ON THE DISCRETE POINCARÉ–FRIEDRICHS INEQUALITIES FOR NONCONFORMING APPROXIMATIONS OF THE SOBOLEV SPACE H^1

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ABSTRACT

We present a direct proof of the discrete Poincaré–Friedrichs inequalities for a class of nonconforming approximations of the Sobolev space $H^1(\Omega)$, indicate optimal values of the constants in these inequalities, and extend the discrete Friedrichs inequality onto domains only bounded in one direction. We consider a polygonal domain Ω in two or three space dimensions and its shape-regular simplicial triangulation. The nonconforming approximations of $H^1(\Omega)$ consist of functions from H^1 on each element such that the mean values of their traces on interelement boundaries coincide. The key idea is to extend the proof of the discrete Poincaré–Friedrichs inequalities for piecewise constant functions used in the finite volume method. The results have applications in the analysis of nonconforming numerical methods, such as nonconforming finite element or discontinuous Galerkin methods.

Key Words: Poincaré–Friedrichs inequalities, Sobolev space H^1, nonconforming approximation

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1 INTRODUCTION

The Friedrichs (also called Poincaré) inequality

$$\int_\Omega g^2(\mathbf{x})\,\mathrm{d}\mathbf{x} \le c_F \int_\Omega |\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \qquad \forall g \in H_0^1(\Omega) \tag{1.1}$$

and the Poincaré (also called mean Poincaré) inequality

$$\int_\Omega g^2(\mathbf{x})\,\mathrm{d}\mathbf{x} \le c_P \int_\Omega |\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \tilde{c}_P\Big(\int_\Omega g(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)^2 \qquad \forall g \in H^1(\Omega) \tag{1.2}$$

(cf. Nečas [9]) play an important role in the theory of partial differential equations. We consider here a polygonal (we use this term for $d = 3$ as well instead of polyhedral) domain (open, bounded, and connected set) $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, $H^1(\Omega)$ is the Sobolev space of $L^2(\Omega)$ functions with square-integrable generalized derivatives, and $H^1_0(\Omega)$ is the subspace of $H^1(\Omega)$ of functions with zero trace on the boundary $\partial\Omega$ of Ω. We refer for instance to Adams [1] for details on the spaces $H^1(\Omega)$, $H^1_0(\Omega)$.

Let $\{\mathcal{T}_h\}_h$ be a shape-regular family of simplicial triangulations of Ω (consisting of triangles in space dimension two and of tetrahedra in space dimension three). Let the spaces $W(\mathcal{T}_h)$ be formed by functions locally in $H^1(K)$ on each $K \in \mathcal{T}_h$ such that the mean values of their traces on interior sides (edges if $d = 2$, faces if $d = 3$) coincide. Finally, let $W_0(\mathcal{T}_h) \subset W(\mathcal{T}_h)$ be such that the mean values of the traces on exterior sides of functions from $W_0(\mathcal{T}_h)$ are equal to zero (precise definitions are given in the next section). These spaces are nonconforming approximations of the continuous ones, i.e. $W_0(\mathcal{T}_h) \not\subset H^1_0(\Omega)$ and $W(\mathcal{T}_h) \not\subset H^1(\Omega)$. We investigate in this paper analogies of (1.1) and (1.2) in the forms

$$\int_\Omega g^2(\mathbf{x})\,\mathrm{d}\mathbf{x} \le C_F \sum_{K\in\mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \qquad \forall g \in W_0(\mathcal{T}_h)\,,\ \forall h > 0\,, \tag{1.3}$$

$$\int_\Omega g^2(\mathbf{x})\,\mathrm{d}\mathbf{x} \le C_P \sum_{K\in\mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \tilde{C}_P\Big(\int_\Omega g(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)^2 \qquad \forall g \in W(\mathcal{T}_h)\,,\ \forall h > 0\,. \tag{1.4}$$

The validity of (1.3) for $W_0(\mathcal{T}_h)$ consisting of piecewise linear functions (used e.g. in the Crouzeix–Raviart finite element method) has been established by Temam in [13, 14, Proposition I.4.13]. Thomas [15, Theorem V.4.3] generalizes this result to higher-order polynomial spaces and, under the condition that the triangulations are not locally refined, to a subspace of functions of $W_0(\mathcal{T}_h)$ only fixed to zero on a part of the boundary, see the proof of Theorem V.4.2 in the same reference. Analogous results for polynomial functions and triangulations that cannot be locally refined have next been established by Stummel [12, Theorems 3.2.(15) and 3.2.(16)] for (1.3) and (1.4), respectively, and by Dolejší *et al* [5] for (1.3). Extensions to $\mathcal{T}_h$ only satisfying the shape regularity (minimal angle) assumption and onto spaces that include $W_0(\mathcal{T}_h)$ are finally given by Knobloch [8] and Brenner [3]. This last paper also shows how to extend the discrete Friedrichs and Poincaré inequalities to general polygonal (nonmatching) partitions of Ω and to functions that do not satisfy the equality of the means of traces on interior sides, provided that (1.3), (1.4) are satisfied.

