A POSTERIORI ERROR ESTIMATES, STOPPING CRITERIA, AND INEXPENSIVE IMPLEMENTATIONS

for error control and efficiency in numerical simulations

Martin Vohralík

Habilitation à diriger des recherches

Paris, December 13, 2010

Outline

Outline

Basic CV

Basic CV

- joined Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions as a “maître de conférences” in *September 2006*
- *January 2005–August 2006*: post-doc at CNRS, Université Paris-Sud
- *December 2004*: Ph.D. at Czech Technical University in Prague & Université Paris-Sud

Topics of the habilitation

Topics of the habilitation

Part 1

- a posteriori error estimates
- guaranteed and robust error control
- unified frameworks

Part 2

- stopping criteria
- equilibration of error components
- adaptive algorithms

Part 3

- a priori error estimates
- inexpensive implementations
- development of scientific calculation codes

Topics of the habilitation

Topics of the habilitation

Part 1

- a posteriori error estimates
- guaranteed and robust error control
- unified frameworks

Part 2

- stopping criteria
- equilibration of error components
- adaptive algorithms

Part 3

- a priori error estimates
- inexpensive implementations
- development of scientific calculation codes

Topics of the habilitation

Topics of the habilitation

Part 1

- a posteriori error estimates
- guaranteed and robust error control
- unified frameworks

Part 2

- stopping criteria
- equilibration of error components
- adaptive algorithms

Part 3

- a priori error estimates
- inexpensive implementations
- development of scientific calculation codes

Background

Background

- GNR MoMaS project *Mathematical Modeling and Numerical Simulation for Nuclear Waste Management Problems*
- ERT project with the French Petroleum Institute *Enhanced oil recovery and geological sequestration of CO_2: mesh adaptivity, a posteriori error control, and other advanced techniques*
- HydroExpert society, simulations of flow and contaminant transport in underground porous media (code *TALISMAN*)

Co-supervision of Ph.D. candidates

Nancy Chalhoub

- framework of the GNR MoMaS project
- co-supervision with Alexandre Ern (ENPC) and Toni Sayah (Université Saint-Joseph, Beirut, Lebanon)
- *general framework for a posteriori error estimation in instationary convection–diffusion–reaction problems*

Soleiman Yousef, Carole Widmer

- framework of the ERT project with the French Petroleum Institute
- co-supervision with Daniele Di Pietro (French Petroleum Institute) and Vivette Girault (LJLL)
- *a posteriori estimates, stopping criteria, and adaptive algorithms for the Stefan problem* (S. Yousef) and *a posteriori estimates and adaptivity for cell-centered finite volume discretizations of two-phase flows* (C. Widmer)

Co-supervision of Ph.D. candidates

Nancy Chalhoub

- framework of the GNR MoMaS project
- co-supervision with Alexandre Ern (ENPC) and Toni Sayah (Université Saint-Joseph, Beirut, Lebanon)
- *general framework for a posteriori error estimation in instationary convection–diffusion–reaction problems*

Soleiman Yousef, Carole Widmer

- framework of the ERT project with the French Petroleum Institute
- co-supervision with Daniele Di Pietro (French Petroleum Institute) and Vivette Girault (LJLL)
- *a posteriori estimates, stopping criteria, and adaptive algorithms for the Stefan problem* (S. Yousef) and *a posteriori estimates and adaptivity for cell-centered finite volume discretizations of two-phase flows* (C. Widmer)

Numerical approximation of a nonlinear, instationary PDE

Exact and approximate solution

- let p be the weak solution of $Ap = F$, A **nonlinear**, **instationary**
- let p_h be its approximate numerical solution, $A_h p_h = F_h$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by t^n, $0 \le n \le N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n, solve a nonlinear algebraic problem $A_h^n p_h^n = F_h^n$

Numerical approximation of a nonlinear, instationary PDE

Exact and approximate solution

- let p be the weak solution of $Ap = F$, A **nonlinear**, **instationary**
- let p_h be its approximate numerical solution, $A_h p_h = F_h$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n, solve a nonlinear algebraic problem $A_h^n p_h^n = F_h^n$

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega \times (0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega \times (0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega \times (0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $\mathbb{A}_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $\mathbb{A}_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$**?**

Iterative linearization and iterative algebraic solvers

Iterative linearization of $A_h^n p_h^n = F_h^n$

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$: discrete Newton or fixed-point linearization
- loop in i
- **when do we stop?**

Iterative algebraic system solution on each t^n and for each i

- $A_{\mathrm{L},h}^{n,(i-1)} p_h^{n,(i)} = F_{\mathrm{L},h}^{n,(i-1)}$ is a linear algebraic system
- we only solve it inexactly by, e.g., some iterative method
- **when do we stop?**

Approximate solution

- the approximate solution p_h^{a} that we have as an outcome does not solve $A_h p_h^{\mathrm{a}} = F_h$
- how big is the **overall error** $\|p - p_h^{\mathrm{a}}\|_{\Omega \times (0,T)}$**?**

A posteriori error estimates: 5 optimal properties

Guaranteed upper bound (global error upper bound)

- $\|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2$
- no undetermined constant: error control

Local efficiency (local error lower bound)

- $\eta_K^n(p_h^{\mathrm{a}})^2 \leq C^2_{\mathrm{eff},K,n} \sum_{L \text{ close to } K} \|p - p_h^{\mathrm{a}}\|^2_{L\times(t^{n-1},t^n)}$
- enables to predict the overall error distribution

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2 / \|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Robustness

- $C_{\mathrm{eff},K,n}$ does not depend on coefficients, their relative size and variation, solution regularity, domain Ω, final time T
- estimators equally good in all situations

Negligible evaluation cost

- estimators can be evaluated locally

A posteriori error estimates: 5 optimal properties

Guaranteed upper bound (global error upper bound)

- $\|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^N \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2$
- no undetermined constant: error control

Local efficiency (local error lower bound)

- $\eta_K^n(p_h^{\mathrm{a}})^2 \leq C^2_{\mathrm{eff},K,n} \sum_{L \, \mathrm{close\ to}\, K} \|p - p_h^{\mathrm{a}}\|^2_{L\times(t^{n-1},t^n)}$
- enables to predict the overall error distribution

Asymptotic exactness

- $\sum_{n=1}^N \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2 / \|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Robustness

- $C_{\mathrm{eff},K,n}$ does not depend on coefficients, their relative size and variation, solution regularity, domain Ω, final time T
- estimators equally good in all situations

Negligible evaluation cost

- estimators can be evaluated locally

A posteriori error estimates: 5 optimal properties

Guaranteed upper bound (global error upper bound)

- $\|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^N \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2$
- no undetermined constant: error control

Local efficiency (local error lower bound)

- $\eta_K^n(p_h^{\mathrm{a}})^2 \leq C^2_{\mathrm{eff},K,n} \sum_{L\,\mathrm{close\ to}\,K} \|p - p_h^{\mathrm{a}}\|^2_{L\times(t^{n-1},t^n)}$
- enables to predict the overall error distribution

Asymptotic exactness

- $\sum_{n=1}^N \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2 / \|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Robustness

- $C_{\mathrm{eff},K,n}$ does not depend on coefficients, their relative size and variation, solution regularity, domain Ω, final time T
- estimators equally good in all situations

Negligible evaluation cost

- estimators can be evaluated locally

A posteriori error estimates: 5 optimal properties

Guaranteed upper bound (global error upper bound)

- $\|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2$
- no undetermined constant: error control

Local efficiency (local error lower bound)

- $\eta_K^n(p_h^{\mathrm{a}})^2 \leq C^2_{\mathrm{eff},K,n} \sum_{L \text{ close to } K} \|p - p_h^{\mathrm{a}}\|^2_{L\times(t^{n-1},t^n)}$
- enables to predict the overall error distribution

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2 / \|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Robustness

- $C_{\mathrm{eff},K,n}$ does not depend on coefficients, their relative size and variation, solution regularity, domain Ω, final time T
- estimators equally good in all situations

Negligible evaluation cost

- estimators can be evaluated locally

A posteriori error estimates: 5 optimal properties

Guaranteed upper bound (global error upper bound)

- $\|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2$
- no undetermined constant: error control

Local efficiency (local error lower bound)

- $\eta_K^n(p_h^{\mathrm{a}})^2 \leq C^2_{\mathrm{eff},K,n} \sum_{L\,\mathrm{close\ to}\,K} \|p - p_h^{\mathrm{a}}\|^2_{L\times(t^{n-1},t^n)}$
- enables to predict the overall error distribution

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(p_h^{\mathrm{a}})^2 / \|p - p_h^{\mathrm{a}}\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Robustness

- $C_{\mathrm{eff},K,n}$ does not depend on coefficients, their relative size and variation, solution regularity, domain Ω, final time T
- estimators equally good in all situations

Negligible evaluation cost

- estimators can be evaluated locally

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed** and **robust** upper bound on the overall error $\|p - p_h^{\mathrm{a}}\|_{\Omega\times(0,T)}$, if possible asymptotically exact
- ensure **local efficiency** (optimal mesh refinement)
- develop **unified frameworks**
- distinguish the algebraic/linearization errors, due to inexact solution of linear/nonlinear problems, and the space and time discretization errors, due to mesh size, time step, and numerical scheme
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Previous results

Continuous finite elements

- Babuška and Rheinboldt (1978), introduction
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates (equality of Prager and Synge (1947))
- Zienkiewicz and Zhu (1987), averaging-based estimates
- Verfürth (1996, book), residual-based estimates
- Repin (1997), functional a posteriori error estimates
- Destuynder and Métivet (1999), equilibrated fluxes estimates
- Ainsworth and Oden (2000, book), equilibrated residual estimates
- Luce and Wohlmuth (2004), equilibrated fluxes estimates
- Braess and Schöberl (2008), equilibrated fluxes estimates

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates
- Achdou, Bernardi, Coquel (2003), links FV–FE
- Nicaise (2005, 2006), postprocessing

Discontinuous Galerkin finite elements

- Becker, Hansbo, Larson (2003), residual-based estimates
- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Mixed finite elements

- Braess and Verfürth (1996)
- Carstensen (1997)
- Hoppe and Wohlmuth (1997)
- Lovadina and Stenberg (2006)
- Kim (2007)

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates
- Achdou, Bernardi, Coquel (2003), links FV–FE
- Nicaise (2005, 2006), postprocessing

Discontinuous Galerkin finite elements

- Becker, Hansbo, Larson (2003), residual-based estimates
- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Mixed finite elements

- Braess and Verfürth (1996)
- Carstensen (1997)
- Hoppe and Wohlmuth (1997)
- Lovadina and Stenberg (2006)
- Kim (2007)

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates
- Achdou, Bernardi, Coquel (2003), links FV–FE
- Nicaise (2005, 2006), postprocessing

Discontinuous Galerkin finite elements

- Becker, Hansbo, Larson (2003), residual-based estimates
- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Mixed finite elements

- Braess and Verfürth (1996)
- Carstensen (1997)
- Hoppe and Wohlmuth (1997)
- Lovadina and Stenberg (2006)
- Kim (2007)

Previous results: inhomogeneous diffusion, reaction

Diffusion with discontinuous coefficients

- Dörfler and Wilderotter, conforming finite elements
- Bernardi and Verfürth (2000), conforming finite elements
- Petzoldt (2002), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Reaction-dominated problems

- Verfürth (1998), residual estimates
- Ainsworth and Babuška (1999), equilibrated residual estimates
- Grosman (2006), equilibrated residual estimates, anisotropic meshes

Previous results: inhomogeneous diffusion, reaction

Diffusion with discontinuous coefficients

- Dörfler and Wilderotter, conforming finite elements
- Bernardi and Verfürth (2000), conforming finite elements
- Petzoldt (2002), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Reaction-dominated problems

- Verfürth (1998), residual estimates
- Ainsworth and Babuška (1999), equilibrated residual estimates
- Grosman (2006), equilibrated residual estimates, anisotropic meshes

Previous results: parabolic problems

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), rigorous analysis
- Picasso (1998), evolving meshes
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Finite volumes

- Ohlberger (2001), non energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Previous results: parabolic problems

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), rigorous analysis
- Picasso (1998), evolving meshes
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Finite volumes

- Ohlberger (2001), non energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Previous results: parabolic problems

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), rigorous analysis
- Picasso (1998), evolving meshes
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Finite volumes

- Ohlberger (2001), non energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Previous results: parabolic problems

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), rigorous analysis
- Picasso (1998), evolving meshes
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Finite volumes

- Ohlberger (2001), non energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Previous results: algebraic error

A posteriori estimates accounting for algebraic error

- Repin (1997)

Stopping criteria for iterative solvers

- Becker, Johnson, and Rannacher (1995)
- Maday and Patera (2000)
- Arioli (2004)
- Meidner, Rannacher, Vihnarev (2009)

Algebraic energy error estimation in the conjugate gradient method

- Meurant (1997)
- Strakoš and Tichý (2002)

Previous results: algebraic error

A posteriori estimates accounting for algebraic error

- Repin (1997)

Stopping criteria for iterative solvers

- Becker, Johnson, and Rannacher (1995)
- Maday and Patera (2000)
- Arioli (2004)
- Meidner, Rannacher, Vihnarev (2009)

Algebraic energy error estimation in the conjugate gradient method

- Meurant (1997)
- Strakoš and Tichý (2002)

Previous results: algebraic error

A posteriori estimates accounting for algebraic error

- Repin (1997)

Stopping criteria for iterative solvers

- Becker, Johnson, and Rannacher (1995)
- Maday and Patera (2000)
- Arioli (2004)
- Meidner, Rannacher, Vihnarev (2009)

Algebraic energy error estimation in the conjugate gradient method

- Meurant (1997)
- Strakoš and Tichý (2002)

Previous results: nonlinear problems

Continuous finite elements

- Han (1994), general framework
- Verfürth (1994), residual estimates
- Barrett and Liu (1994), quasi-norm estimates
- Liu and Yan (2001), quasi-norm estimates
- Veeser (2002), convergence p-Laplacian
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors (only fixed-point, one linearized problem (not an iterative loop))
- Diening and Kreuzer (2008), linear cvg p-Laplacian

Other methods

- Liu and Yan (2001), quasi-norm estimates for the nonconforming finite element method
- Kim (2007), guaranteed estimates for locally conservative methods

Previous results: nonlinear problems

Continuous finite elements

- Han (1994), general framework
- Verfürth (1994), residual estimates
- Barrett and Liu (1994), quasi-norm estimates
- Liu and Yan (2001), quasi-norm estimates
- Veeser (2002), convergence p-Laplacian
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors (only fixed-point, one linearized problem (not an iterative loop))
- Diening and Kreuzer (2008), linear cvg p-Laplacian

Other methods

- Liu and Yan (2001), quasi-norm estimates for the nonconforming finite element method
- Kim (2007), guaranteed estimates for locally conservative methods

Previous results: various

Variational inequalities

- Hlaváček, Haslinger, Nečas, and Lovíšek (1982)
- Ainsworth, Oden, and Lee (1993)
- Chen and Nochetto (2000)
- Wohlmuth (2007)

Stokes problem

- Verfürth (1989)
- Dörfler and Ainsworth (2005)

Unified frameworks

- Ainsworth (2005)
- Carstensen (2005–2009)

Previous results: various

Variational inequalities

- Hlaváček, Haslinger, Nečas, and Lovíšek (1982)
- Ainsworth, Oden, and Lee (1993)
- Chen and Nochetto (2000)
- Wohlmuth (2007)

Stokes problem

- Verfürth (1989)
- Dörfler and Ainsworth (2005)

Unified frameworks

- Ainsworth (2005)
- Carstensen (2005–2009)

Previous results: various

Variational inequalities

- Hlaváček, Haslinger, Nečas, and Lovíšek (1982)
- Ainsworth, Oden, and Lee (1993)
- Chen and Nochetto (2000)
- Wohlmuth (2007)

Stokes problem

- Verfürth (1989)
- Dörfler and Ainsworth (2005)

Unified frameworks

- Ainsworth (2005)
- Carstensen (2005–2009)

Previous results: error components equilibration

Error components equilibration

- engineering literature, since 1950's
- Ladevèze (since 1980's)
- Verfürth (2003), space and time error equilibration
- Braack and Ern (2003), estimation of model errors
- Babuška, Oden (2004), verification and validation
- ...

Previous results: implementations, relations

Implementations, mixed methods

- Arnold and Brezzi (1985)
- Arbogast and Chen (1995)
- Cockburn and Gopalakrishnan (2004, 2005)

Relations between different methods

- Russell and Wheeler (1983)
- Younès, Ackerer, Chavent (1999–2004)

Previous results: implementations, relations

Implementations, mixed methods

- Arnold and Brezzi (1985)
- Arbogast and Chen (1995)
- Cockburn and Gopalakrishnan (2004, 2005)

Relations between different methods

- Russell and Wheeler (1983)
- Younès, Ackerer, Chavent (1999–2004)

Papers of this habilitation (a posteriori estimates)

Co-authors	No.	Year	Journal	Problem	Num. meth.	Main results
—	[A13]	2010	JSC	D	FE, VCFV, CCFV, FD	guar. & rob. estimates w.r.t. the jumps in dif. coef.
Cheddadi, Fučík, Prieto	[A4]	2009	M2AN	RD	FE, VCFV	guar. & rob. estimates w.r.t. the reaction
Ern, Stephansen	[A6]	2010	JCAM	CRD	DG	guar. & rob. estimates w.r.t. the convection and reaction
—	[A11]	2007	SINUM	CRD	MFE	guar. estimates
—	[A12]	2008	NM	CRD	FV	guar. estimates
Ern	[A7]	2010	SINUM	heat	DG, MFE, VCFV, CCFV FCFV, FE, NCFE	unified framework
Hilhorst	[A9]	2010	CMAME	CRD instat.	VCFV	guar. (& rob.) estimates
Hannukainen, Stenberg	[B2]	2010	submt.	Stokes	DG, MFE, FV FE, FES, NCFE	unified framework
Pencheva, Wheeler, Wildey	[B3]	2010	submt.	D	DG, MFE, FV	extension to multiscale, multinumerics, and mortars
Ben Belgacem, Bernardi, Blouza	[B1]	2010	submt.	syst. var. ineq.	FE	optimal a posteriori estimates

Papers of this habilitation (stopping criteria)

Co-authors	No.	Year	Journal	Problem	Num. meth.	Main results
Jiránek, Strakoš	[A10]	2010	SISC	D	CCFV, MFE	algebraic error, stopping criteria
El Alaoui, Ern	[A5]	2010	CMAME	monot. nonlin.	FE	guar. & rob. estimates w.r.t. the nonlinearity; linealization error, stoping criteria
—	[B4]	2010	prep.	two-phase	DG, MFE, VCFV CCFV, FCFV	a posteriori estimates, stoping criteria

Papers of this habilitation (implementations, relations between methods, postprocessing, a priori estimates)

Co-authors	No.	Year	Journal	Problem	Num. meth.	Main results
Eymard, Hilhorst	[A8]	2010	NMPDE	CRD par. deg.	VCFV	convergence proof
Ben Belgacem, Bernardi, Blouza	[A1]	2009	M2AN	syst. var. ineq.	FE	well-posedness, a priori estimates, residual a posteriori estimates
Ben Belgacem, Bernardi, Blouza	[A2]	2009	MMNP	syst. var. ineq.	FE	inexpensive implementation
—	[A14]	2010	MC	D	MFE	unified a priori and a posteriori analysis
Wohlmuth	[B5]	2010	submt.	D	MFE	inexpensive implementation, relation to FV methods

Outline

Outline

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $\mathbf{S}$ is a piecewise constant scalar, inhomogeneous

Energy norm

$$
|||\varphi|||^2 := \|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)
$$

VOHRALÍK
Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients
J. Sci. Comput. 2010 [A13]

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $\mathbf{S}$ is a piecewise constant scalar, inhomogeneous

Energy norm

$$
|||\varphi|||^2 := \|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)
$$

VOHRALÍK
Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients
J. Sci. Comput. 2010 [A13]

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $\mathbf{S}$ is a piecewise constant scalar, inhomogeneous

Energy norm

$$|||\varphi|||^2 := \|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)$$

VOHRALÍK
Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients
J. Sci. Comput. 2010 [A13]

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

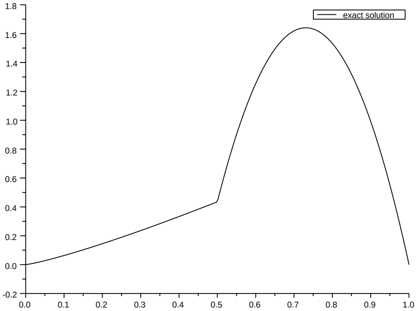
- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $\mathbf{S}$ is a piecewise constant scalar, inhomogeneous

