(K) \quad \forall K \in \mathcal{T}^n\}$$

- $q$ -degree **discontinuous** piecewise polynomials in **time**

$$\mathcal{Q}_{q_n}(I_n; V) := \{V\text{-valued pols of degree at most } q_n \text{ over } I_n\}$$

## High-order discretization

Find  $u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n}(I_n; V_h^n)$  with  $u_{h\tau}(0) = \Pi_h u_0$  such that

$$\begin{aligned} & \int_{I_n} (\partial_t u_{h\tau}, v_{h\tau}) + (\nabla u_{h\tau}, \nabla v_{h\tau}) dt - ((u_{h\tau})_{n-1}, v_{h\tau}(t_{n-1}^+)) \\ &= \int_{I_n} (f, v_{h\tau}) dt \quad \forall v_{h\tau} \in \mathcal{Q}_{q_n}(I_n; V_h^n) \quad \forall 1 \leq n \leq N. \end{aligned}$$

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## Efficiency

For suitable  $\sigma_{h\tau}$ , there holds

$$\int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|_{\Omega}^2 dt \leq C_{\text{eff}}^2 \|u - \mathcal{I}u_{h\tau}\|_{Y(I_n)} \quad \forall 1 \leq n \leq N.$$

- ✗ local-in-time but **global-in-space** only (as in Verfürth & Bergam–Bernardi–Mghazli)

## Reason

- ✗  $\mathcal{I}u_{h\tau}$  misses the Galerkin orthogonality:

$$\int_{I_n} (f, v_{h\tau}) - (\partial_t \mathcal{I}u_{h\tau}, v_{h\tau}) - (\nabla \mathcal{I}u_{h\tau}, \nabla v_{h\tau}) dt$$

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- ✓ the misfit is known:  $u_{h\tau} - \mathcal{I}u_{h\tau}$

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## Augmented norm

- augment the norm:  $\|v\|_{\mathcal{E}_Y}^2 := \|\mathcal{I}v\|_Y^2 + \|v - \mathcal{I}v\|_X^2$ ,  $v \in Y + V_{h\tau}$
- $\mathcal{I}u = u \Rightarrow$

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 = \|u - \mathcal{I}u_{h\tau}\|_Y^2 + \underbrace{\|u_{h\tau} - \mathcal{I}u_{h\tau}\|_X^2}_{\text{known, computable}}$$

- we are adding to  $Y$  norm the time jumps in  $X$  norm (Schötzau–Wihler):

$$\begin{aligned}\|u_{h\tau} - \mathcal{I}u_{h\tau}\|_{X(I_n)}^2 &= \int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 dt \\ &= \frac{\tau_n(q_n+1)}{(2q_n+1)(2q_n+3)} \|\nabla(u_{h\tau})_{n-1}\|^2\end{aligned}$$

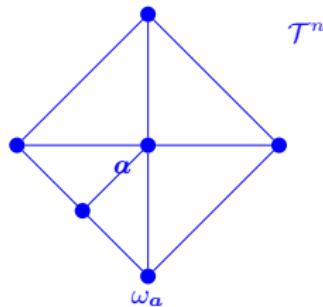
## Theorem (Global equivalence)

Suppose *no source term oscillation* or *no coarsening*. Then there holds

$$\|u - \mathcal{I}u_{h\tau}\|_{\mathcal{Y}} \leq \|u - u_{h\tau}\|_{\mathcal{E}_Y} \leq 3\|u - \mathcal{I}u_{h\tau}\|_{\mathcal{Y}}$$

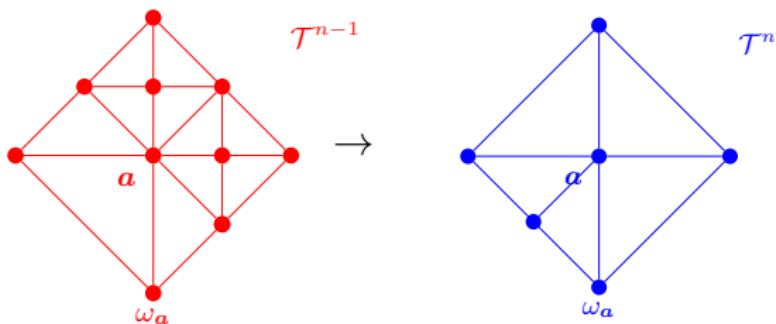
- the two norms  $\|\cdot\|_{\mathcal{Y}}$  and  $\|\cdot\|_{\mathcal{E}_Y}$  still may differ locally
- in general, an additional source term oscillation or coarsening term appears

# Handling mesh adaptivity



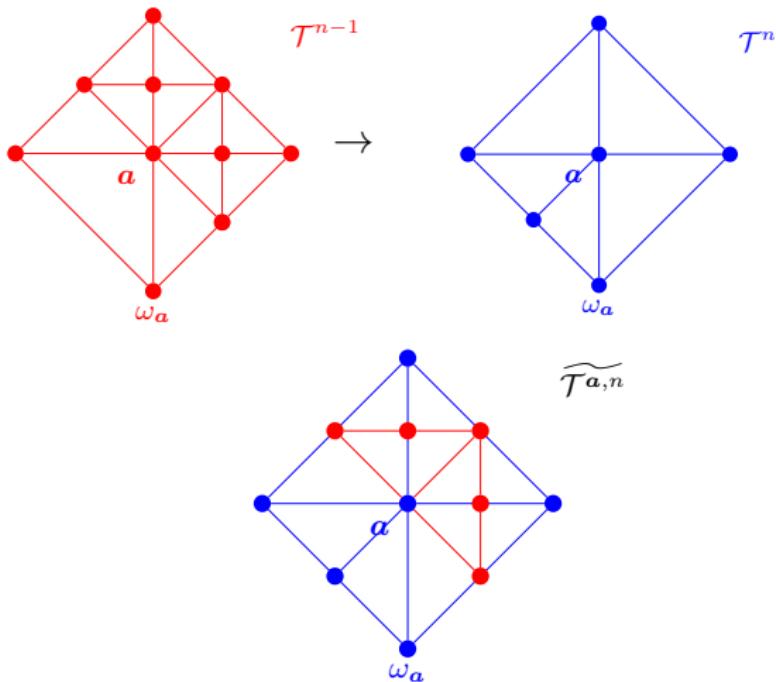
- refinement & coarsening can also involve changing polynomial degrees

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