Energy norm

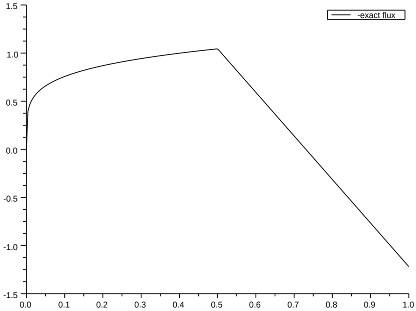
$$|||\varphi|||^2 := \|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)$$

VOHRALÍK
Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients
J. Sci. Comput. 2010 [A13]

Properties of the weak solution

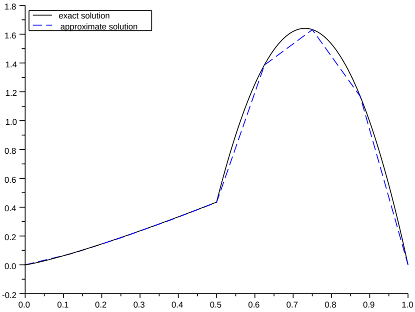


Solution p is in $H_0^1(\Omega)$

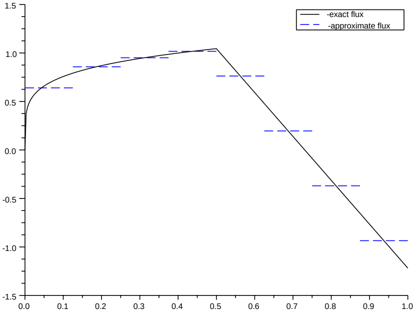


Flux $-\mathbf{S}\nabla p$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate solution and approximate flux

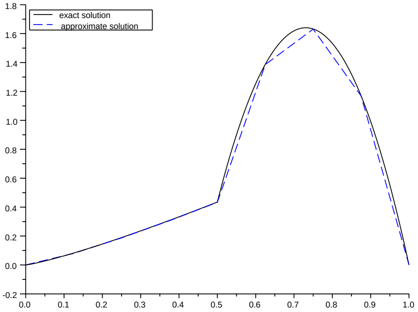


Approximate solution p_h is in $H_0^1(\Omega)$

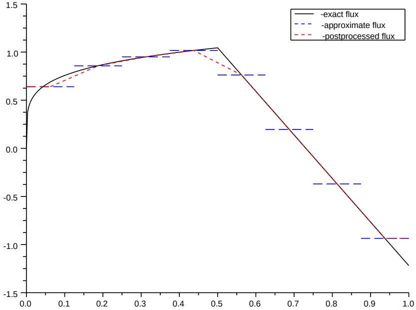


Approximate flux $-\mathbf{S}\nabla p_h$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate solution and postprocessed flux



Approximate solution p_h is in $H_0^1(\Omega)$



Construct a postprocessed flux $\mathbf{t}_h$ in $\mathbf{H}(\mathrm{div}, \Omega)$ (Prager–Synge (1947))

A posteriori error estimate for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Theorem (A posteriori error estimate)

Let

- *p be the weak solution,*
- *$p_h \in H_0^1(\Omega)$ be arbitrary,*
- *$\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ be a partition of Ω,*
- *$\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ be arbitrary but such that*

$$(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D \qquad \text{for all } D \in \mathcal{D}_h^{\mathrm{int}}.$$

Then

$$|||p - p_h||| \le \left\{ \sum_{D\in\mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2}.$$

A posteriori error estimate for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,\mathbf{S}}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{P},D} h_D{}^2 / c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{F},D} h_D{}^2 / c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{\mathbf{S},D}$ is the smallest value of $\mathbf{S}$ on D
  - $C_{\mathrm{P},D}$ and $C_{\mathrm{F},D}$ can be replaced by $1/\pi^2$ for appropriate construction of $\mathbf{t}_h$ (always done in practice)

A posteriori error estimate for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,\mathbf{S}}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{P},D}{h_D}^2/c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{F},D}{h_D}^2/c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{\mathbf{S},D}$ is the smallest value of $\mathbf{S}$ on D
  - $C_{\mathrm{P},D}$ and $C_{\mathrm{F},D}$ can be replaced by $1/\pi^2$ for appropriate construction of $\mathbf{t}_h$ (always done in practice)

A posteriori error estimate for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,\mathbf{S}}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{P},D}{h_D}^2/c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{F},D}{h_D}^2/c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{\mathbf{S},D}$ is the smallest value of $\mathbf{S}$ on D
  - $C_{\mathrm{P},D}$ and $C_{\mathrm{F},D}$ can be replaced by $1/\pi^2$ for appropriate construction of $\mathbf{t}_h$ (always done in practice)

A posteriori error estimate for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,\mathbf{S}}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{P},D} {h_D}^2 / c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,\mathbf{S}}^2 := C_{\mathrm{F},D} {h_D}^2 / c_{\mathbf{S},D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{\mathbf{S},D}$ is the smallest value of $\mathbf{S}$ on D
  - $C_{\mathrm{P},D}$ and $C_{\mathrm{F},D}$ can be replaced by $1/\pi^2$ for appropriate construction of $\mathbf{t}_h$ (always done in practice)

Main steps of the proof

Main steps of the proof, cf. Prager–Synge equality (1947).

- energy norm characterization:

$$|||p - p_h||| = \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} (\mathbf{S}\nabla(p - p_h), \nabla\varphi)$$

- adding and subtracting $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, Green theorem:

$$|||p - p_h||| = \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{|(f - \nabla\cdot\mathbf{t}_h, \varphi)| + |(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)|\}$$

- Cauchy–Schwarz inequality:

$$|(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)| \le \left\|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\right\| = \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{DF},D}^2\right\}^{1/2}$$

- local conservation property $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D \ \forall D \in \mathcal{D}_h^{\mathrm{int}}$, Poincaré and Friedrichs inequalities:

$$|(f - \nabla\cdot\mathbf{t}_h, \varphi)| \le \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{R},D}^2\right\}^{1/2}$$

Main steps of the proof

Main steps of the proof, cf. Prager–Synge equality (1947).

- energy norm characterization:

$$|||p - p_h||| = \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} (\mathbf{S}\nabla(p - p_h), \nabla\varphi)$$

- adding and subtracting $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, Green theorem:

$$|||p-p_h||| = \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{|(f-\nabla\cdot\mathbf{t}_h, \varphi)| + |(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)|\}$$

- Cauchy–Schwarz inequality:

$$|(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)| \le \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\| = \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{DF},D}^2\right\}^{1/2}$$

- local conservation property $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D \ \forall D \in \mathcal{D}_h^{\mathrm{int}}$, Poincaré and Friedrichs inequalities:

$$|(f - \nabla\cdot\mathbf{t}_h, \varphi)| \le \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{R},D}^2\right\}^{1/2}$$

Main steps of the proof

Main steps of the proof, cf. Prager–Synge equality (1947).

- energy norm characterization:

$$|||p - p_h||| = \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} (\mathbf{S}\nabla(p - p_h), \nabla\varphi)$$

- adding and subtracting $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, Green theorem:

$$|||p-p_h||| = \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{|(f-\nabla\cdot\mathbf{t}_h,\varphi)|+|(\mathbf{S}\nabla p_h+\mathbf{t}_h,\nabla\varphi)|\}$$

- Cauchy–Schwarz inequality:

$$|(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)| \leq \left\|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\right\| = \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{DF},D}^2\right\}^{1/2}$$

- local conservation property $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D \; \forall D \in \mathcal{D}_h^{\mathrm{int}}$, Poincaré and Friedrichs inequalities:

$$|(f - \nabla\cdot\mathbf{t}_h, \varphi)| \leq \left\{\sum_{D \in \mathcal{D}_h} \eta_{\mathrm{R},D}^2\right\}^{1/2}$$

Main steps of the proof

Main steps of the proof, cf. Prager–Synge equality (1947).

- energy norm characterization:

$$|||p - p_h||| = \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} (\mathbf{S}\nabla(p - p_h), \nabla\varphi)$$

- adding and subtracting $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, Green theorem:

$$|||p-p_h||| = \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{|(f-\nabla\cdot\mathbf{t}_h,\varphi)|+|(\mathbf{S}\nabla p_h+\mathbf{t}_h,\nabla\varphi)|\}$$

- Cauchy–Schwarz inequality:

$$|(\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)| \le \left\|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\right\| = \left\{\sum_{D\in\mathcal{D}_h} \eta_{\mathrm{DF},D}^2\right\}^{1/2}$$

- local conservation property $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D \ \forall D \in \mathcal{D}_h^{\mathrm{int}}$, Poincaré and Friedrichs inequalities:

$$|(f - \nabla\cdot\mathbf{t}_h, \varphi)| \le \left\{\sum_{D\in\mathcal{D}_h} \eta_{\mathrm{R},D}^2\right\}^{1/2}$$

Construction of $\mathbf{t}_h$

Practical construction of $\mathbf{t}_h$

- relies on the local conservation property of the given numerical method
- uses Raviart–Thomas–Nédélec spaces, two types
  - direct prescription of the degrees of freedom by averaging the normal fluxes $\nabla p_h \cdot \mathbf{n}$
  - solution of local Neumann problems by mixed finite elements (Bank and Weiser (1985), Ern and Vohralík (2009))

Local efficiency of the estimates for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Theorem (Local efficiency)

Suppose that

- $\mathbf{t}_h$ *is constructed from* p_h *by direct prescription or solution of local Neumann MFE problems*
- *the discontinuities are aligned with the dual mesh* $\mathcal{D}_h$*;*
- *harmonic averaging was used in the scheme;*
- *harmonic averaging was used in the construction of* $\mathbf{t}_h$*.*

Then

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C|||p - p_h|||_{\mathcal{T}_{V_D}},$$

where C *depends only on the space dimension* d*, on the shape regularity parameter* $\kappa_{\mathcal{T}}$*, and on the polynomial degree* m *of* f*.*

Local efficiency of the estimates for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Theorem (Local efficiency)

Suppose that

- $\mathbf{t}_h$ *is constructed from* p_h *by direct prescription or solution of local Neumann MFE problems*
- *the discontinuities are aligned with the dual mesh* $\mathcal{D}_h$*;*
- *harmonic averaging was used in the scheme;*
- *harmonic averaging was used in the construction of* $\mathbf{t}_h$*.*

Then

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C |||p - p_h|||_{\mathcal{T}_{V_D}},$$

where C depends only on the space dimension d, on the shape regularity parameter $\kappa_{\mathcal{T}}$*, and on the polynomial degree m of f.*

Local efficiency of the estimates for $-\nabla\cdot(\mathbf{S}\nabla p) = f$

Theorem (Local efficiency)

Suppose that

- $\mathbf{t}_h$ *is constructed from* p_h *by direct prescription or solution of local Neumann MFE problems*
- *the discontinuities are aligned with the dual mesh* $\mathcal{D}_h$*;*
- *harmonic averaging was used in the scheme;*
- *harmonic averaging was used in the construction of* $\mathbf{t}_h$*.*

Then

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C|||p - p_h|||_{\mathcal{T}_{V_D}},$$

where C depends only on the space dimension d, on the shape regularity parameter $\kappa_{\mathcal{T}}$*, and on the polynomial degree m of f.*

Main steps of the proof

Main steps of the proof.

- $$\eta_{\mathrm{DF},D} \le C \left\{ \sum_{\sigma \in \mathcal{E}_D} \| [\![\mathbf{S} \nabla p_h \cdot \mathbf{n}]\!] \|_\sigma^2 \right\}^{\frac{1}{2}}$$

- $$\eta_{\mathrm{R},D} \le C(|||p - p_h|||_D + \eta_{\mathrm{DF},D})$$

Philosophy

- our estimates are, up to a generic constant, a lower bound for the residual ones
- we then can use the results for the residual estimates (R. Verfürth)

Main steps of the proof

Main steps of the proof.

- $$\eta_{\mathrm{DF},D} \leq C \left\{ \sum_{\sigma \in \mathcal{E}_D} \| [\![\mathbf{S} \nabla p_h \cdot \mathbf{n}]\!] \|_\sigma^2 \right\}^{\frac{1}{2}}$$
- $$\eta_{\mathrm{R},D} \leq C(|||p - p_h|||_D + \eta_{\mathrm{DF},D})$$

Philosophy

- our estimates are, up to a generic constant, a lower bound for the residual ones
- we then can use the results for the residual estimates (R. Verfürth)

Main steps of the proof

Main steps of the proof.

- $$\eta_{\mathrm{DF},D} \leq C \left\{ \sum_{\sigma \in \mathcal{E}_D} \| [\![\mathbf{S} \nabla p_h \cdot \mathbf{n}]\!] \|_\sigma^2 \right\}^{\frac{1}{2}}$$
- $$\eta_{\mathrm{R},D} \leq C(|||p - p_h|||_D + \eta_{\mathrm{DF},D})$$

Philosophy

- our estimates are, up to a generic constant, a lower bound for the residual ones
- we then can use the results for the residual estimates (R. Verfürth)

Estimates for $-\nabla\cdot(\mathbf{S}\nabla p) = f$, FE, VCFV, CCFV, FD

Properties

- **guaranteed upper bound**
- local efficiency
- **full robustness**, singular cases included (no monotonicity-like assumption as in Dörfler and Wilderotter (2000), Bernardi and Verfürth (2000), Petzoldt (2002), Ainsworth (2005), Chen and Dai (2002), or Cai and Zhang (2009))
- almost asymptotically exact
- negligible evaluation cost

Vertex-centered finite volumes in 1D

Model problem

$$\begin{aligned} -p'' &= \pi^2 sin(\pi x) \quad \text{in }]0,1[, \\ p &= 0 \quad \text{in } 0,1 \end{aligned}$$

Exact solution

$$p(x) = sin(\pi x)$$

Vertex-centered finite volumes in 1D

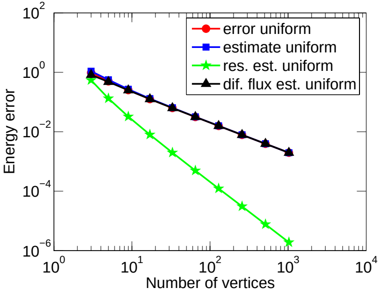
Model problem

$$\begin{aligned} -p'' &= \pi^2 sin(\pi x) \quad \text{in }]0,1[, \\ p &= 0 \quad \text{in } 0,1 \end{aligned}$$

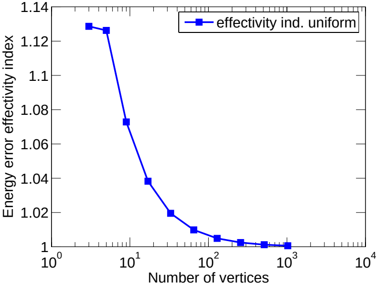
Exact solution

$$p(x) = sin(\pi x)$$

Estimated and actual errors



Actual error and estimator and its components



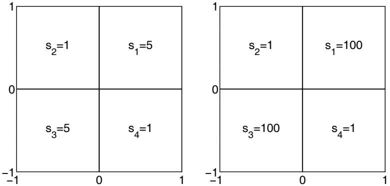
Effectivity index

Discontinuous diffusion tensor and vertex-centered finite volumes

- consider the pure diffusion equation

$$-\nabla\cdot(\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

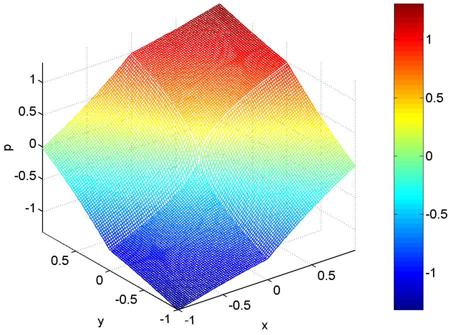


- analytical solution: singularity at the origin

$$p(r,\theta)|_{\Omega_i} = r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

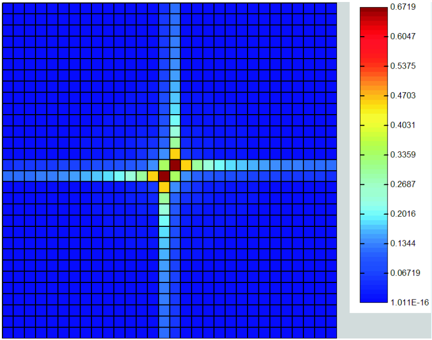
  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

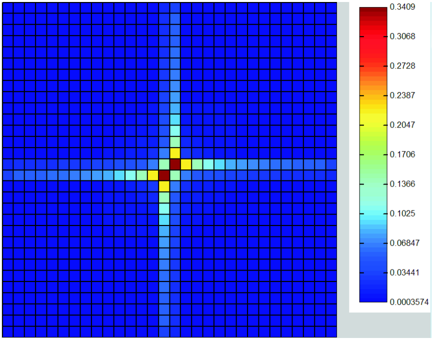


Case 1 Case 2

Error distribution on a uniformly ref. mesh, case 1

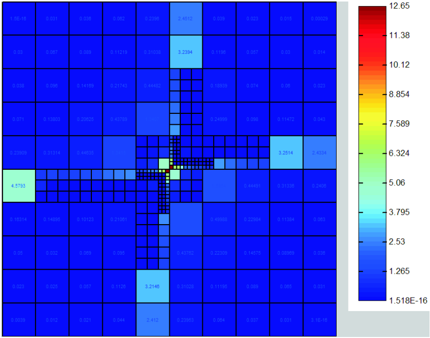


Estimated error distribution

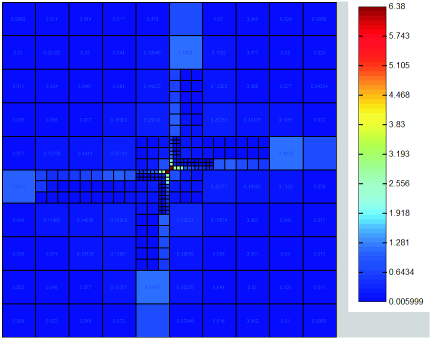


Exact error distribution

Error distribution on an adaptively ref. mesh, case 2

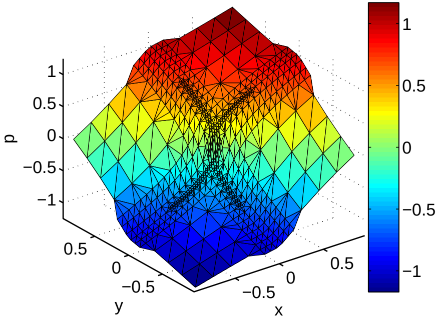


Estimated error distribution

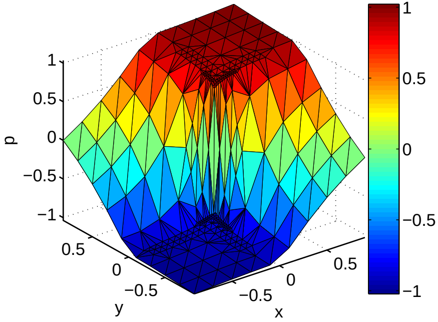


Exact error distribution

Approximate solutions on adaptively refined meshes

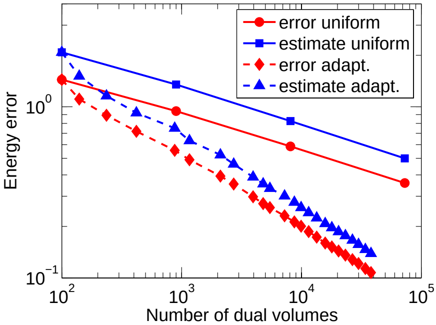


Case 1

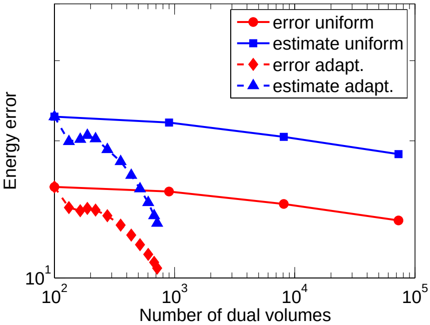


Case 2

Estimated and actual errors in uniformly/adaptively refined meshes

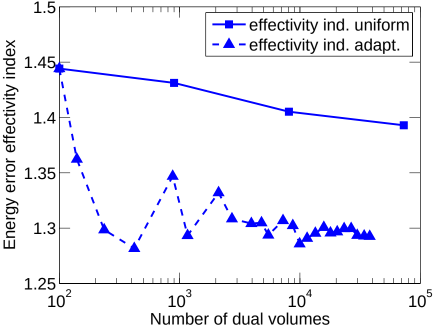


Case 1

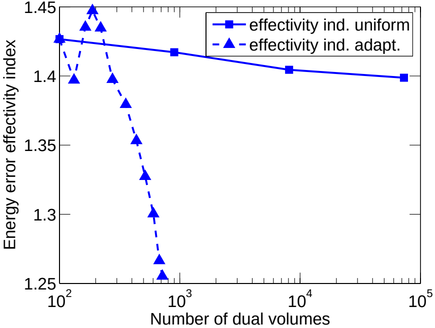


Case 2

Effectivity indices in uniformly/adaptively refined meshes



Case 1



Case 2

Outline

A reaction–diffusion problem

Problem

$$\begin{aligned} -\triangle p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $r \in L^\infty(\Omega)$ such that for each $D \in \mathcal{D}_h$, $0 \le c_{r,D} \le r \le C_{r,D}$, a.e. in D

Energy norm

$$|||\varphi|||_\Omega^2 := \|\nabla\varphi\|^2 + \|r^{1/2}\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)$$

Cheddadi, Fučík, Prieto, Vohralík
Guaranteed and robust a posteriori error estimates for singularly perturbed reaction–diffusion problems
M2AN Math. Model. Numer. Anal. 2009 [A4]

A reaction–diffusion problem

Problem

$$\begin{aligned} -\triangle p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $r \in L^\infty(\Omega)$ such that for each $D \in \mathcal{D}_h$, $0 \leq c_{r,D} \leq r \leq C_{r,D}$, a.e. in D

Energy norm

$$|||\varphi|||_\Omega^2 := \|\nabla\varphi\|^2 + \|r^{1/2}\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)$$

Cheddadi, Fučík, Prieto, Vohralík
Guaranteed and robust a posteriori error estimates for singularly perturbed reaction–diffusion problems
M2AN Math. Model. Numer. Anal. 2009 [A4]

A reaction–diffusion problem

Problem

$$
\begin{aligned}
-\triangle p + rp &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $r \in L^\infty(\Omega)$ such that for each $D \in \mathcal{D}_h$, $0 \leq c_{r,D} \leq r \leq C_{r,D}$, a.e. in D

Energy norm

$$
|||\varphi|||_\Omega^2 := \|\nabla\varphi\|^2 + \|r^{1/2}\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)
$$

CHEDDADI, FUČÍK, PRIETO, VOHRALÍK
Guaranteed and robust a posteriori error estimates for singularly perturbed reaction–diffusion problems
M2AN Math. Model. Numer. Anal. 2009 [A4]

A reaction–diffusion problem

Problem

$$\begin{aligned} -\triangle p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- $r \in L^\infty(\Omega)$ such that for each $D \in \mathcal{D}_h$, $0 \leq c_{r,D} \leq r \leq C_{r,D}$, a.e. in D

Energy norm

$$|||\varphi|||_\Omega^2 := \|\nabla \varphi\|^2 + \|r^{1/2}\varphi\|^2, \qquad \varphi \in H_0^1(\Omega)$$

Cheddadi, Fučík, Prieto, Vohralík
Guaranteed and robust a posteriori error estimates for singularly perturbed reaction–diffusion problems
M2AN Math. Model. Numer. Anal. 2009 [A4]

A posteriori error estimate for $-\triangle p + rp = f$

Theorem (A posteriori error estimate)

Let

- p *be the weak solution,*
- $p_h \in H_0^1(\Omega)$ *be arbitrary,*
- $\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ *be a partition of* Ω*,*
- $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be arbitrary but such that*

$$(\nabla \cdot \mathbf{t}_h + rp_h, 1)_D = (f, 1)_D \qquad \text{for all } D \in \mathcal{D}_h^{\mathrm{int}}.$$

Then

$$|||p - p_h||| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2}.$$

Residual and diffusive flux estimators

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D} := m_D \|f - \nabla\cdot\mathbf{t}_h - rp_h\|_D$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D} := \min\left\{\eta_{\mathrm{DF},D}^{(1)}, \eta_{\mathrm{DF},D}^{(2)}\right\}$$

$$\eta_{\mathrm{DF},D}^{(1)} := \|\nabla p_h + \mathbf{t}_h\|_D$$

$$\eta_{\mathrm{DF},D}^{(2)} := \left\{\sum_{K\in\mathcal{S}_D}\left(m_K\|\triangle p_h + \nabla\cdot\mathbf{t}_h - (\triangle p_h + \nabla\cdot\mathbf{t}_h)_K\|_K + \widetilde{m}_K^{\frac{1}{2}} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{G}_h^{\mathrm{int}}} C_{\mathrm{t},K,\sigma}^{\frac{1}{2}} \|(\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}\|_\sigma\right)^2\right\}^{\frac{1}{2}}$$

- m_D, m_K, $\widetilde{m}_K$: cutoff factors as in Verfürth (1998)

Residual and diffusive flux estimators

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D} := m_D \|f - \nabla\cdot\mathbf{t}_h - rp_h\|_D$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D} := \min\left\{\eta^{(1)}_{\mathrm{DF},D}, \eta^{(2)}_{\mathrm{DF},D}\right\}$$

$$\eta^{(1)}_{\mathrm{DF},D} := \|\nabla p_h + \mathbf{t}_h\|_D$$

$$\eta^{(2)}_{\mathrm{DF},D} := \left\{\sum_{K\in\mathcal{S}_D}\left(m_K\|\triangle p_h + \nabla\cdot\mathbf{t}_h - (\triangle p_h + \nabla\cdot\mathbf{t}_h)_K\|_K + \widetilde{m}_K^{\frac{1}{2}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{G}_h^{\mathrm{int}}} C^{\frac{1}{2}}_{\mathrm{t},K,\sigma}\|(\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}\|_\sigma\right)^2\right\}^{\frac{1}{2}}$$

- m_D, m_K, $\widetilde{m}_K$: cutoff factors as in Verfürth (1998)

Local efficiency of the estimates for $-\triangle p + rp = f$

Theorem (Local efficiency)

Suppose that $\mathbf{t}_h$ was constructed from p_h by direct prescription. Then there holds

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C|||p - p_h|||_D,$$

where C depends only on d, $\kappa_{\mathcal{T}}$, m, and $C_{r,D}/c_{r,D}$.

Properties

- **guaranteed upper bound**
- local efficiency
- **robustness**
- almost asymptotically exact
- negligible evaluation cost

Local efficiency of the estimates for $-\triangle p + rp = f$

Theorem (Local efficiency)

Suppose that $\mathbf{t}_h$ was constructed from p_h by direct prescription. Then there holds

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C|||p - p_h|||_D,$$

where C depends only on d, $\kappa_{\mathcal{T}}$, m, and $C_{r,D}/c_{r,D}$.

Properties

- **guaranteed upper bound**
- local efficiency
- **robustness**
- almost asymptotically exact
- negligible evaluation cost

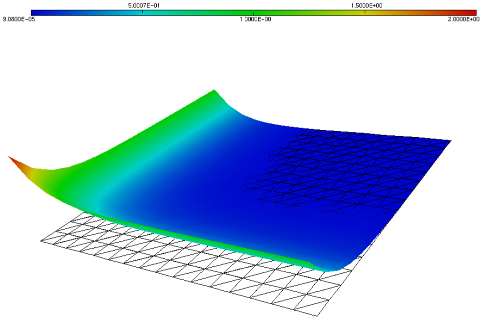
Problem and exact solution

Problem

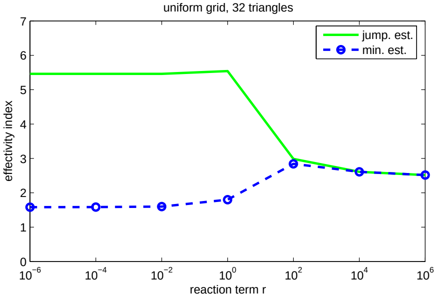
$$-\triangle p + rp = 0 \qquad \text{in } \Omega,$$
$$p = p_0 \qquad \text{on } \partial\Omega$$

Solution

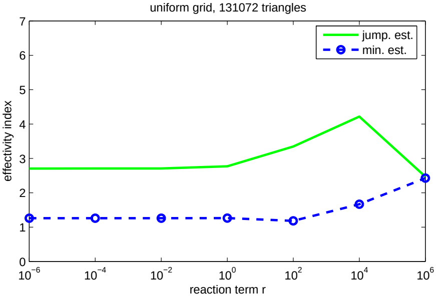
$$p_0(x,y) = e^{-\sqrt{r}x} + e^{-\sqrt{r}y}$$



Effectivity indices in dependence on r

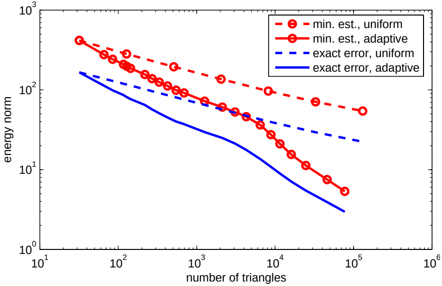


Mesh with 32 triangles

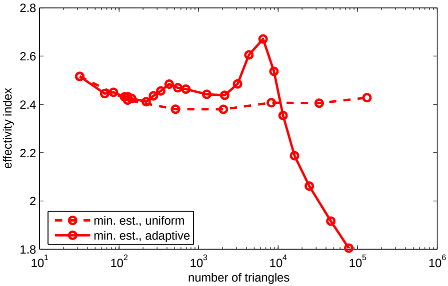


Mesh with 131072 triangles

Estimated and actual errors in uniformly/adaptively refined meshes and effectivity indices

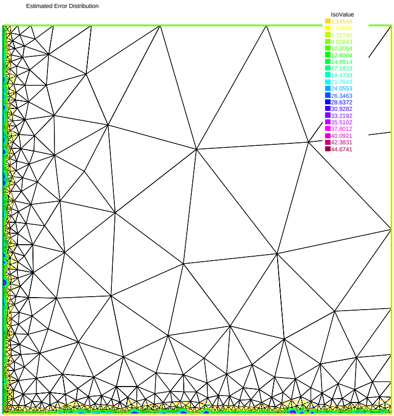


Est. and act. errors, $r = 10^6$

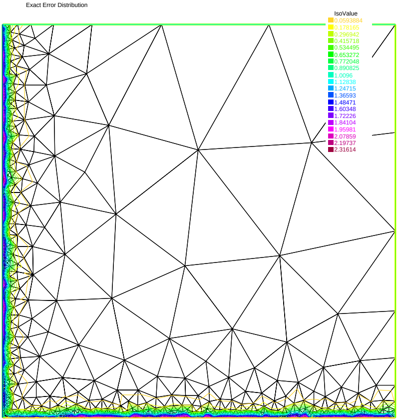


Effectivity indices, $r = 10^6$

Error distribution on an adaptively refined mesh, $r = 10^6$



Estimated error distribution



Exact error distribution

Outline

A convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Energy norm

Set $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$\mathcal{B}_{\mathrm{S}}(p,\varphi) := (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right),$$
$$\mathcal{B}_{\mathrm{A}}(p,\varphi) := \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right)$$

- $\mathcal{B}_{\mathrm{S}}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_{\mathrm{S}}(\varphi,\varphi)$
- $\mathcal{B}_{\mathrm{A}}$ is skew-symmetric on $H^1_0(\Omega)$

Ern, Stephansen, Vohralík
Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems
J. Comput. Appl. Math. 2010 [A6]

A convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Energy norm

Set $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$\begin{aligned} \mathcal{B}_{\mathrm{S}}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right), \\ \mathcal{B}_{\mathrm{A}}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right) \end{aligned}$$

- $\mathcal{B}_{\mathrm{S}}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_{\mathrm{S}}(\varphi,\varphi)$
- $\mathcal{B}_{\mathrm{A}}$ is skew-symmetric on $H_0^1(\Omega)$

ERN, STEPHANSEN, VOHRALÍK
Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems
J. Comput. Appl. Math. 2010 [A6]

A convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Energy norm

Set $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$\begin{aligned} \mathcal{B}_{\mathrm{S}}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right), \\ \mathcal{B}_{\mathrm{A}}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right) \end{aligned}$$

- $\mathcal{B}_{\mathrm{S}}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_{\mathrm{S}}(\varphi,\varphi)$
- $\mathcal{B}_{\mathrm{A}}$ is skew-symmetric on $H_0^1(\Omega)$

ERN, STEPHANSEN, VOHRALÍK
Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems
J. Comput. Appl. Math. 2010 [A6]

A dual norm augmented by the convective derivative

- for $p, \varphi \in H^1(\mathcal{T}_h)$ define

$$\mathcal{B}_{\mathrm{D}}(p, \varphi) := -\sum_{\sigma \in \mathcal{E}_h} \langle \mathbf{w}\cdot\mathbf{n}[\![p]\!], \{\!\{\Pi_0 \varphi\}\!\}\rangle_\sigma$$

- introduce the **augmented norm**

$$|||p|||_{\oplus} := |||p||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(p, \varphi) + \mathcal{B}_{\mathrm{D}}(p, \varphi)\}$$

- when $p \in H_0^1(\Omega)$ and $\nabla\cdot\mathbf{w} = 0$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- for $p, \varphi \in H^1(\mathcal{T}_h)$ define

$$\mathcal{B}_{\mathrm{D}}(p, \varphi) := - \sum_{\sigma \in \mathcal{E}_h} \langle \mathbf{w} \cdot \mathbf{n} [\![p]\!], \{\!\{ \Pi_0 \varphi \}\!\} \rangle_\sigma$$

- introduce the **augmented norm**

$$|\!|\!| p |\!|\!|_{\oplus} := |\!|\!| p |\!|\!| + \sup_{\varphi \in H_0^1(\Omega), |\!|\!| \varphi |\!|\!| = 1} \{ \mathcal{B}_{\mathrm{A}}(p, \varphi) + \mathcal{B}_{\mathrm{D}}(p, \varphi) \}$$

- when $p \in H_0^1(\Omega)$ and $\nabla \cdot \mathbf{w} = 0$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- for $p, \varphi \in H^1(\mathcal{T}_h)$ define

$$\mathcal{B}_{\mathrm{D}}(p,\varphi) := -\sum_{\sigma\in\mathcal{E}_h} \langle \mathbf{w}\cdot\mathbf{n}[\![p]\!], \{\!\{\Pi_0\varphi\}\!\}\rangle_\sigma$$

- introduce the **augmented norm**

$$|||p|||_\oplus := |||p||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(p,\varphi) + \mathcal{B}_{\mathrm{D}}(p,\varphi)\}$$

- when $p \in H_0^1(\Omega)$ and $\nabla\cdot\mathbf{w} = 0$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- for $p, \varphi \in H^1(\mathcal{T}_h)$ define

$$\mathcal{B}_{\mathrm{D}}(p, \varphi) := -\sum_{\sigma \in \mathcal{E}_h} \langle \mathbf{w} \cdot \mathbf{n} [\![p]\!], \{\!\{ \Pi_0 \varphi \}\!\} \rangle_\sigma$$

- introduce the **augmented norm**

$$|||p|||_\oplus := |||p||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(p, \varphi) + \mathcal{B}_{\mathrm{D}}(p, \varphi)\}$$

- when $p \in H_0^1(\Omega)$ and $\nabla \cdot \mathbf{w} = 0$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$
\begin{aligned}
|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} &\leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \\
&\leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).
\end{aligned}
$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$
\begin{aligned}
|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} &\leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \\
&\leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).
\end{aligned}
$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \le \eta + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\le C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

Let p be the weak solution and p_h its DG approximation. Then there holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \eta + |||p_h|||_{\#,\mathcal{E}_h} \leq C(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- η: fully computable estimate
- $|||\cdot|||_{\#,\mathcal{E}_h}$: jump seminorm
- **fully robust** with respect to **convection** or **reaction dominance**
- nonconforming setting
- only global efficiency
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same and hence the adaptive strategies are the same

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
&|||p - p_h|||_\oplus \\
&\leq 2 \inf_{s_h \in H_0^1(\Omega)} \Bigg\{ |||p_h - s_h||| + \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla s_h - r s_h, \varphi) \\
&\quad - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s_h - p_h), \varphi\big)\} \Bigg\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
&\quad \Bigg\{ \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \{(f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla p_h - r p_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi)\} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Bigg\} \\
&\leq 5 |||p - p_h|||_\oplus .
\end{aligned}
$$

Comments

- characterization of the error
- distance of potentials to $H_0^1(\Omega)$
- distance of fluxes to $\mathbf{H}(\mathrm{div}, \Omega)$

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
& |||p - p_h|||_{\oplus} \\
& \leq 2 \inf_{s_h \in H^1_0(\Omega)} \Bigg\{ |||p_h - s_h||| + \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||=1} \big\{ (f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla s_h - r s_h, \varphi) \\
& \quad - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) + \big(\big(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\big)(s_h - p_h), \varphi\big) \big\} \Bigg\} + \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||=1} \\
& \quad \Bigg\{ \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla p_h - r p_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Bigg\} \\
& \leq 5 |||p - p_h|||_{\oplus}.
\end{aligned}
$$

Comments

- characterization of the error
- distance of potentials to $H^1_0(\Omega)$
- distance of fluxes to $\mathbf{H}(\mathrm{div}, \Omega)$

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
& |||p - p_h|||_\oplus \\
& \leq 2 \inf_{s_h \in H_0^1(\Omega)} \Bigg\{ |||p_h - s_h||| + \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ (f - \nabla{\cdot}\mathbf{t}_h - \mathbf{w}{\cdot}\nabla s_h - r s_h, \varphi) \\
& \quad - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla{\cdot}\mathbf{w})(s_h - p_h), \varphi \big) \} \Bigg\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
& \quad \Bigg\{ \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla{\cdot}\mathbf{t}_h - \mathbf{w}{\cdot}\nabla p_h - r p_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Bigg\} \\
& \leq 5 |||p - p_h|||_\oplus .
\end{aligned}
$$

Comments

- characterization of the error
- distance of potentials to $H_0^1(\Omega)$
- distance of fluxes to $\mathbf{H}(\mathrm{div}, \Omega)$

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

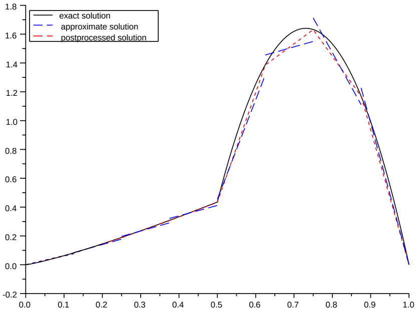
Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\begin{aligned} &|||p - p_h|||_\oplus \\ &\leq 2 \inf_{s_h \in H_0^1(\Omega)} \Bigg\{ |||p_h - s_h||| + \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \Big\{ (f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla s_h - r s_h, \varphi) \\ &\quad - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s_h - p_h), \varphi\big) \Big\} \Bigg\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\ &\quad \Bigg\{ \inf_{\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla\cdot\mathbf{t}_h - \mathbf{w}\cdot\nabla p_h - r p_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}_h, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Bigg\} \\ &\leq 5 |||p - p_h|||_\oplus. \end{aligned}$$

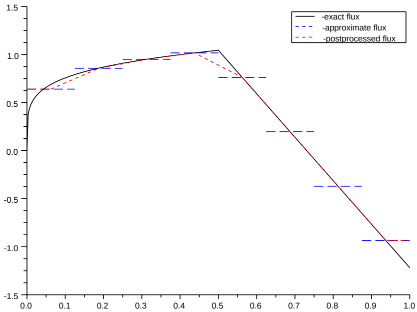
Comments

- characterization of the error
- distance of potentials to $H_0^1(\Omega)$
- distance of fluxes to $\mathbf{H}(\mathrm{div}, \Omega)$

Potential and flux reconstructions



A postprocessed potential s_h is in $H_0^1(\Omega)$



A postprocessed flux $\mathbf{t}_h$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

The heat equation

The heat equation

$$\begin{aligned} \partial_t p - \Delta p &= f && \text{a.e. in } Q := \Omega \times (0, T), \\ p &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\ p(\cdot, 0) &= p_0 && \text{a.e. in } \Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain
- $T > 0$ is the final simulation time

ERN, VOHRALÍK
A posteriori error estimation based on potential and flux reconstruction for the heat equation
SIAM J. Numer. Anal. 2010 [A7]

The heat equation

The heat equation

$$\begin{aligned} \partial_t p - \Delta p &= f && \text{a.e. in } Q := \Omega \times (0,T), \\ p &= 0 && \text{a.e. on } \partial\Omega \times (0,T), \\ p(\cdot,0) &= p_0 && \text{a.e. in } \Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain
- $T > 0$ is the final simulation time

ERN, VOHRALÍK
A posteriori error estimation based on potential and flux reconstruction for the heat equation
SIAM J. Numer. Anal. 2010 [A7]

The heat equation

The heat equation

$$\begin{aligned} \partial_t p - \Delta p &= f && \text{a.e. in } Q := \Omega \times (0, T), \\ p &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\ p(\cdot, 0) &= p_0 && \text{a.e. in } \Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain
- $T > 0$ is the final simulation time

ERN, VOHRALÍK
A posteriori error estimation based on potential and flux reconstruction for the heat equation
SIAM J. Numer. Anal. 2010 [A7]

The heat equation: the setting

Spaces

- $X := L^2(0,T;H^1_0(\Omega))$
- $X' = L^2(0,T,H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

Norms

- energy norm $\|y\|_X^2 := \int_0^T \|\nabla y\|^2(t)\,\mathrm{d}t$
- dual norm $\|y\|_Y := \|y\|_X + \|\partial_t y\|_{X'}$, following Verfürth (2003)

$$\|\partial_t y\|_{X'} = \left\{\int_0^T \|\partial_t y\|^2_{H^{-1}}(t)\,\mathrm{d}t\right\}^{1/2}$$

The heat equation: the setting

Spaces

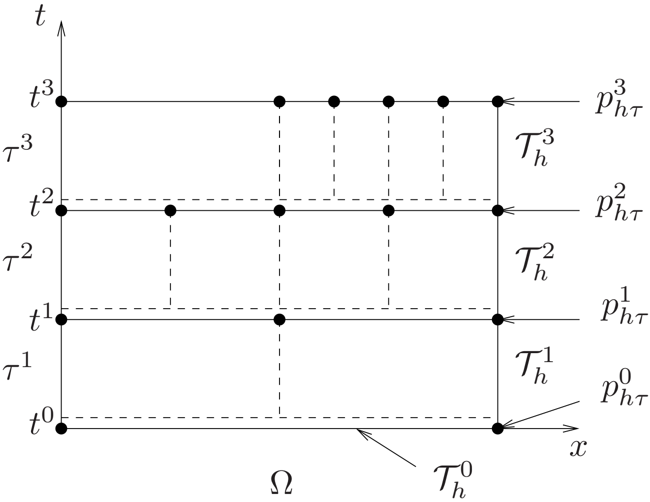
- $X := L^2(0, T; H_0^1(\Omega))$
- $X' = L^2(0, T, H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

Norms

- energy norm $\|y\|_X^2 := \int_0^T \|\nabla y\|^2(t)\,\mathrm{d}t$
- dual norm $\|y\|_Y := \|y\|_X + \|\partial_t y\|_{X'}$, following Verfürth (2003)

$$\|\partial_t y\|_{X'} = \left\{ \int_0^T \|\partial_t y\|_{H^{-1}}^2(t)\,\mathrm{d}t \right\}^{1/2}$$

Time-dependent meshes and discrete solutions



Time-dependent meshes and discrete solutions

Potential and flux reconstructions

General form

- potential reconstruction $s_{h\tau}$ is continuous and piecewise affine in time with $s^n_{h\tau} \in H^1_0(\Omega)$ for all $0 \le n \le N$
- flux reconstruction $\mathbf{t}_{h\tau}$ is piecewise constant in time with $(\mathbf{t}_{h\tau})|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \le n \le N$

Two additional assumptions

- $\mathbf{t}^n_{h\tau}$ satisfies a **local conservation property**

$$(\tilde{f}^n - \partial_t p^n_{h\tau} - \nabla \cdot \mathbf{t}^n_{h\tau}, 1)_K = 0 \qquad \forall K \in \mathcal{T}^n_h$$

- $s^n_{h\tau}$ **preserves the mean values** of $p^n_{h\tau}$

$$(s^n_{h\tau}, 1)_K = (p^n_{h\tau}, 1)_K \qquad \forall K \in \mathcal{T}^{n,n+1}_h$$

Potential and flux reconstructions

General form

- potential reconstruction $s_{h\tau}$ is continuous and piecewise affine in time with $s_{h\tau}^n \in H_0^1(\Omega)$ for all $0 \leq n \leq N$
- flux reconstruction $\mathbf{t}_{h\tau}$ is piecewise constant in time with $(\mathbf{t}_{h\tau})|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \leq n \leq N$

Two additional assumptions

- $\mathbf{t}_{h\tau}^n$ satisfies a **local conservation property**

$$(\tilde{f}^n - \partial_t p_{h\tau}^n - \nabla \cdot \mathbf{t}_{h\tau}^n, 1)_K = 0 \qquad \forall K \in \mathcal{T}_h^n$$

- $s_{h\tau}^n$ **preserves the mean values** of $p_{h\tau}^n$

$$(s_{h\tau}^n, 1)_K = (p_{h\tau}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^{n,n+1}$$

Potential and flux reconstructions

General form

- potential reconstruction $s_{h\tau}$ is continuous and piecewise affine in time with $s_{h\tau}^n \in H_0^1(\Omega)$ for all $0 \leq n \leq N$
- flux reconstruction $\mathbf{t}_{h\tau}$ is piecewise constant in time with $(\mathbf{t}_{h\tau})|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \leq n \leq N$

Two additional assumptions

- $\mathbf{t}_{h\tau}^n$ satisfies a **local conservation property**

$$(\tilde{f}^n - \partial_t p_{h\tau}^n - \nabla \cdot \mathbf{t}_{h\tau}^n, 1)_K = 0 \qquad \forall K \in \mathcal{T}_h^n$$

- $s_{h\tau}^n$ **preserves the mean values** of $p_{h\tau}^n$

$$(s_{h\tau}^n, 1)_K = (p_{h\tau}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^{n,n+1}$$

Potential and flux reconstructions

General form

- potential reconstruction $s_{h\tau}$ is continuous and piecewise affine in time with $s_{h\tau}^n \in H_0^1(\Omega)$ for all $0 \le n \le N$
- flux reconstruction $\mathbf{t}_{h\tau}$ is piecewise constant in time with $(\mathbf{t}_{h\tau})|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \le n \le N$

Two additional assumptions

- $\mathbf{t}_{h\tau}^n$ satisfies a **local conservation property**

$$(\tilde{f}^n - \partial_t p_{h\tau}^n - \nabla \cdot \mathbf{t}_{h\tau}^n, 1)_K = 0 \qquad \forall K \in \mathcal{T}_h^n$$

- $s_{h\tau}^n$ **preserves the mean values** of $p_{h\tau}^n$

$$(s_{h\tau}^n, 1)_K = (p_{h\tau}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^{n,n+1}$$

A posteriori error estimate

Theorem (A posteriori error estimate)

Let

- *p be the weak solution*
- *$p_{h\tau}^n \in H^1(\mathcal{T}_h^n)$ be arbitrary*
- *$s_{h\tau}$ be the mean values-preserving potential reconstruction and $\mathbf{t}_{h\tau}$ locally conservative flux reconstruction.*

Then

$$\|p - p_{h\tau}\|_Y \le 3\left\{\sum_{n=1}^N \int_{I_n} \sum_{K\in\mathcal{T}_h^n} (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K}(t))^2 \,\mathrm{d}t\right\}^{1/2} + \left\{\sum_{n=1}^N \int_{I_n} \sum_{K\in\mathcal{T}_h^n} (\eta_{\mathrm{NC1},K}^n)^2(t)\,\mathrm{d}t\right\}^{1/2} + \left\{\sum_{n=1}^N \tau^n \sum_{K\in\mathcal{T}_h^n} (\eta_{\mathrm{NC2},K}^n)^2\right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \widetilde{f}\|_{X'}.$$

Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},K}(t) := \|\nabla s_{h\tau}(t) + \mathbf{t}_{h\tau}^n\|_K, \quad t \in I_n$
- *residual estimator*
  - $\eta_{\mathrm{R},K} := 1/\pi h_K \|\tilde{f}^n - \partial_t s_{h\tau}^n - \nabla\cdot\mathbf{t}_{h\tau}^n\|_K$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},K}^n(t) := \|\nabla(s_{h\tau} - p_{h\tau})(t)\|_K, \quad t \in I_n$
  - $\eta_{\mathrm{NC2},K}^n := 1/\pi h_K \|\partial_t(s_{h\tau} - p_{h\tau})^n\|_K$
  - penalize the fact that $p_{h\tau}^n \notin H_0^1(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - p^0\|$
- *data oscillation estimator*
  - $\|f - \tilde{f}\|_{X'}$

Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},K}(t) := \|\nabla s_{h\tau}(t) + \mathbf{t}_{h\tau}^n\|_K, \quad t \in I_n$
- *residual estimator*
  - $\eta_{\mathrm{R},K} := 1/\pi h_K \|\tilde{f}^n - \partial_t s_{h\tau}^n - \nabla \cdot \mathbf{t}_{h\tau}^n\|_K$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},K}^n(t) := \|\nabla(s_{h\tau} - p_{h\tau})(t)\|_K, \quad t \in I_n$
  - $\eta_{\mathrm{NC2},K}^n := 1/\pi h_K \|\partial_t(s_{h\tau} - p_{h\tau})^n\|_K$
  - penalize the fact that $p_{h\tau}^n \notin H_0^1(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - p^0\|$
- *data oscillation estimator*
  - $\|f - \tilde{f}\|_{X'}$

Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},K}(t) := \|\nabla s_{h\tau}(t) + \mathbf{t}^n_{h\tau}\|_K, \quad t \in I_n$
- *residual estimator*
  - $\eta_{\mathrm{R},K} := 1/\pi h_K \|\tilde{f}^n - \partial_t s^n_{h\tau} - \nabla\cdot\mathbf{t}^n_{h\tau}\|_K$
- *nonconformity estimators*
  - $\eta^n_{\mathrm{NC1},K}(t) := \|\nabla(s_{h\tau} - p_{h\tau})(t)\|_K, \quad t \in I_n$
  - $\eta^n_{\mathrm{NC2},K} := 1/\pi h_K \|\partial_t(s_{h\tau} - p_{h\tau})^n\|_K$
  - penalize the fact that $p^n_{h\tau} \notin H^1_0(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - p^0\|$
- *data oscillation estimator*
  - $\|f - \tilde{f}\|_{X'}$

Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},K}(t) := \|\nabla s_{h\tau}(t) + \mathbf{t}_{h\tau}^n\|_K, \quad t \in I_n$
- *residual estimator*
  - $\eta_{\mathrm{R},K} := 1/\pi h_K \|\widetilde{f}^n - \partial_t s_{h\tau}^n - \nabla\cdot\mathbf{t}_{h\tau}^n\|_K$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},K}^n(t) := \|\nabla(s_{h\tau} - p_{h\tau})(t)\|_K, \quad t \in I_n$
  - $\eta_{\mathrm{NC2},K}^n := 1/\pi h_K \|\partial_t(s_{h\tau} - p_{h\tau})^n\|_K$
  - penalize the fact that $p_{h\tau}^n \notin H_0^1(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - p^0\|$
- *data oscillation estimator*
  - $\|f - \widetilde{f}\|_{X'}$

Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},K}(t) := \|\nabla s_{h\tau}(t) + \mathbf{t}^n_{h\tau}\|_K, \quad t \in I_n$
- *residual estimator*
  - $\eta_{\mathrm{R},K} := 1/\pi h_K \|\widetilde{f}^n - \partial_t s^n_{h\tau} - \nabla\cdot\mathbf{t}^n_{h\tau}\|_K$
- *nonconformity estimators*
  - $\eta^n_{\mathrm{NC1},K}(t) := \|\nabla(s_{h\tau} - p_{h\tau})(t)\|_K, \quad t \in I_n$
  - $\eta^n_{\mathrm{NC2},K} := 1/\pi h_K \|\partial_t(s_{h\tau} - p_{h\tau})^n\|_K$
  - penalize the fact that $p^n_{h\tau} \notin H^1_0(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - p^0\|$
- *data oscillation estimator*
  - $\|f - \widetilde{f}\|_{X'}$

Separating the space and time errors

Separating the space and time errors

- triangle inequality: time-dependent $\times$ time-independent terms (similarly to Picasso (1998), Verfürth (2003), Bergam, Bernardi, Mghazli (2005), but no unknown constant)

Corollary (Estimate separating the space and time errors)

$$\|p - p_{h\tau}\|_Y \leq \left\{\sum_{n=1}^{N} (\eta_{\mathrm{sp}}^n)^2\right\}^{1/2} + \left\{\sum_{n=1}^{N} (\eta_{\mathrm{tm}}^n)^2\right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \tilde{f}\|_{X'}$$

Efficiency

Approximation property

$$\|\nabla p_{h\tau}^n + \mathbf{t}_{h\tau}^n\|_K \leq C \left\{ \sum_{L \in \mathcal{T}_K} h_L^2 \|\widetilde{f}^n - \partial_t p_{h\tau}^n + \Delta p_{h\tau}^n\|_L^2 \right\}^{1/2} + |[\![\nabla p_{h\tau}^n \cdot \mathbf{n}]\!]|_{+\frac{1}{2}, \mathfrak{E}_K^{\mathrm{int},n}} + |[\![p_{h\tau}^n]\!]|_{-\frac{1}{2}, \mathfrak{E}_K^n}$$

Theorem (Efficiency)

Under the approximation property, there holds, for all n,

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|p - p_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(p_{h\tau}) + \mathcal{E}_f^n$$

Efficiency

Approximation property

$$\|\nabla p_{h\tau}^n + \mathbf{t}_{h\tau}^n\|_K \leq C \left\{ \sum_{L \in \mathcal{T}_K} h_L^2 \|\widetilde{f}^n - \partial_t p_{h\tau}^n + \Delta p_{h\tau}^n\|_L^2 \right\}^{1/2} + |[\![\nabla p_{h\tau}^n \cdot \mathbf{n}]\!]|_{+\frac{1}{2}, \mathfrak{E}_K^{\mathrm{int},n}} + |[\![p_{h\tau}^n]\!]|_{-\frac{1}{2}, \mathfrak{E}_K^n}$$

Theorem (Efficiency)

Under the approximation property, there holds, for all n,

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|p - p_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(p_{h\tau}) + \mathcal{E}_f^n$$

A space-time adaptive time-marching algorithm

Achieving a given relative precision ε

- we want to satisfy

$$\frac{\|p - p_{h\tau}\|_Y}{\|p_{h\tau}\|_Z} \leq \varepsilon$$

- we achieve it by

$$\frac{\sum_{n=1}^{N} \left\{ (\eta_{\rm sp}^n)^2 + (\eta_{\rm tm}^n)^2 \right\}}{\sum_{n=1}^{N} \|p_{h\tau}\|_{Z(I_n)}^2} \leq \varepsilon^2$$

Algorithm

1. Initialization
   1. choose an initial mesh $\mathcal{T}_h^0$;
   2. select an initial time step τ^0;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}_h^{n*} := \mathcal{T}_h^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $p_{h\tau}^{n*} := \mathrm{Sol}(p_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}_h^{n*})$;
   3. estimate the space and time errors by $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$;
   4. equilibrate $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$ to $\varepsilon\|p_{h\tau}\|_{Z(I_n)}/\sqrt{2}$

A space-time adaptive time-marching algorithm

Achieving a given relative precision ε

- we want to satisfy
$$\frac{\|p - p_{h\tau}\|_Y}{\|p_{h\tau}\|_Z} \leq \varepsilon$$
- we achieve it by
$$\frac{\sum_{n=1}^{N}\left\{(\eta_{\rm sp}^n)^2 + (\eta_{\rm tm}^n)^2\right\}}{\sum_{n=1}^{N}\|p_{h\tau}\|^2_{Z(I_n)}} \leq \varepsilon^2$$

Algorithm

1. Initialization
   1. choose an initial mesh $\mathcal{T}_h^0$;
   2. select an initial time step τ^0;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}_h^{n*} := \mathcal{T}_h^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $p_{h\tau}^{n*} := \mathrm{Sol}(p_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}_h^{n*})$;
   3. estimate the space and time errors by $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$;
   4. equilibrate $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$ to $\varepsilon\|p_{h\tau}\|_{Z(I_n)}/\sqrt{2}$

A space-time adaptive time-marching algorithm

Achieving a given relative precision ε

- we want to satisfy

$$\frac{\|p - p_{h\tau}\|_Y}{\|p_{h\tau}\|_Z} \leq \varepsilon$$

- we achieve it by

$$\frac{\sum_{n=1}^{N} \{(\eta_{\mathrm{sp}}^n)^2 + (\eta_{\mathrm{tm}}^n)^2\}}{\sum_{n=1}^{N} \|p_{h\tau}\|_{Z(I_n)}^2} \leq \varepsilon^2$$

Algorithm

1. Initialization
   1. choose an initial mesh $\mathcal{T}_h^0$;
   2. select an initial time step τ^0;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}_h^{n*} := \mathcal{T}_h^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $p_{h\tau}^{n*} := \mathrm{Sol}(p_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}_h^{n*})$;
   3. estimate the space and time errors by η_{sp}^n and η_{tm}^n;
   4. equilibrate η_{sp}^n and η_{tm}^n to $\varepsilon \|p_{h\tau}\|_{Z(I_n)}/\sqrt{2}$

A space-time adaptive time-marching algorithm

Achieving a given relative precision ε

- we want to satisfy
$$\frac{\|p - p_{h\tau}\|_Y}{\|p_{h\tau}\|_Z} \leq \varepsilon$$
- we achieve it by
$$\frac{\sum_{n=1}^N \{(\eta_{\rm sp}^n)^2 + (\eta_{\rm tm}^n)^2\}}{\sum_{n=1}^N \|p_{h\tau}\|^2_{Z(I_n)}} \leq \varepsilon^2$$

Algorithm

1. Initialization
   1. choose an initial mesh $\mathcal{T}_h^0$;
   2. select an initial time step τ^0;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}_h^{n*} := \mathcal{T}_h^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $p_{h\tau}^{n*} := \mathrm{Sol}(p_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}_h^{n*})$;
   3. estimate the space and time errors by $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$;
   4. equilibrate $\eta_{\rm sp}^n$ and $\eta_{\rm tm}^n$ to $\varepsilon\|p_{h\tau}\|_{Z(I_n)}/\sqrt{2}$

Discontinuous Galerkin method

Definition (DG method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_k(\mathcal{T}_h^n)$, $k \geq 1$, such that

$$(\partial_t p_{h\tau}^n, v_h) - \sum_{\sigma \in \mathcal{E}_h^n} \{\langle \mathbf{n} \cdot \{\!\{\nabla p_{h\tau}^n\}\!\}, [\![v_h]\!]\rangle_\sigma + \theta \langle \mathbf{n} \cdot \{\!\{\nabla v_h\}\!\}, [\![p_{h\tau}^n]\!]\rangle_\sigma\}$$

$$+ (\nabla p_{h\tau}^n, \nabla v_h) + \sum_{\sigma \in \mathcal{E}_h^n} \langle \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], [\![v_h]\!]\rangle_\sigma = (\tilde{f}^n, v_h) \quad \forall v_h \in V_h^n.$$

- jump operator $[\![v_h]\!] = v_h^- - v_h^+$
- average operator $\{\!\{v_h\}\!\} = \frac{1}{2}(v_h^- + v_h^+)$
- θ: different scheme types (SIPG/NIPG/IIPG)
- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

Discontinuous Galerkin method

Definition (DG method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_k(\mathcal{T}_h^n)$, $k \ge 1$, such that

$$(\partial_t p_{h\tau}^n, v_h) - \sum_{\sigma\in\mathcal{E}_h^n} \{\langle \mathbf{n}\cdot\{\!\{\nabla p_{h\tau}^n\}\!\}, [\![v_h]\!]\rangle_\sigma + \theta\langle \mathbf{n}\cdot\{\!\{\nabla v_h\}\!\}, [\![p_{h\tau}^n]\!]\rangle_\sigma\}$$

$$+(\nabla p_{h\tau}^n, \nabla v_h) + \sum_{\sigma\in\mathcal{E}_h^n} \langle \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], [\![v_h]\!]\rangle_\sigma = (\tilde{f}^n, v_h) \quad \forall v_h \in V_h^n.$$

- jump operator $[\![v_h]\!] = v_h^- - v_h^+$
- average operator $\{\!\{v_h\}\!\} = \frac{1}{2}(v_h^- + v_h^+)$
- θ: different scheme types (SIPG/NIPG/IIPG)
- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

Discontinuous Galerkin method

Definition (DG method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_k(\mathcal{T}_h^n)$, $k \geq 1$, such that

$$(\partial_t p_{h\tau}^n, v_h) - \sum_{\sigma \in \mathcal{E}_h^n} \{ \langle \mathbf{n} \cdot \{\!\{ \nabla p_{h\tau}^n \}\!\}, [\![v_h]\!] \rangle_\sigma + \theta \langle \mathbf{n} \cdot \{\!\{ \nabla v_h \}\!\}, [\![p_{h\tau}^n]\!] \rangle_\sigma \}$$

$$+ (\nabla p_{h\tau}^n, \nabla v_h) + \sum_{\sigma \in \mathcal{E}_h^n} \langle \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], [\![v_h]\!] \rangle_\sigma = (\tilde{f}^n, v_h) \quad \forall v_h \in V_h^n.$$

- jump operator $[\![v_h]\!] = v_h^- - v_h^+$
- average operator $\{\!\{ v_h \}\!\} = \frac{1}{2}(v_h^- + v_h^+)$
- θ: different scheme types (SIPG/NIPG/IIPG)
- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

DG flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h^n)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $l = 1$

Flux reconstruction $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_l(\mathcal{T}_h^n)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, q_h \rangle_\sigma = \langle -\mathbf{n} \cdot \{\!\{ \nabla p_{h\tau}^n \}\!\} + \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], q_h \rangle_\sigma$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)$,
  $(\mathbf{t}_{h\tau}^n, \mathbf{r}_h)_K = -(\nabla p_{h\tau}^n, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K^n} \omega_\sigma \langle \mathbf{n} \cdot \mathbf{r}_h, [\![p_{h\tau}^n]\!] \rangle_\sigma$

Reconstructed flux property

For $l = k$ and when $\mathcal{T}_h^{n-1} = \mathcal{T}_h^n$

$$\partial_t p_{h\tau}^n + \nabla \cdot \mathbf{t}_{h\tau}^n = \Pi_{V_h^n} \widetilde{f}^n$$

DG flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h^n)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $\qquad$ $l = 1$

Flux reconstruction $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_l(\mathcal{T}_h^n)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, q_h \rangle_\sigma = \langle -\mathbf{n} \cdot \{\!\{\nabla p_{h\tau}^n\}\!\} + \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], q_h \rangle_\sigma$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)$,
$(\mathbf{t}_{h\tau}^n, \mathbf{r}_h)_K = -(\nabla p_{h\tau}^n, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K^n} \omega_\sigma \langle \mathbf{n} \cdot \mathbf{r}_h, [\![p_{h\tau}^n]\!] \rangle_\sigma$

Reconstructed flux property

For $l = k$ and when $\mathcal{T}_h^{n-1} = \mathcal{T}_h^n$

$$\partial_t p_{h\tau}^n + \nabla \cdot \mathbf{t}_{h\tau}^n = \Pi_{V_h^n} \widetilde{f}^n$$

DG flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h^n)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $l = 1$

Flux reconstruction $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_l(\mathcal{T}_h^n)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, q_h \rangle_\sigma = \langle -\mathbf{n} \cdot \{\!\{ \nabla p_{h\tau}^n \}\!\} + \alpha_\sigma h_\sigma^{-1} [\![p_{h\tau}^n]\!], q_h \rangle_\sigma$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)$,
$(\mathbf{t}_{h\tau}^n, \mathbf{r}_h)_K = -(\nabla p_{h\tau}^n, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K^n} \omega_\sigma \langle \mathbf{n} \cdot \mathbf{r}_h, [\![p_{h\tau}^n]\!] \rangle_\sigma$

Reconstructed flux property

For $l = k$ and when $\mathcal{T}_h^{n-1} = \mathcal{T}_h^n$

$$\partial_t p_{h\tau}^n + \nabla \cdot \mathbf{t}_{h\tau}^n = \Pi_{V_h^n} \widetilde{f}^n$$

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $\bar{p}_{h\tau}^n \in \bar{V}_h^n := \mathbb{P}_0(\mathcal{T}_h^n)$ such that

$$\frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, 1)_K + \sum_{\sigma \in \mathcal{E}_K^n} S_{K,\sigma}^n = (\tilde{f}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{T}_h^n)$

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, 1 \rangle_\sigma := S_{K,\sigma}^n$$

Postprocessing of the potential

- $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ not suitable for energy error estimates ($\nabla \bar{p}_{h\tau}^n = 0$)
- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_1(\mathcal{T}_h^n)$ enriched elementwise by parabolas
- $$-\nabla p_{h\tau}^n = \mathbf{t}_{h\tau}^n,$$ $$(p_{h\tau}^n, 1)_K = (\bar{p}_{h\tau}^n, 1)_K$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $\bar{p}_{h\tau}^n \in \bar{V}_h^n := \mathbb{P}_0(\mathcal{T}_h^n)$ such that

$$\frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, 1)_K + \sum_{\sigma \in \mathcal{E}_K^n} S_{K,\sigma}^n = (\tilde{f}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{T}_h^n)$

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, 1 \rangle_\sigma := S_{K,\sigma}^n$$

Postprocessing of the potential

- $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ not suitable for energy error estimates ($\nabla \bar{p}_{h\tau}^n = 0$)
- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_1(\mathcal{T}_h^n)$ enriched elementwise by parabolas
- $$-\nabla p_{h\tau}^n = \mathbf{t}_{h\tau}^n,$$ $$(p_{h\tau}^n, 1)_K = (\bar{p}_{h\tau}^n, 1)_K$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $\bar{p}_{h\tau}^n \in \bar{V}_h^n := \mathbb{P}_0(\mathcal{T}_h^n)$ such that

$$\frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, 1)_K + \sum_{\sigma \in \mathcal{E}_K^n} S_{K,\sigma}^n = (\tilde{f}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{T}_h^n)$

$$\langle \mathbf{t}_{h\tau}^n \cdot \mathbf{n}, 1 \rangle_\sigma := S_{K,\sigma}^n$$

Postprocessing of the potential

- $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ not suitable for energy error estimates ($\nabla \bar{p}_{h\tau}^n = 0$)
- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_1(\mathcal{T}_h^n)$ enriched elementwise by parabolas
- $$-\nabla p_{h\tau}^n = \mathbf{t}_{h\tau}^n,$$ $$(p_{h\tau}^n, 1)_K = (\bar{p}_{h\tau}^n, 1)_K$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ such that

$$(\boldsymbol{\sigma}_{h\tau}^n, \mathbf{w}_h) - (\bar{p}_{h\tau}^n, \nabla\cdot\mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}_h^n,$$

$$(\nabla\cdot\boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{W}_h^n$

$\mathbf{t}_{h\tau}^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}_h^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}_h^n}(-\nabla p_{h\tau}^n) = \boldsymbol{\sigma}_{h\tau}^n,$$ $$\Pi_{\bar{V}_h^n}(p_{h\tau}^n) = \bar{p}_{h\tau}^n$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ such that

$$(\boldsymbol{\sigma}_{h\tau}^n, \mathbf{w}_h) - (\bar{p}_{h\tau}^n, \nabla \cdot \mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}_h^n,$$

$$(\nabla \cdot \boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{W}_h^n$

$\mathbf{t}_{h\tau}^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}_h^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}_h^n}(-\nabla p_{h\tau}^n) = \boldsymbol{\sigma}_{h\tau}^n,$$
  $$\Pi_{\bar{V}_h^n}(p_{h\tau}^n) = \bar{p}_{h\tau}^n$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{p}_{h\tau}^n \in \bar{V}_h^n$ such that

$$(\boldsymbol{\sigma}_{h\tau}^n, \mathbf{w}_h) - (\bar{p}_{h\tau}^n, \nabla\cdot\mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}_h^n,$$

$$(\nabla\cdot\boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{p}_{h\tau}^n - p_{h\tau}^{n-1}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{W}_h^n$

$\mathbf{t}_{h\tau}^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $p_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}_h^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}_h^n}(-\nabla p_{h\tau}^n) = \boldsymbol{\sigma}_{h\tau}^n,$$ $$\Pi_{\bar{V}_h^n}(p_{h\tau}^n) = \bar{p}_{h\tau}^n$$
- diffusive flux estimator $\eta_{\mathrm{DF},K,1}^n$ vanishes (flux-conf. method)

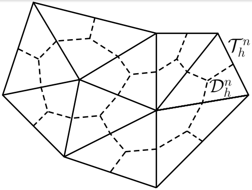
Vertex-centered finite volume method

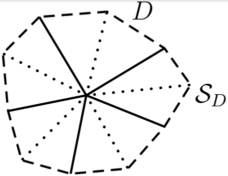
Definition (VCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_1(\mathcal{T}_h^n) \cap H_0^1(\Omega)$ s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Setting





- $s = p_{h\tau}$, $\eta_{\mathrm{NC1},K}^n$, $\eta_{\mathrm{NC2},K}^n$ vanish (conforming method)

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by MFE sol. of local Neumann problems on patches $\mathcal{S}_D$:

$$(\mathbf{t}_{h\tau}^n + \nabla u_{h\tau}^n, \mathbf{v}_h)_D - (q_h, \nabla \cdot \mathbf{v}_h)_D = 0 \qquad \forall \mathbf{v}_h \in \mathbf{RTN}_0^{\mathrm{N},0}(\mathcal{S}_D),$$
$$(\nabla \cdot \mathbf{t}_{h\tau}^n, \phi_h)_D - (\tilde{f}^n - \partial_t u_{h\tau}^n, \phi_h)_D = 0 \qquad \forall \phi_h \in \mathbb{P}_0^*(\mathcal{S}_D).$$

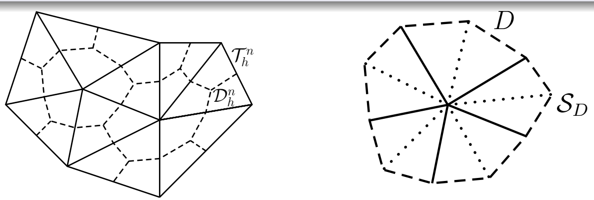
Vertex-centered finite volume method

Definition (VCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_1(\mathcal{T}_h^n) \cap H_0^1(\Omega)$ s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Setting



- $s = p_{h\tau}$, $\eta_{\mathrm{NC1},K}^n$, $\eta_{\mathrm{NC2},K}^n$ vanish (conforming method)

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by MFE sol. of local Neumann problems on patches $\mathcal{S}_D$:

$$(\mathbf{t}_{h\tau}^n + \nabla u_{h\tau}^n, \mathbf{v}_h)_D - (q_h, \nabla \cdot \mathbf{v}_h)_D = 0 \qquad \forall \mathbf{v}_h \in \mathbf{RTN}_0^{\mathrm{N},0}(\mathcal{S}_D),$$
$$(\nabla \cdot \mathbf{t}_{h\tau}^n, \phi_h)_D - (\tilde{f}^n - \partial_t u_{h\tau}^n, \phi_h)_D = 0 \qquad \forall \phi_h \in \mathbb{P}_0^*(\mathcal{S}_D).$$

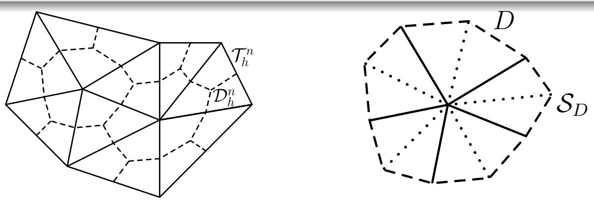
Vertex-centered finite volume method

Definition (VCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $p_{h\tau}^n \in V_h^n := \mathbb{P}_1(\mathcal{T}_h^n) \cap H_0^1(\Omega)$ s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Setting



- $s = p_{h\tau}$, $\eta_{\mathrm{NC1},K}^n$, $\eta_{\mathrm{NC2},K}^n$ vanish (conforming method)

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by MFE sol. of local Neumann problems on patches $\mathcal{S}_D$:

$$(\mathbf{t}_{h\tau}^n + \nabla u_{h\tau}^n, \mathbf{v}_h)_D - (q_h, \nabla \cdot \mathbf{v}_h)_D = 0 \qquad \forall \mathbf{v}_h \in \mathbf{RTN}_0^{\mathrm{N},0}(\mathcal{S}_D),$$
$$(\nabla \cdot \mathbf{t}_{h\tau}^n, \phi_h)_D - (\tilde{f}^n - \partial_t u_{h\tau}^n, \phi_h)_D = 0 \qquad \forall \phi_h \in \mathbb{P}_0^*(\mathcal{S}_D).$$

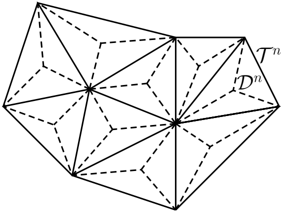
Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $p_{h\tau}^n \in V_h^n$ (Crouzeix–Raviart sp.) s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\widetilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Setting



- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by solution of local Neumann problems on patches $\mathcal{S}_D$

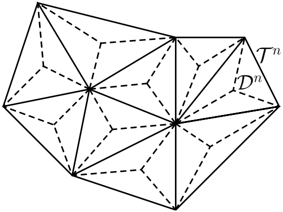
Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \le n \le N$, find $p_{h\tau}^n \in V_h^n$ (Crouzeix–Raviart sp.) s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

Setting



- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by solution of local Neumann problems on patches $\mathcal{S}_D$

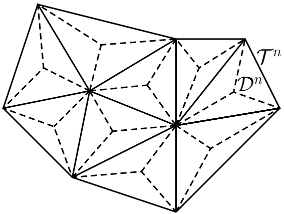
Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}_h^n$, $1 \leq n \leq N$, find $p_{h\tau}^n \in V_h^n$ (Crouzeix–Raviart sp.) s.t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1 \rangle_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\rm int}.$$

Setting

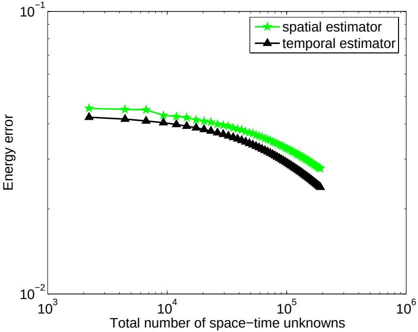


- $p_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla p_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

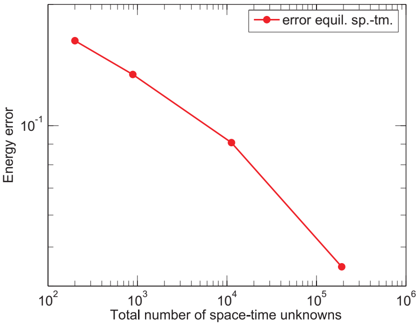
Flux $\mathbf{t}_{h\tau}^n \in \mathbf{RTN}_0(\mathcal{S}_h)$

- by prescription: $\mathbf{t}_{h\tau}^n \cdot \mathbf{n}_\sigma|_\sigma := -\{\!\{\nabla p_{h\tau}^n \cdot \mathbf{n}_\sigma\}\!\}$ on faces σ of $\mathcal{S}_h$
- by solution of local Neumann problems on patches $\mathcal{S}_D$

Spatial and temporal estimators equilibrated

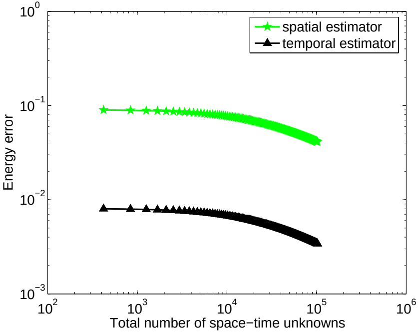


Spatial estimators η_{sp}^{n} and temporal estimators η_{tm}^{n}

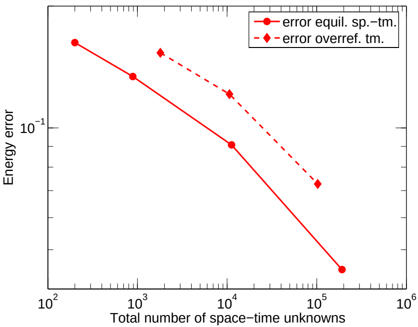


Equilibrated case

Overrefinement in time

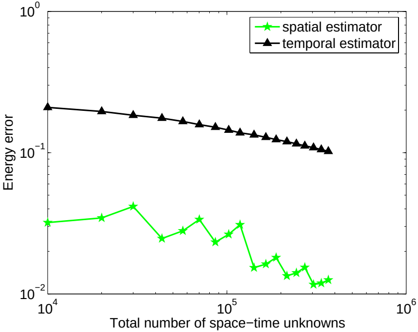


Spatial estimators η_{sp}^n and temporal estimators η_{tm}^n

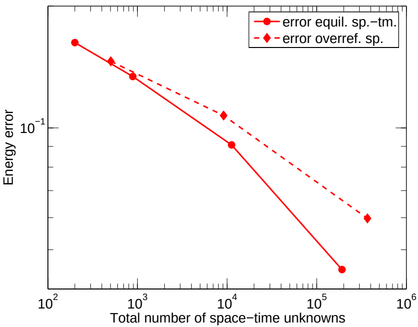


Comparison with the equilibrated case

Overrefinement in space

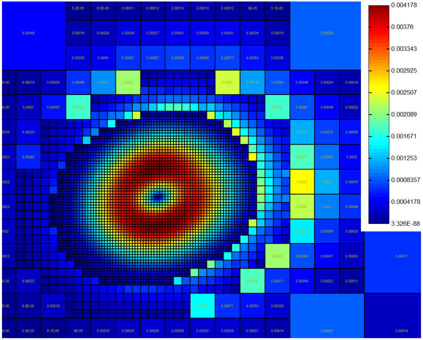


Spatial estimators η_{sp}^n and temporal estimators η_{tm}^n

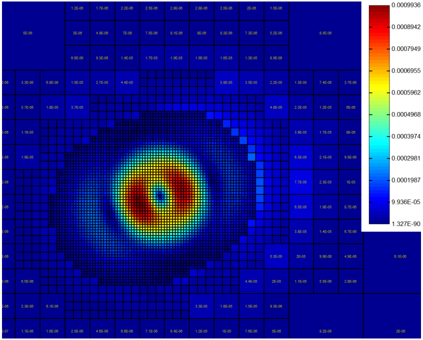


Comparison with the equilibrated case

Error distributions

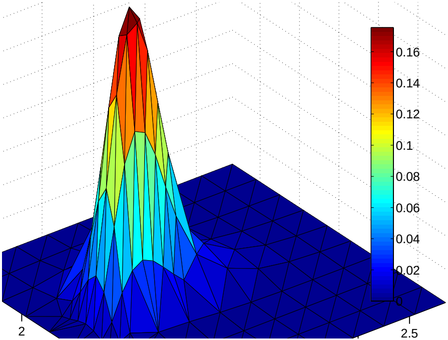


Estimated error distribution, convection dominance

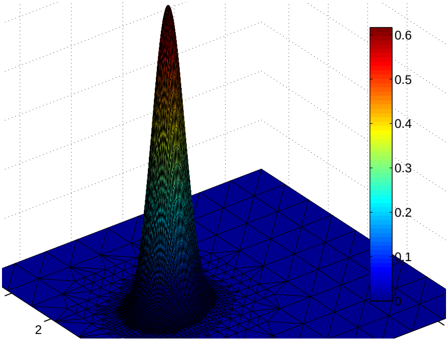


Exact error distribution, convection dominance

Adaptive refinement approximate solutions



Approximate solutions, convection dominance, two levels of refinement



Approximate solutions, convection dominance, four levels of refinement

Outline

The Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega,\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega,\\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$
- β–the constant from the inf–sup condition

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in (H^1_0(\Omega))^d} \frac{(q, \nabla\cdot\mathbf{v})}{|||\mathbf{v}|||\,\|q\|} \geq \beta$$

- C_S–the constant from the stability estimate

$$\inf_{(\mathbf{v},q) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \sup_{(\mathbf{z},r) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \frac{(\nabla \mathbf{v}, \nabla \mathbf{z}) - (q, \nabla\cdot\mathbf{z}) - (r, \nabla\cdot\mathbf{v})}{|||(\mathbf{z}, r)|||\,|||(\mathbf{v}, q)|||} \geq C_S$$

HANNUKAINEN, STENBERG, VOHRALÍK
A unified framework for a posteriori error estimation for the Stokes problem
submitted [B2]

The Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$
- β–the constant from the inf–sup condition

$$\inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in (H_0^1(\Omega))^d} \frac{(q, \nabla\cdot \mathbf{v})}{|||\mathbf{v}|||\, \|q\|} \geq \beta$$

- C_S–the constant from the stability estimate

$$\inf_{(\mathbf{v},q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)} \sup_{(\mathbf{z},r) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)} \frac{(\nabla \mathbf{v}, \nabla \mathbf{z}) - (q, \nabla\cdot \mathbf{z}) - (r, \nabla\cdot \mathbf{v})}{|||(\mathbf{z}, r)|||\, |||(\mathbf{v}, q)|||} \geq C_S$$

Hannukainen, Stenberg, Vohralík
A unified framework for a posteriori error estimation for the Stokes problem
submitted [B2]

The Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$
- β–the constant from the inf–sup condition

$$\inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in (H_0^1(\Omega))^d} \frac{(q, \nabla \cdot \mathbf{v})}{|||\mathbf{v}|||\, \|q\|} \geq \beta$$

- C_S–the constant from the stability estimate

$$\inf_{(\mathbf{v},q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)} \sup_{(\mathbf{z},r) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)} \frac{(\nabla \mathbf{v}, \nabla \mathbf{z}) - (q, \nabla \cdot \mathbf{z}) - (r, \nabla \cdot \mathbf{v})}{|||(\mathbf{z}, r)|||\, |||(\mathbf{v}, q)|||} \geq C_S$$

Hannukainen, Stenberg, Vohralík
A unified framework for a posteriori error estimation for the Stokes problem
submitted [B2]

The Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in} \quad \Omega, \\ \mathbf{u} &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$
- β–the constant from the inf–sup condition

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in (H^1_0(\Omega))^d} \frac{(q, \nabla\cdot\mathbf{v})}{|||\mathbf{v}|||\, \|q\|} \geq \beta$$

- C_{S}–the constant from the stability estimate

$$\inf_{(\mathbf{v},q) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \sup_{(\mathbf{z},r) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \frac{(\nabla \mathbf{v}, \nabla \mathbf{z}) - (q, \nabla\cdot\mathbf{z}) - (r, \nabla\cdot\mathbf{v})}{|||(\mathbf{z}, r)|||\, |||(\mathbf{v}, q)|||} \geq C_{\mathrm{S}}$$

Hannukainen, Stenberg, Vohralík
A unified framework for a posteriori error estimation for the Stokes problem
submitted [B2]

The Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in} \quad \Omega, \\ \mathbf{u} &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

Setting

- $|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$
- β–the constant from the inf–sup condition

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in (H^1_0(\Omega))^d} \frac{(q, \nabla \cdot \mathbf{v})}{|||\mathbf{v}||| \, \|q\|} \geq \beta$$

- C_{S}–the constant from the stability estimate

$$\inf_{(\mathbf{v},q) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \sup_{(\mathbf{z},r) \in (H^1_0(\Omega))^d \times L^2_0(\Omega)} \frac{(\nabla \mathbf{v}, \nabla \mathbf{z}) - (q, \nabla \cdot \mathbf{z}) - (r, \nabla \cdot \mathbf{v})}{|||(\mathbf{z}, r)||| \, |||(\mathbf{v}, q)|||} \geq C_{\mathrm{S}}$$

HANNUKAINEN, STENBERG, VOHRALÍK
A unified framework for a posteriori error estimation for the Stokes problem
submitted [B2]

The Stokes problem

Theorem (A posteriori error estimate for the Stokes problem)

There holds

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K}+\eta_{\mathrm{DF},K})^2 + \eta_{\mathrm{D},K}^2 \right\}^{\frac{1}{2}}.$$

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_K$$

- *residual estimator*

$$\eta_{\mathrm{R},K} := 1/\pi h_K \|\nabla\cdot\mathbb{t}_h + \mathbf{f}\|_K$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},K} := \|\nabla\mathbf{s}_h - p_h\mathbb{I} - \mathbb{t}_h\|_K$$

- *divergence estimator*

$$\eta_{\mathrm{D},K} := \frac{\|\nabla\cdot\mathbf{s}_h\|_K}{\beta}$$

Properties

- **loc. eff.**, **unified framework** (DG, MFE, FV, FE/S, NCFE)

The Stokes problem

Theorem (A posteriori error estimate for the Stokes problem)

There holds

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq \left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{\frac{1}{2}} + \frac{1}{C_{\mathrm{S}}}\left\{\sum_{K\in\mathcal{T}_h}(\eta_{\mathrm{R},K}+\eta_{\mathrm{DF},K})^2+\eta_{\mathrm{D},K}^2\right\}^{\frac{1}{2}}.$$

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_K$$

- *residual estimator*

$$\eta_{\mathrm{R},K} := 1/\pi h_K \|\nabla\cdot\mathbb{t}_h + \mathbf{f}\|_K$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},K} := \|\nabla \mathbf{s}_h - p_h\mathbb{I} - \mathbb{t}_h\|_K$$

- *divergence estimator*

$$\eta_{\mathrm{D},K} := \frac{\|\nabla\cdot\mathbf{s}_h\|_K}{\beta}$$

Properties

- **loc. eff.**, **unified framework** (DG, MFE, FV, FE/S, NCFE)

The Stokes problem

Theorem (A posteriori error estimate for the Stokes problem)

There holds

$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \leq \left\{\sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2\right\}^{\frac{1}{2}} + \frac{1}{C_{\mathrm{S}}}\left\{\sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K}+\eta_{\mathrm{DF},K})^2 + \eta_{\mathrm{D},K}^2\right\}^{\frac{1}{2}}.$$

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_K$$

- *residual estimator*

$$\eta_{\mathrm{R},K} := 1/\pi h_K \|\nabla\cdot \mathbb{t}_h + \mathbf{f}\|_K$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},K} := \|\nabla \mathbf{s}_h - p_h \mathbb{I} - \mathbb{t}_h\|_K$$

- *divergence estimator*

$$\eta_{\mathrm{D},K} := \frac{\|\nabla\cdot \mathbf{s}_h\|_K}{\beta}$$

Properties

- **loc. eff.**, **unified framework** (DG, MFE, FV, FE/S, NCFE)

Outline

Multiscale, multinumerics, and mortars

Model problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Multiscale

- decomposition of the problem into h-scale subdomain problems and H-scale interface problems

Multinumerics

- different numerical methods in different parts of the domain

Mortar technique

- flux normal components conservativity imposed weakly
- following Bernardi, Maday, Patera (1994)

PENCHEVA, VOHRALÍK, WHEELER, WILDEY
Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling
submitted [B3]

Multiscale, multinumerics, and mortars

Model problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Multiscale

- decomposition of the problem into h-scale subdomain problems and H-scale interface problems

Multinumerics

- different numerical methods in different parts of the domain

Mortar technique

- flux normal components conservativity imposed weakly
- following Bernardi, Maday, Patera (1994)

PENCHEVA, VOHRALÍK, WHEELER, WILDEY
Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling
submitted [B3]

Multiscale, multinumerics, and mortars

Model problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Multiscale

- decomposition of the problem into h-scale subdomain problems and H-scale interface problems

Multinumerics

- different numerical methods in different parts of the domain

Mortar technique

- flux normal components conservativity imposed weakly
- following Bernardi, Maday, Patera (1994)

PENCHEVA, VOHRALÍK, WHEELER, WILDEY
Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling
submitted [B3]

Multiscale, multinumerics, and mortars

Model problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Multiscale

- decomposition of the problem into h-scale subdomain problems and H-scale interface problems

Multinumerics

- different numerical methods in different parts of the domain

Mortar technique

- flux normal components conservativity imposed weakly
- following Bernardi, Maday, Patera (1994)

PENCHEVA, VOHRALÍK, WHEELER, WILDEY
Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling
submitted [B3]

Multiscale, multinumerics, and mortars

Model problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Multiscale

- decomposition of the problem into h-scale subdomain problems and H-scale interface problems

Multinumerics

- different numerical methods in different parts of the domain

Mortar technique

- flux normal components conservativity imposed weakly
- following Bernardi, Maday, Patera (1994)

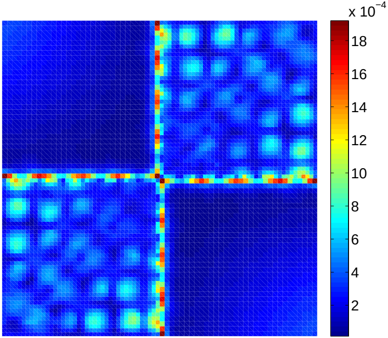
PENCHEVA, VOHRALÍK, WHEELER, WILDEY
Robust a posteriori error control and adaptivity for multiscale, multinumerics, and mortar coupling
submitted [B3]

Multiscale, multinumerics, and mortars

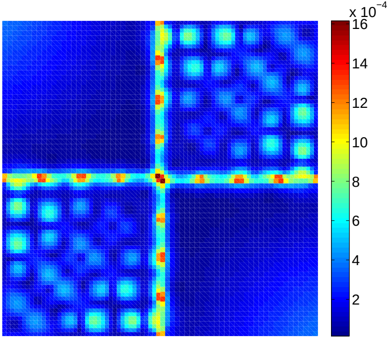
A posteriori error estimates

- potential reconstruction
- flux reconstruction
  - direct prescription
  - h-grid-size low order local Neumann MFE problems
  - H-grid-size high order local Neumann MFE problems
- **guaranteed**, locally efficient, **robust** with respect to the ratio H/h for sufficiently regular p

Multiscale, multinumerics, and mortars—num. exp.

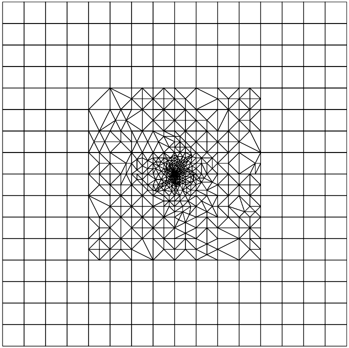


Estimated error distribution inside the subdomains and along mortar interfaces

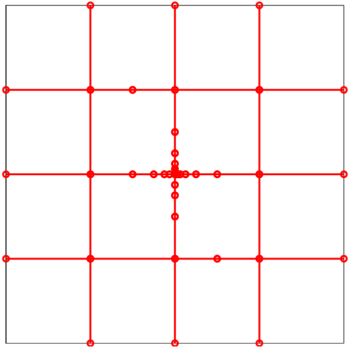


Exact error distribution inside the subdomains and along mortar interfaces

Multiscale, multinumerics, and mortars—num. exp.



Adapted mesh in multinumerics DG–MFE discretization



Corresponding adapted mortar mesh

Outline

A system of variational inequalities

Contact between two membranes

$$\begin{aligned}
-\mu_1 \Delta p_1 - \lambda &= f_1 \quad \text{in } \Omega\\
-\mu_2 \Delta p_2 + \lambda &= f_2 \quad \text{in } \Omega\\
p_1 - p_2 &\geq 0, \quad \lambda \geq 0, \quad (p_1 - p_2)\lambda = 0 \quad \text{in } \Omega\\
p_1 &= 0 \quad \text{on } \partial\Omega,\\
p_2 &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

Ben Belgacem, Bernardi, Blouza, Vohralík
On the unilateral contact between membranes. Part 2: A posteriori analysis and numerical experiments
submitted [B1]

A system of variational inequalities

Contact between two membranes

$$
\begin{aligned}
-\mu_1 \Delta p_1 - \lambda &= f_1 \quad \text{in } \Omega \\
-\mu_2 \Delta p_2 + \lambda &= f_2 \quad \text{in } \Omega \\
p_1 - p_2 &\geq 0, \quad \lambda \geq 0, \quad (p_1 - p_2)\lambda = 0 \quad \text{in } \Omega \\
p_1 &= 0 \quad \text{on } \partial\Omega, \\
p_2 &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

BEN BELGACEM, BERNARDI, BLOUZA, VOHRALÍK
On the unilateral contact between membranes. Part 2: A posteriori analysis and numerical experiments
submitted [B1]

A system of variational inequalities

Theorem (A posteriori error estimate for the contact between two membranes)

Let p_i be the weak solutions, $p_{h,i}$ the finite element approximations, and f_i pw. constant. Then

$$\left\{\sum_{i=1}^{2} \mu_i \|\nabla(p_i - p_{h,i})\|^2\right\}^{\frac{1}{2}} \leq \left\{\sum_{D\in\mathcal{D}_h}\left(\sum_{i=1}^{2}\left(\eta_{\mathrm{DF},D,i} + \eta_{\mathrm{R},D,i}\right)^2 + \eta_{\mathrm{C},D}\right)\right\}^{\frac{1}{2}}.$$

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D,i} := m_{D,i}\|f_i - \nabla\cdot\mathbf{t}_{h,i} - (-1)^i\lambda_h\|_D$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D,i} := \|\mu_i^{\frac{1}{2}}\nabla p_{h,i} + \mu_i^{-\frac{1}{2}}\mathbf{t}_{h,i}\|_D$$

- *contact estimator*

$$\eta_{\mathrm{C},D} := 2(p_{1,h} - p_{2,h}, \lambda_h)_D$$

Properties

- guaranteed, **locally efficient**, **optimal** (up to $\eta_{\mathrm{C},D}$)

A system of variational inequalities

Theorem (A posteriori error estimate for the contact between two membranes)

Let p_i be the weak solutions, $p_{h,i}$ the finite element approximations, and f_i pw. constant. Then

$$\left\{\sum_{i=1}^{2}\mu_i\|\nabla(p_i-p_{h,i})\|^2\right\}^{\frac{1}{2}}\leq\left\{\sum_{D\in\mathcal{D}_h}\left(\sum_{i=1}^{2}(\eta_{\mathrm{DF},D,i}+\eta_{\mathrm{R},D,i})^2+\eta_{\mathrm{C},D}\right)\right\}^{\frac{1}{2}}.$$

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D,i} := m_{D,i}\|f_i-\nabla\cdot\mathbf{t}_{h,i}-(-1)^i\lambda_h\|_D$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D,i} := \|\mu_i^{\frac{1}{2}}\nabla p_{h,i}+\mu_i^{-\frac{1}{2}}\mathbf{t}_{h,i}\|_D$$

- *contact estimator*

$$\eta_{\mathrm{C},D} := 2(p_{1,h}-p_{2,h},\lambda_h)_D$$

Properties

- guaranteed, **locally efficient**, **optimal** (up to $\eta_{\mathrm{C},D}$)

A system of variational inequalities

Theorem (A posteriori error estimate for the contact between two membranes)

Let p_i be the weak solutions, $p_{h,i}$ the finite element approximations, and f_i pw. constant. Then

$$\left\{\sum_{i=1}^{2} \mu_i \|\nabla(p_i - p_{h,i})\|^2\right\}^{\frac{1}{2}} \leq \left\{\sum_{D\in\mathcal{D}_h} \left(\sum_{i=1}^{2} \left(\eta_{\mathrm{DF},D,i} + \eta_{\mathrm{R},D,i}\right)^2 + \eta_{\mathrm{C},D}\right)\right\}^{\frac{1}{2}}.$$

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D,i} := m_{D,i} \|f_i - \nabla\cdot\mathbf{t}_{h,i} - (-1)^i \lambda_h\|_D$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D,i} := \|\mu_i^{\frac{1}{2}} \nabla p_{h,i} + \mu_i^{-\frac{1}{2}} \mathbf{t}_{h,i}\|_D$$

- *contact estimator*

$$\eta_{\mathrm{C},D} := 2(p_{1,h} - p_{2,h}, \lambda_h)_D$$

Properties

- guaranteed, **locally efficient**, **optimal** (up to $\eta_{\mathrm{C},D}$)

Outline

Outline

Nonlinear diffusion problem

Nonlinear diffusion problem

$$-\nabla\cdot\boldsymbol{\sigma}(\nabla p) = f \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega,$$

where $\forall \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\sigma}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi}$

Fixed-point linearization

$$\sigma_{\mathrm{L}}(\xi) := a(|\nabla p_0|)\xi$$

Newton linearization

$$\sigma_{\mathrm{L}}(\xi) := a(|\nabla p_0|)\xi + a'(|\nabla p_0|)\frac{1}{|\nabla p_0|}(\nabla p_0 \otimes \nabla p_0)(\xi - \nabla p_0)$$

El Alaoui, Ern, Vohralík
Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems
Comput. Methods Appl. Mech. Engrg. 2010 [A5]

Nonlinear diffusion problem

Nonlinear diffusion problem

$$-\nabla\cdot\boldsymbol{\sigma}(\nabla p) = f \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega,$$

where $\forall \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\sigma}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi}$

Fixed-point linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi}$$

Newton linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi} + a'(|\nabla p_0|)\frac{1}{|\nabla p_0|}(\nabla p_0 \otimes \nabla p_0)(\boldsymbol{\xi} - \nabla p_0)$$

El Alaoui, Ern, Vohralík
Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems
Comput. Methods Appl. Mech. Engrg. 2010 [A5]

Nonlinear diffusion problem

Nonlinear diffusion problem

$$-\nabla\cdot\boldsymbol{\sigma}(\nabla p) = f \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega,$$

where $\forall \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\sigma}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi}$

Fixed-point linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi}$$

Newton linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi} + a'(|\nabla p_0|)\frac{1}{|\nabla p_0|}(\nabla p_0 \otimes \nabla p_0)(\boldsymbol{\xi} - \nabla p_0)$$

El Alaoui, Ern, Vohralík
Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems
Comput. Methods Appl. Mech. Engrg. 2010 [A5]

Nonlinear diffusion problem

Nonlinear diffusion problem

$$\begin{aligned} -\nabla\cdot\boldsymbol{\sigma}(\nabla p) &= f && \text{in } \Omega, \\ p &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\forall \boldsymbol{\xi} \in \mathbb{R}^d$, $\boldsymbol{\sigma}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi}$

Fixed-point linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi}$$

Newton linearization

$$\boldsymbol{\sigma}_{\mathrm{L}}(\boldsymbol{\xi}) := a(|\nabla p_0|)\boldsymbol{\xi} + a'(|\nabla p_0|)\frac{1}{|\nabla p_0|}(\nabla p_0 \otimes \nabla p_0)(\boldsymbol{\xi} - \nabla p_0)$$

EL ALAOUI, ERN, VOHRALÍK
Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems
Comput. Methods Appl. Mech. Engrg. 2010 [A5]

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

Error measure

Nonlinear operator $A : V := W_0^{1,q}(\Omega) \to V'$

$$\langle Ap, \varphi \rangle_{V',V} := (\boldsymbol{\sigma}(\nabla p), \nabla \varphi)$$

Error measure

$$\mathcal{J}_p(p_{\mathrm{L},h}) := \|Ap - Ap_{\mathrm{L},h}\|_{V'} = \sup_{\varphi \in V \setminus \{0\}} \frac{(\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)}{\|\nabla \varphi\|_q}$$

- based on the difference of the fluxes
- dual norm of the residual
- inspired from Angermann (1995), Verfürth (2005), Chaillou and Suri (2006, 2007)
- avoids any appearance of the ratio continuity constant / monotonicity constant

A posteriori error estimate

Theorem (A posteriori error estimate)

Let

- $p \in V$ *be the weak solution,*
- $p_{\mathrm{L},h} \in V$ *be arbitrary,*
- $\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ *be a partition of* Ω*,*
- $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div}, \Omega)$ *be arbitrary but such that* $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ *for all* $D \in \mathcal{D}_h^{\mathrm{int}}$.

Then there holds

$$\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta := \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^r \right\}^{1/r} + \left\{ \sum_{D \in \mathcal{D}_h} \eta_{\mathrm{L},D}^r \right\}^{1/r}.$$

A posteriori error estimate

Theorem (A posteriori error estimate)

Let

- $p \in V$ *be the weak solution,*
- $p_{\mathrm{L},h} \in V$ *be arbitrary,*
- $\mathcal{D}_h = \mathcal{D}_h^{\mathrm{int}} \cup \mathcal{D}_h^{\mathrm{ext}}$ *be a partition of* Ω,
- $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div}, \Omega)$ *be arbitrary but such that* $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ *for all* $D \in \mathcal{D}_h^{\mathrm{int}}$.

Then there holds

$$\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta := \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^r \right\}^{1/r} + \left\{ \sum_{D \in \mathcal{D}_h} \eta_{\mathrm{L},D}^r \right\}^{1/r}.$$

Estimators

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D} := C_{\mathrm{P/F},q,D} h_D \|f - \nabla\cdot \mathbf{t}_h\|_{r,D}$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D} := \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_{r,D}$$

- *linearization estimator*

$$\eta_{\mathrm{L},D} := \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_{r,D}$$

Estimators

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D} := C_{\mathrm{P/F},q,D} h_D \|f - \nabla\cdot\mathbf{t}_h\|_{r,D}$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D} := \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_{r,D}$$

- *linearization estimator*

$$\eta_{\mathrm{L},D} := \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_{r,D}$$

Estimators

Estimators

- *residual estimator*

$$\eta_{\mathrm{R},D} := C_{\mathrm{P/F},q,D} h_D \|f - \nabla\cdot \mathbf{t}_h\|_{r,D}$$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},D} := \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_{r,D}$$

- *linearization estimator*

$$\eta_{\mathrm{L},D} := \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_{r,D}$$

Proof

Proof.

- take $\varphi \in V$ with $\|\nabla\varphi\|_q = 1$
- note that, for an arbitrary $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div}, \Omega)$ by the Green thm
  $(f, \varphi) - (\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi) = (f - \nabla\cdot\mathbf{t}_h, \varphi) - (\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi)$
- $(\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi) \le \|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r$ by the Hölder in.
- $\|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r \le \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_r$
  $+ \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_r$ by the triangle inequality
- for a locally conservative $\mathbf{t}_h$, $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1_D)$ $\forall D \in \mathcal{D}_h^{\mathrm{int}}$,

$$(f - \nabla\cdot\mathbf{t}_h, \varphi) = \sum_{D\in\mathcal{D}_h^{\mathrm{int}}} (f - \nabla\cdot\mathbf{t}_h, \varphi - \varphi_D)_D + \sum_{D\in\mathcal{D}_h^{\mathrm{ext}}} (f - \nabla\cdot\mathbf{t}_h, \varphi)_D$$

$$\le \sum_{D\in\mathcal{D}_h} C_{\mathrm{P/F},q,D} h_D \|f - \nabla\cdot\mathbf{t}_h\|_{r,D} \|\nabla\varphi\|_{q,D},$$

using the Hölder inequality and the generalized Poincaré and Friedrichs inequalities

Proof

Proof.

- take $\varphi \in V$ with $\|\nabla \varphi\|_q = 1$
- note that, for an arbitrary $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div}, \Omega)$ by the Green thm
  $$(f, \varphi) - (\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi) = (f - \nabla \cdot \mathbf{t}_h, \varphi) - (\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi)$$
- $(\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla \varphi) \leq \|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r$ by the Hölder in.
- $\|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r \leq \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_r$ $+ \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_r$ by the triangle inequality
- for a locally conservative $\mathbf{t}_h$, $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1_D)$ $\forall D \in \mathcal{D}_h^{\mathrm{int}}$,
  $$(f - \nabla \cdot \mathbf{t}_h, \varphi) = \sum_{D \in \mathcal{D}_h^{\mathrm{int}}} (f - \nabla \cdot \mathbf{t}_h, \varphi - \varphi_D)_D + \sum_{D \in \mathcal{D}_h^{\mathrm{ext}}} (f - \nabla \cdot \mathbf{t}_h, \varphi)_D$$
  $$\leq \sum_{D \in \mathcal{D}_h} C_{\mathrm{P/F},q,D} h_D \|f - \nabla \cdot \mathbf{t}_h\|_{r,D} \|\nabla \varphi\|_{q,D},$$
  using the Hölder inequality and the generalized Poincaré and Friedrichs inequalities

Proof

Proof.

- take $\varphi \in V$ with $\|\nabla\varphi\|_q = 1$
- note that, for an arbitrary $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div},\Omega)$ by the Green thm
$$(f,\varphi)-(\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi) = (f-\nabla\cdot\mathbf{t}_h,\varphi)-(\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi)$$
- $(\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi) \le \|\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r$ by the Hölder in.
- $\|\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r \le \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})+\mathbf{t}_h\|_r$ $+\|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})-\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_r$ by the triangle inequality
- for a locally conservative $\mathbf{t}_h$, $(\nabla\cdot\mathbf{t}_h,1)_D = (f,1_D)\ \forall D\in\mathcal{D}_h^{\mathrm{int}}$,
$$(f-\nabla\cdot\mathbf{t}_h,\varphi)=\sum_{D\in\mathcal{D}_h^{\mathrm{int}}}(f-\nabla\cdot\mathbf{t}_h,\varphi-\varphi_D)_D+\sum_{D\in\mathcal{D}_h^{\mathrm{ext}}}(f-\nabla\cdot\mathbf{t}_h,\varphi)_D$$
$$\le\sum_{D\in\mathcal{D}_h} C_{\mathrm{P/F},q,D}h_D\|f-\nabla\cdot\mathbf{t}_h\|_{r,D}\|\nabla\varphi\|_{q,D},$$
using the Hölder inequality and the generalized Poincaré and Friedrichs inequalities

Proof

Proof.

- take $\varphi \in V$ with $\|\nabla\varphi\|_q = 1$
- note that, for an arbitrary $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div}, \Omega)$ by the Green thm
  $$(f, \varphi) - (\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi) = (f - \nabla{\cdot}\mathbf{t}_h, \varphi) - (\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi)$$
- $(\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}), \nabla\varphi) \leq \|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r$ by the Hölder in.
- $\|\mathbf{t}_h + \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r \leq \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) + \mathbf{t}_h\|_r$ $+ \|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}) - \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_r$ by the triangle inequality
- for a locally conservative $\mathbf{t}_h$, $(\nabla{\cdot}\mathbf{t}_h, 1)_D = (f, 1_D)$ $\forall D \in \mathcal{D}_h^{\mathrm{int}}$,
  $$(f - \nabla{\cdot}\mathbf{t}_h, \varphi) = \sum_{D\in\mathcal{D}_h^{\mathrm{int}}} (f - \nabla{\cdot}\mathbf{t}_h, \varphi - \varphi_D)_D + \sum_{D\in\mathcal{D}_h^{\mathrm{ext}}} (f - \nabla{\cdot}\mathbf{t}_h, \varphi)_D$$
  $$\leq \sum_{D\in\mathcal{D}_h} C_{\mathrm{P/F},q,D} h_D \|f - \nabla{\cdot}\mathbf{t}_h\|_{r,D} \|\nabla\varphi\|_{q,D},$$
  using the Hölder inequality and the generalized Poincaré and Friedrichs inequalities

Proof

Proof.

- take $\varphi \in V$ with $\|\nabla\varphi\|_q = 1$
- note that, for an arbitrary $\mathbf{t}_h \in \mathbf{H}^r(\mathrm{div},\Omega)$ by the Green thm
  $$(f,\varphi)-(\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi) = (f-\nabla{\cdot}\mathbf{t}_h,\varphi)-(\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi)$$
- $(\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h}),\nabla\varphi) \le \|\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r$ by the Hölder in.
- $\|\mathbf{t}_h+\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_r \le \|\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})+\mathbf{t}_h\|_r$
  $+\|\boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})-\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h})\|_r$ by the triangle inequality
- for a locally conservative $\mathbf{t}_h$, $(\nabla{\cdot}\mathbf{t}_h,1)_D = (f,1_D)\ \forall D\in\mathcal{D}_h^{\mathrm{int}}$,
  $$(f-\nabla{\cdot}\mathbf{t}_h,\varphi)=\sum_{D\in\mathcal{D}_h^{\mathrm{int}}}(f-\nabla{\cdot}\mathbf{t}_h,\varphi-\varphi_D)_D+\sum_{D\in\mathcal{D}_h^{\mathrm{ext}}}(f-\nabla{\cdot}\mathbf{t}_h,\varphi)_D$$
  $$\le\sum_{D\in\mathcal{D}_h} C_{\mathrm{P/F},q,D}h_D\|f-\nabla{\cdot}\mathbf{t}_h\|_{r,D}\|\nabla\varphi\|_{q,D},$$
  using the Hölder inequality and the generalized Poincaré and Friedrichs inequalities

Balancing the discretization and linearization errors

Global linearization stopping criterion

- stop the Newton (or fixed-point) linearization whenever

$$\eta_{\mathrm{L}} \leq \gamma\, \eta_{\mathrm{D}},$$

where

$$\eta_{\mathrm{L}} := \left\{ \sum_{D \in \mathcal{D}_h} \eta_{\mathrm{L},D}^r \right\}^{1/r} \quad \text{linearization error},$$

$$\eta_{\mathrm{D}} := \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^r \right\}^{1/r} \quad \text{discretization error}$$

Local linearization stopping criterion

- stop the Newton (or fixed-point) linearization whenever

$$\eta_{\mathrm{L},D} \leq \gamma_D\, (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D}) \qquad \forall D \in \mathcal{D}_h$$

Balancing the discretization and linearization errors

Global linearization stopping criterion

- stop the Newton (or fixed-point) linearization whenever

$$\eta_{\mathrm{L}} \leq \gamma\, \eta_{\mathrm{D}},$$

where

$$\eta_{\mathrm{L}} := \left\{ \sum_{D \in \mathcal{D}_h} \eta_{\mathrm{L},D}^r \right\}^{1/r} \quad \text{linearization error},$$

$$\eta_{\mathrm{D}} := \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^r \right\}^{1/r} \quad \text{discretization error}$$

Local linearization stopping criterion

- stop the Newton (or fixed-point) linearization whenever

$$\eta_{\mathrm{L},D} \leq \gamma_D \left(\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D}\right) \qquad \forall D \in \mathcal{D}_h$$

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criterion, with γ_D small enough, hold. Then

$$\eta_{\mathrm{L},D} + \eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \le C \|\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_{r,D},$$

where the constant C is independent of $\mathbf{a}$ and q.

- local efficiency, but in a different norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criterion, with γ_D small enough, hold. Then

$$\eta_{\mathrm{L},D} + \eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C \|\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_{r,D},$$

where the constant C is independent of a and q.

- local efficiency, but in a different norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criterion, with γ_D small enough, hold. Then

$$\eta_{\mathrm{L},D} + \eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \leq C\|\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_{r,D},$$

where the constant C is independent of a and q.

- local efficiency, but in a different norm

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criterion, with γ small enough, hold. Recall that $\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta$. Then

$$\eta \leq C \mathcal{J}_p(p_{\mathrm{L},h}),$$

where the constant C is independent of a and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criterion, with γ small enough, hold. Recall that $\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta$. Then

$$\eta \leq C \mathcal{J}_p(p_{\mathrm{L},h}),$$

where the constant C is independent of a and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criterion, with γ small enough, hold. Recall that $\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta$. Then

$$\eta \leq C\mathcal{J}_p(p_{\mathrm{L},h}),$$

where the constant C is independent of a and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criterion, with γ small enough, hold. Recall that $\mathcal{J}_p(p_{\mathrm{L},h}) \leq \eta$. Then

$$\eta \leq C\mathcal{J}_p(p_{\mathrm{L},h}),$$

where the constant C is independent of a and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm

Main elements of the proof

Lemma (Approximation property of the reconstructed flux)

There holds

$$\eta_{\mathrm{DF},D} \lesssim \eta_{\mathrm{res},D}$$

$$:= \left\{ \sum_{K \in \mathcal{S}_D} h_K^r \| f + \nabla \cdot \boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) \|_{r,K}^r + \sum_{F \in \mathcal{G}_D^{\mathcal{T}}} h_F \| [\![\boldsymbol{\sigma}_{\mathrm{L}}(\nabla p_{\mathrm{L},h}) \cdot \mathbf{n}]\!] \|_{r,F}^r \right\}^{\frac{1}{r}}$$

- uses the construction of $\mathbf{t}_h$ from $p_{\mathrm{L},h}$
- properties of Raviart–Thomas–Nédélec spaces, scaling arguments, equivalence of norms on finite-dimensions spaces
- q-independent inverse inequalities
- local postprocessing of potentials from MFE
- Hölder inequality, generalized discrete Poincaré and Friedrichs inequalities

Main elements of the proof

Lemma (Local efficiency of residual estimators)

There holds

$$\eta_{\mathrm{res},D} \lesssim \|\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_{r,D} + \eta_{\mathrm{L},D}.$$

- element and face bubble functions, extension operators
- q-independent estimates
- Green theorem, Hölder inequality, duality

Lemma (Global efficiency of residual estimators)

There holds

$$\eta_{\mathrm{res}} \lesssim \|Ap - Ap_{\mathrm{L},h}\|_{V'} + \eta_{\mathrm{L}}.$$

- extension operators
- dual norms

Main elements of the proof

Lemma (Local efficiency of residual estimators)

There holds

$$\eta_{\mathrm{res},D} \lesssim \|\boldsymbol{\sigma}(\nabla p) - \boldsymbol{\sigma}(\nabla p_{\mathrm{L},h})\|_{r,D} + \eta_{\mathrm{L},D}.$$

- element and face bubble functions, extension operators
- q-independent estimates
- Green theorem, Hölder inequality, duality

Lemma (Global efficiency of residual estimators)

There holds

$$\eta_{\mathrm{res}} \lesssim \|Ap - Ap_{\mathrm{L},h}\|_{V'} + \eta_{\mathrm{L}}.$$

- extension operators
- dual norms

Adaptive strategy

Adaptive strategy

- choose an initial mesh $\mathcal{T}_h^0$ and an initial guess $p^0_{\mathrm{L},h} \in V_h(\mathcal{T}_h^0)$
- on the mesh $\mathcal{T}_h^j$, $j \geq 0$, for $i \geq 1$, do the iterative loop:
  1) linearize the flux function at $p^{i-1}_{\mathrm{L},h}$
  2) solve the discrete linearized problem for $p^i_{\mathrm{L},h}$
  3) if the linearization stopping criterion is reached, then stop the linearization, else set $i \leftarrow (i+1)$ and go to step 1)
- evaluate the overall a posteriori error estimate η
- if the desired overall **precision is reached**, then **stop**, else refine the mesh adaptively, interpolate to it the current solution, $j \leftarrow (j+1)$, and go to the second step

Adaptive strategy

Adaptive strategy

- choose an initial mesh $\mathcal{T}_h^0$ and an initial guess $p_{\mathrm{L},h}^0 \in V_h(\mathcal{T}_h^0)$
- on the mesh $\mathcal{T}_h^j$, $j \geq 0$, for $i \geq 1$, do the iterative loop:
  1) linearize the flux function at $p_{\mathrm{L},h}^{i-1}$
  2) solve the discrete linearized problem for $p_{\mathrm{L},h}^i$
  3) if the linearization stopping criterion is reached, then stop the linearization, else set $i \leftarrow (i+1)$ and go to step 1)
- evaluate the overall a posteriori error estimate η
- if the desired overall **precision is reached**, then **stop**, else refine the mesh adaptively, interpolate to it the current solution, $j \leftarrow (j+1)$, and go to the second step

Adaptive strategy

Adaptive strategy

- choose an initial mesh $\mathcal{T}_h^0$ and an initial guess $p_{\mathrm{L},h}^0 \in V_h(\mathcal{T}_h^0)$
- on the mesh $\mathcal{T}_h^j$, $j \geq 0$, for $i \geq 1$, do the iterative loop:
  1) linearize the flux function at $p_{\mathrm{L},h}^{i-1}$
  2) solve the discrete linearized problem for $p_{\mathrm{L},h}^i$
  3) if the linearization stopping criterion is reached, then stop the linearization, else set $i \leftarrow (i+1)$ and go to step 1)
- evaluate the overall a posteriori error estimate η
- if the desired overall **precision is reached**, then **stop**, else refine the mesh adaptively, interpolate to it the current solution, $j \leftarrow (j+1)$, and go to the second step

Adaptive strategy

Adaptive strategy

- choose an initial mesh $\mathcal{T}_h^0$ and an initial guess $p_{\mathrm{L},h}^0 \in V_h(\mathcal{T}_h^0)$
- on the mesh $\mathcal{T}_h^j$, $j \geq 0$, for $i \geq 1$, do the iterative loop:
  1) linearize the flux function at $p_{\mathrm{L},h}^{i-1}$
  2) solve the discrete linearized problem for $p_{\mathrm{L},h}^i$
  3) if the linearization stopping criterion is reached, then stop the linearization, else set $i \leftarrow (i+1)$ and go to step 1)
- evaluate the overall a posteriori error estimate η
- if the desired overall **precision is reached**, then **stop**, else refine the mesh adaptively, interpolate to it the current solution, $j \leftarrow (j+1)$, and go to the second step

Numerical experiment I

Model problem

- q-Laplacian

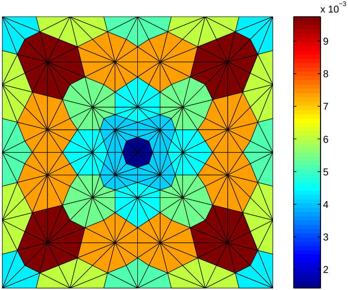
$$\begin{aligned} \nabla\cdot(|\nabla p|^{q-2}\nabla p) &= f && \text{in } \Omega, \\ p &= p_0 && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose a Dirichlet BC)

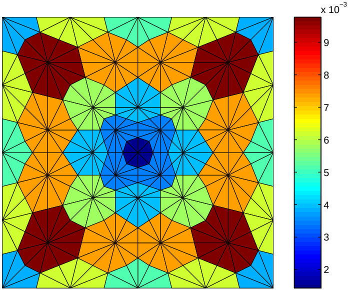
$$p(x,y) = -\frac{q-1}{q}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{q}{2(q-1)}} + \frac{q-1}{q}\left(\frac{1}{2}\right)^{\frac{q}{q-1}}$$

- tested values $q = 1.4, 3, 10, 50$

Error distribution on a uniformly refined mesh, $q = 3$

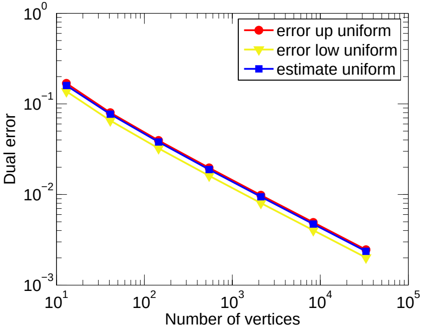


Estimated error distribution

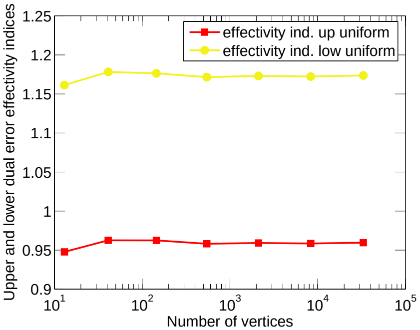


Exact error distribution

Estimated and actual errors and the eff. index, $q = 1.4$

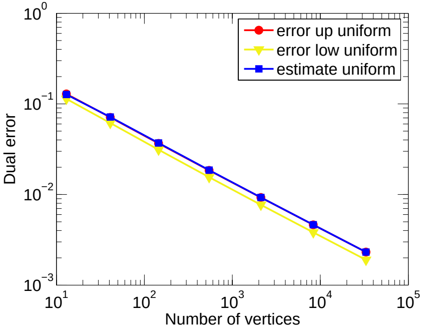


Estimated and actual errors

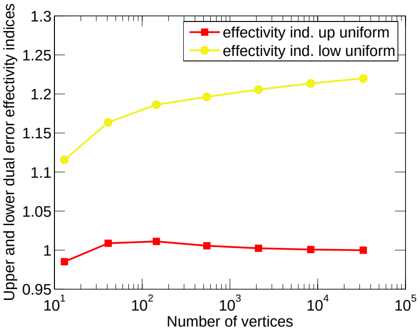


Effectivity index

Estimated and actual errors and the eff. index, $q = 3$

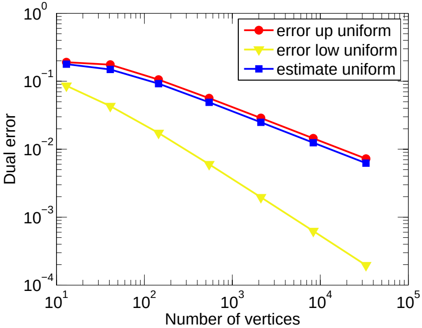


Estimated and actual errors

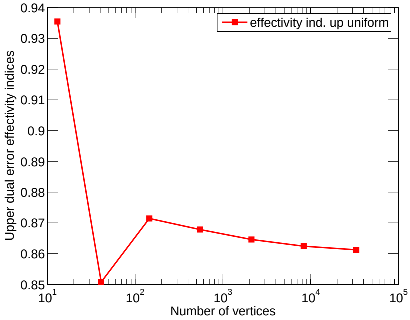


Effectivity index

Estimated and actual errors and the eff. index, $q = 10$

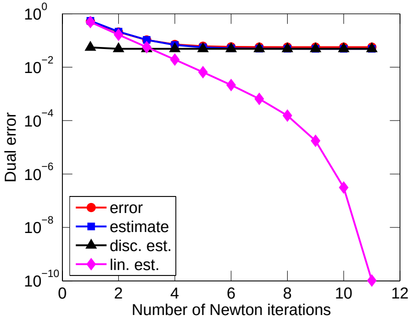


Estimated and actual errors

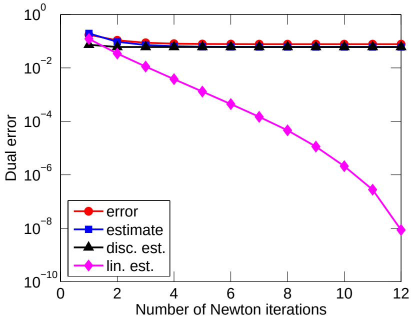


Effectivity index

Discretization and linearization components

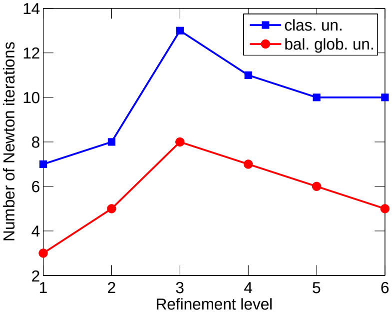


Case $q = 10$

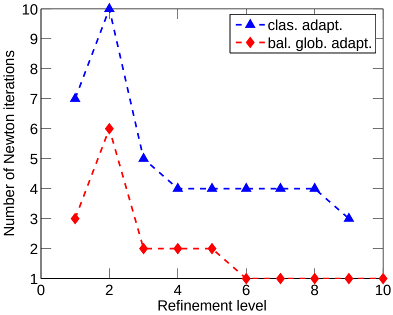


Case $q = 50$

Evolution of Newton iterations



Classical versus balanced Newton, uniform refinement



Classical versus balanced Newton, adaptive ref.

Numerical experiment II

Model problem

- q-Laplacian

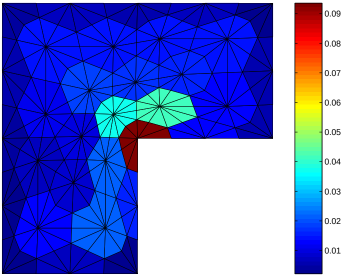
$$\nabla\cdot(|\nabla p|^{q-2}\nabla p) = f \qquad \text{in } \Omega,$$
$$p = p_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose a Dirichlet BC)

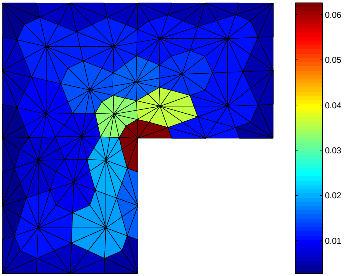
$$p_0(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $q = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))

Error distribution on a uniformly refined mesh

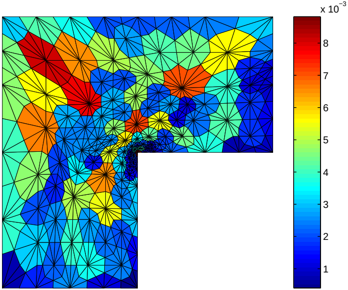


Estimated error distribution

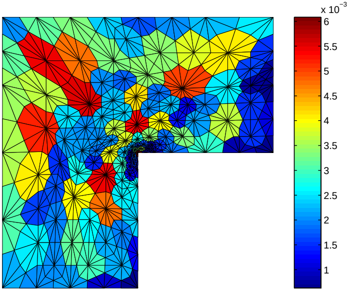


Exact error distribution

Error distribution on an adaptively refined mesh



Estimated error distribution



Exact error distribution

Outline

A model diffusion problem

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

JIRÁNEK, STRAKOŠ, VOHRALÍK
A posteriori error estimates including algebraic error and stopping criteria for iterative solvers
SIAM J. Sci. Comput. 2010 [A10]

A model diffusion problem

A model diffusion problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

JIRÁNEK, STRAKOŠ, VOHRALÍK
A posteriori error estimates including algebraic error and stopping criteria for iterative solvers
SIAM J. Sci. Comput. 2010 [A10]

A model diffusion problem

A model diffusion problem

$$-\nabla\cdot(\mathbf{S}\nabla p) = f \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

JIRÁNEK, STRAKOŠ, VOHRALÍK
A posteriori error estimates including algebraic error and stopping criteria for iterative solvers
SIAM J. Sci. Comput. 2010 [A10]

A model diffusion problem

A model diffusion problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

JIRÁNEK, STRAKOŠ, VOHRALÍK
A posteriori error estimates including algebraic error and stopping criteria for iterative solvers
SIAM J. Sci. Comput. 2010 [A10]

A model diffusion problem

A model diffusion problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Jiránek, Strakoš, Vohralík
A posteriori error estimates including algebraic error and stopping criteria for iterative solvers
SIAM J. Sci. Comput. 2010 [A10]

A posteriori estimate including the algebraic error

Theorem (Estimate including the algebraic error, FV/MFE)

There holds

$$|||p - \tilde{p}_h^{\mathrm{a}}||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^{\mathrm{a}} - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^{\mathrm{a}})|||_K$
  - reflects the departure of $\tilde{p}_h^{\mathrm{a}}$ from $H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2}}{c_{\mathbf{S},K}^{1/2}} h_K \|f - f_K\|_K$
  - reflects data oscillation
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{q}_h\right\|_K$
  - $\mathbf{q}_h = \arg\inf_{\substack{\mathbf{r}_h \in \mathbf{RTN}(\mathcal{T}_h) \\ \nabla\cdot\mathbf{r}_h|_K = R_K/|K|}} \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{r}_h\right\|$
  - measures the algebraic error

A posteriori estimate including the algebraic error

Theorem (Estimate including the algebraic error, FV/MFE)

There holds

$$|||p - \tilde{p}_h^{\mathrm{a}}||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^{\mathrm{a}} - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^{\mathrm{a}})|||_K$
  - reflects the departure of $\tilde{p}_h^{\mathrm{a}}$ from $H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2}}{c_{\mathbf{S},K}^{1/2}} h_K \|f - f_K\|_K$
  - reflects data oscillation
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \left\|\mathbf{S}^{-\frac{1}{2}}\mathbf{q}_h\right\|_K$
  - $\mathbf{q}_h = \arg\inf_{\substack{\mathbf{r}_h \in \mathbf{RTN}(\mathcal{T}_h) \\ \nabla\cdot\mathbf{r}_h|_K = R_K/|K|}} \left\|\mathbf{S}^{-\frac{1}{2}}\mathbf{r}_h\right\|$
  - measures the algebraic error

A posteriori estimate including the algebraic error

Theorem (Estimate including the algebraic error, FV/MFE)

There holds

$$|||p - \tilde{p}_h^{\mathrm{a}}||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^{\mathrm{a}} - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^{\mathrm{a}})|||_K$
  - reflects the departure of $\tilde{p}_h^{\mathrm{a}}$ from $H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2}}{c_{\mathbf{S},K}^{1/2}} h_K \|f - f_K\|_K$
  - reflects data oscillation
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{q}_h\right\|_K$
  - $\mathbf{q}_h = \arg\inf_{\substack{\mathbf{r}_h \in \mathbf{RTN}(\mathcal{T}_h) \\ \nabla\cdot\mathbf{r}_h|_K = R_K/|K|}} \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{r}_h\right\|$
  - measures the algebraic error

A posteriori estimate including the algebraic error

Theorem (Estimate including the algebraic error, FV/MFE)

There holds

$$|||p - \tilde{p}_h^{\mathrm{a}}||| \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^{\mathrm{a}} - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^{\mathrm{a}})|||_K$
  - reflects the departure of $\tilde{p}_h^{\mathrm{a}}$ from $H_0^1(\Omega)$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := \frac{C_{\mathrm{P}}^{1/2}}{c_{\mathbf{S},K}^{1/2}} h_K \|f - f_K\|_K$
  - reflects data oscillation
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{q}_h\right\|_K$
  - $\mathbf{q}_h = \arg\inf_{\substack{\mathbf{r}_h \in \mathbf{RTN}(\mathcal{T}_h) \\ \nabla \cdot \mathbf{r}_h|_K = R_K/|K|}} \left\|\mathbf{S}^{-\frac{1}{2}} \mathbf{r}_h\right\|$
  - measures the algebraic error

Stopping criteria for iterative solvers

Global stopping criterion (global efficiency)

- stop the iterative solver whenever

$$\eta_{\mathrm{AE}} \leq \gamma \, \eta_{\mathrm{NC}},$$

where

$$\eta_{\mathrm{AE}} = \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}, \quad \eta_{\mathrm{NC}} = \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}}$$

Local stopping criterion (local efficiency)

- stop the iterative solver whenever

$$\eta_{\mathrm{AE},K} \leq \gamma_K \, \eta_{\mathrm{NC},K} \qquad \forall K \in \mathcal{T}_h$$

Stopping criteria for iterative solvers

Global stopping criterion (global efficiency)

- stop the iterative solver whenever

$$\eta_{\mathrm{AE}} \leq \gamma\, \eta_{\mathrm{NC}},$$

where

$$\eta_{\mathrm{AE}} = \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}, \quad \eta_{\mathrm{NC}} = \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}}$$

Local stopping criterion (local efficiency)

- stop the iterative solver whenever

$$\eta_{\mathrm{AE},K} \leq \gamma_K\, \eta_{\mathrm{NC},K} \qquad \forall K \in \mathcal{T}_h$$

Local efficiency

Theorem (Local efficiency of the a posteriori error estimate)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criterion hold. Then

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{AE},K} \leq (1 + \gamma_K)(C C_{\mathbf{S},K}^{\frac{1}{2}} c_{\mathbf{S},\mathcal{T}_K}^{-\frac{1}{2}} |||p - \tilde{p}_h^{\mathrm{a}}|||_{\mathcal{T}_K} + \text{ h.o.t.}),$$

where the constant C depends only on the space dimension and on the shape regularity parameter.

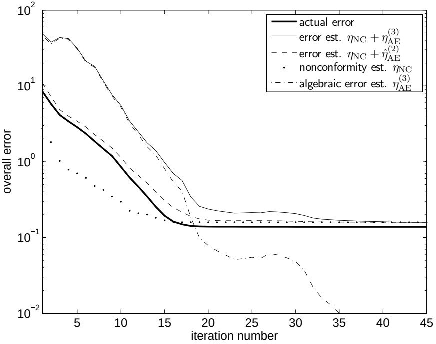
Global efficiency

Theorem (Global efficiency of the a posteriori error estimate)

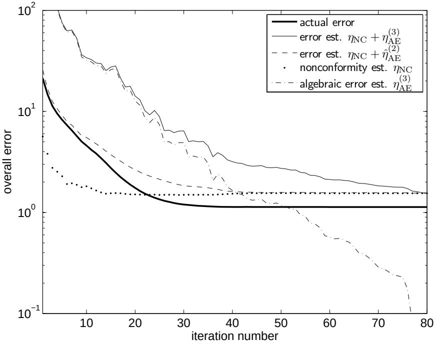
Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criterion hold. Then

$$\eta_{\mathrm{NC}} + \eta_{\mathrm{AE}} \le C(1+\gamma)(|||p - \tilde{p}_h^{\mathrm{a}}||| + \text{ h.o.t.}).$$

Overall error and overall error estimators

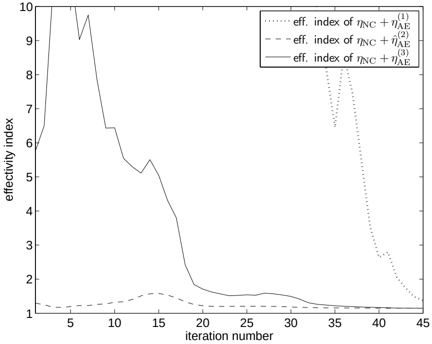


Case 1

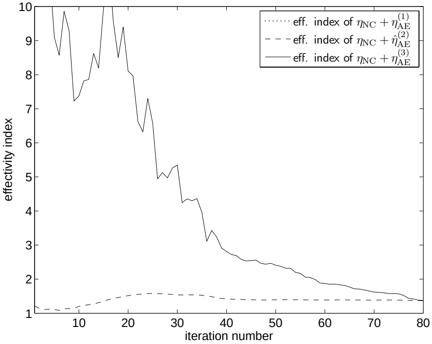


Case 2

Effectivity indices of the overall error estimators



Case 1



Case 2

Outline

Two-phase flows

Two-phase flow problem

$$
\begin{aligned}
\partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha && \text{in } \Omega \times (0,T),\ \alpha \in \{\mathrm{o}, \mathrm{w}\},\\
-\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha && \text{in } \Omega \times (0,T),\ \alpha \in \{\mathrm{o}, \mathrm{w}\},\\
s_\mathrm{o} + s_\mathrm{w} &= 1 && \text{in } \Omega \times (0,T),\\
p_\mathrm{o} - p_\mathrm{w} &= p_c(s_\mathrm{w}) && \text{in } \Omega \times (0,T)
\end{aligned}
$$

VOHRALÍK
A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows
CRAS note in preparation [B4]

CANCÈS, POP, VOHRALÍK
Rigorous a posteriori error estimates for two-phase flows
in preparation

Two-phase flows

Two-phase flow problem

$$\begin{aligned} \partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha && \text{in } \Omega\times(0,T),\ \alpha\in\{\mathrm{o},\mathrm{w}\},\\ -\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha + \rho_\alpha g\nabla z) &= \mathbf{u}_\alpha && \text{in } \Omega\times(0,T),\ \alpha\in\{\mathrm{o},\mathrm{w}\},\\ s_\mathrm{o} + s_\mathrm{w} &= 1 && \text{in } \Omega\times(0,T),\\ p_\mathrm{o} - p_\mathrm{w} &= p_c(s_\mathrm{w}) && \text{in } \Omega\times(0,T) \end{aligned}$$

VOHRALÍK
A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows
CRAS note in preparation [B4]

CANCÈS, POP, VOHRALÍK
Rigorous a posteriori error estimates for two-phase flows
in preparation

Two-phase flows

Theorem (A posteriori error estimate for two-phase flows)

$$|||(s_\alpha - s_{\alpha,h\tau}, p_\alpha - p_{\alpha,h\tau})||| \leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta^n_{\mathrm{R},K,\alpha} + \eta^n_{\mathrm{DF},K,\alpha}(t))^2 \mathrm{d}t \right\}^{\frac{1}{2}}$$

$$+ \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta^n_{\mathrm{NC},K,\alpha}(t))^2 \, \mathrm{d}t \right\}^{\frac{1}{2}}.$$

Two-phase flows

Theorem (A posteriori error estimate distinguishing the different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

and the corresponding approximations $(s_{\alpha,h\tau}^{k,i}, p_{\alpha,h\tau}^{k,i})$. *Then*

$$|||(s_\alpha - s_{\alpha,h\tau}^{k,i}, p_\alpha - p_{\alpha,h\tau}^{k,i})|||_{I_n} \leq \eta_{\mathrm{sp},\alpha}^{n,k,i} + \eta_{\mathrm{tm},\alpha}^{n,k,i} + \eta_{\mathrm{lin},\alpha}^{n,k,i} + \eta_{\mathrm{alg},\alpha}^{n,k,i}.$$

Estimators

- $\eta_{\mathrm{sp},\alpha}^{n,k,i}$: *spatial estimator,*
- $\eta_{\mathrm{tm},\alpha}^{n,k,i}$: *temporal estimator*
- $\eta_{\mathrm{lin},\alpha}^{n,k,i}$: *linearization estimator*
- $\eta_{\mathrm{alg},\alpha}^{n,k,i}$: *algebraic estimator*

Two-phase flows

Theorem (A posteriori error estimate distinguishing the different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

and the corresponding approximations $(s_{\alpha,h\tau}^{k,i}, p_{\alpha,h\tau}^{k,i})$*. Then*

$$|||(s_\alpha - s_{\alpha,h\tau}^{k,i}, p_\alpha - p_{\alpha,h\tau}^{k,i})|||_{I_n} \leq \eta_{\mathrm{sp},\alpha}^{n,k,i} + \eta_{\mathrm{tm},\alpha}^{n,k,i} + \eta_{\mathrm{lin},\alpha}^{n,k,i} + \eta_{\mathrm{alg},\alpha}^{n,k,i}.$$

Estimators

- $\eta_{\mathrm{sp},\alpha}^{n,k,i}$: *spatial estimator,*
- $\eta_{\mathrm{tm},\alpha}^{n,k,i}$: *temporal estimator*
- $\eta_{\mathrm{lin},\alpha}^{n,k,i}$: *linearization estimator*
- $\eta_{\mathrm{alg},\alpha}^{n,k,i}$: *algebraic estimator*

Outline

Outline

Primal formulation-based a priori analysis of MFE

A model diffusion problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Mixed finite element method

Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \Phi_h$ such that

$$
\begin{aligned}
(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) = 0 &\qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h,\\
(\nabla\cdot\mathbf{u}_h, \phi_h) = (f, \phi_h) &\qquad \forall \phi_h \in \Phi_h
\end{aligned}
$$

VOHRALÍK
Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods
Math. Comp. 2010 [A14]

Primal formulation-based a priori analysis of MFE

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \Phi_h$ such that

$$\begin{aligned} (\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) &= 0 && \forall\, \mathbf{v}_h \in \mathbf{V}_h,\\ (\nabla\cdot\mathbf{u}_h, \phi_h) &= (f, \phi_h) && \forall \phi_h \in \Phi_h \end{aligned}$$

VOHRALÍK
Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods
Math. Comp. 2010 [A14]

Primal formulation-based a priori analysis of MFE

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \Phi_h$ such that

$$\begin{aligned} (\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) &= 0 && \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla\cdot\mathbf{u}_h, \phi_h) &= (f, \phi_h) && \forall \phi_h \in \Phi_h \end{aligned}$$

VOHRALÍK
Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods
Math. Comp. 2010 [A14]

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- prove uniform-in-h discrete inf–sup condition
- get

$$\|p - p_h\| \leq C\|p - I(p)\|$$

Present a priori analysis

- get

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)

$$|||\mathbf{u} - \mathbf{u}_h|||_* \le |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- prove uniform-in-h discrete inf–sup condition
- get

$$\|p - p_h\| \le C\|p - I(p)\|$$

Present a priori analysis

- get

$$|||\mathbf{u} - \mathbf{u}_h|||_* \le |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- prove uniform-in-h discrete inf–sup condition
- get

$$\|p - p_h\| \leq C\|p - I(p)\|$$

Present a priori analysis

- get

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- prove uniform-in-h discrete inf–sup condition
- get

$$\|p - p_h\| \leq C\|p - I(p)\|$$

Present a priori analysis

- get

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- prove uniform-in-h discrete inf–sup condition
- get

$$\|p - p_h\| \leq C\|p - I(p)\|$$

Present a priori analysis

- get

$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$

- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Traditional a priori analysis

- get (with $|||\mathbf{v}|||_* := \|\mathbf{S}^{-\frac{1}{2}}\mathbf{v}\|$)
$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$
- prove uniform-in-h discrete inf–sup condition
- get
$$\|p - p_h\| \leq C\|p - I(p)\|$$

Present a priori analysis

- get
$$|||\mathbf{u} - \mathbf{u}_h|||_* \leq |||\mathbf{u} - \mathbf{I}_{\mathbf{V}_h}(\mathbf{u})|||_*$$
- use local postprocessing $\tilde{p}_h$: $P_{\mathbf{V}_h}(\tilde{p}_h) = \mathbf{u}_h$, $P_{\Phi_h}(\tilde{p}_h) = p_h$ (Arnold and Brezzi (1985), Arbogast and Chen (1995))
- get optimal estimate for $|||p - \tilde{p}_h|||$ from that on $|||\mathbf{u} - \mathbf{u}_h|||_*$
- get optimal estimate for $\|p - \tilde{p}_h\|$ by the discrete Friedrichs inequality
- recover estimates for $\|p - p_h\|$

Primal formulation-based a priori analysis of MFE

Extensions

- unified a priori and a posteriori analysis
- optimal a posteriori error estimates
- the weak solution is the orthogonal projection of the postprocessed mixed finite element approximation onto $H_0^1(\Omega)$
- links between mixed finite element approximations and some generalized weak solutions

Outline

Inexpensive implementations of MFE, their link to FV

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- matrix indefinite, of saddle-point-type

Finite volume methods

$$\mathbb{S}P = H$$

VOHRALÍK, WOHLMUTH
Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods
submitted [B5]

Inexpensive implementations of MFE, their link to FV

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- matrix indefinite, of saddle-point-type

Finite volume methods

$$\mathbb{S}P = H$$

VOHRALÍK, WOHLMUTH
Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods
submitted [B5]

Inexpensive implementations of MFE, their link to FV

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- matrix indefinite, of saddle-point-type

Finite volume methods

$$\mathbb{S}P = H$$

VOHRALÍK, WOHLMUTH
Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods
submitted [B5]

Inexpensive implementations of MFE, their link to FV

A model diffusion problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Mixed finite element method

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- matrix indefinite, of saddle-point-type

Finite volume methods

$$\mathbb{S}P = H$$

VOHRALÍK, WOHLMUTH
Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods
submitted [B5]

Inexpensive implementations of MFE, their link to FV

Goals

- reduce equivalently MFE to FV
- get local flux expressions for MFE
- establish links between MFEs and FVs
- achieve, if possible, symmetric and positive definite matrix $\mathbb{S}$
- give a unified framework for such approaches
- achieve inexpensive implementations

Outline

Conclusions

A posteriori error estimates of this habilitation

- satisfy as much as possible the **five optimal properties**
- are typically simultaneously **guaranteed** and **robust**
- are derived in **unified frameworks** for various numerical methods
- lead to **adaptive algorithms** yielding **efficient calculations** through **stopping criteria** and **error components equilibration**

Other contributions

- inexpensive implementations
- relations between different numerical methods
- improvement of approximate solutions by local postprocessing

Conclusions

A posteriori error estimates of this habilitation

- satisfy as much as possible the **five optimal properties**
- are typically simultaneously **guaranteed** and **robust**
- are derived in **unified frameworks** for various numerical methods
- lead to **adaptive algorithms** yielding **efficient calculations** through **stopping criteria** and **error components equilibration**

Other contributions

- inexpensive implementations
- relations between different numerical methods
- improvement of approximate solutions by local postprocessing

Future directions

Future directions

- nonlinear evolutive problems (Stefan problem, two-phase flows, . . .)
- coupled systems
- error control and efficiency for real-life problems