It was shown in [8] and in [3] that the constants C_F, C_P only depend on the domain Ω and on the shape regularity of the meshes. We establish in this paper the exact dependence of C_F, C_P on these parameters. We show that in space dimension two C_F only depends on the area of Ω and that in space dimension two or three C_F only depends on the square of the infimum over the thickness of Ω in one direction. For convex domains, C_P only depends on the square of the diameter of Ω and on the ratio between the area of the circumscribed ball and the area of Ω. For nonconvex domains, our results involve a more complicated dependence of C_P on Ω. The above-mentioned dependencies are optimal in the sense that they coincide with the dependencies of c_F, c_P on Ω in the continuous case. The dependence of C_F on Ω also allows for the extension of the discrete Friedrichs inequality to domains only bounded in one direction. We finally show that C_F depends, in space dimension two and provided that it is expressed using the area of Ω, on the square of a parameter describing the shape regularity of the meshes given in the next section. This dependence still holds true for C_F in space dimension two or three and expressed using the square of the infimum over the thickness of Ω in one direction and also for C_P, provided that the mesh is not locally refined.

We present an example showing that this dependence is optimal. For locally refined meshes, our results involve a more complicated dependence on the shape regularity parameter.

Our proof of the discrete Friedrichs and Poincaré inequalities on the spaces $W_0(\mathcal{T}_h)$, $W(\mathcal{T}_h)$ respectively is more direct than those presented in [8] and in [3]; in particular, all the necessary intermediate results are proved here. In [8] the author uses a Clément-type interpolation operator (cf. [4]) mapping the space $W_0(\mathcal{T}_h)$ to $H_0^1(\Omega)$. In [3] the key idea is to construct nonconforming P_1 interpolants of functions from $W(\mathcal{T}_h)$ and to connect the nonconforming P_1 finite elements and conforming P_2 finite elements (in space dimension two) or conforming P_3 finite elements (in space dimension three). In both cases one finally makes use of the continuous inequalities (1.1), (1.2). Our main idea is to construct a piecewise constant interpolant and to extend the discrete Poincaré–Friedrichs inequalities for piecewise constant functions known from finite volume methods, see Eymard *et al* [6, 7]. In particular, we do not make use of the continuous inequalities; since $H_0^1(\Omega) \subset W_0(\mathcal{T}_h)$ and $H^1(\Omega) \subset W(\mathcal{T}_h)$, we rather prove them. The established inequalities are necessary in the analysis of nonconforming numerical methods, such as nonconforming finite element or discontinuous Galerkin methods.

The structure of the paper is as follows. In Section 2 we describe the assumptions on $\mathcal{T}_h$, define a dual mesh $\mathcal{D}_h$ where the dual elements are associated with the sides of $\mathcal{T}_h$, define the function spaces used in the sequel, and introduce the interpolation operator. In Section 3 we give the discrete Friedrichs inequality for piecewise constant functions on $\mathcal{D}_h$. In Section 4 we prove some interpolation estimates on functions from $H^1(K)$, where K is a simplex in two or three space dimensions. In Section 5 we prove the discrete Friedrichs inequality for functions from $W_0(\mathcal{T}_h)$, using their interpolation by piecewise constant functions on $\mathcal{D}_h$. In Section 6 we show how this proof simplifies for Crouzeix–Raviart finite elements in two space dimensions. Finally, Section 7 is devoted to the proof of the discrete Poincaré inequality for piecewise constant functions on $\mathcal{D}_h$ and Section 8 to the extension of this result to functions from $W(\mathcal{T}_h)$.

2 NOTATION AND ASSUMPTIONS

Throughout this paper, we shall mean by "segment" a segment of a straight line. Let us consider a domain $K \subset \mathbb{R}^d$, $d = 2, 3$. We denote by $\|\cdot\|_{0,K}$ the norm on $L^2(K)$, $\|g\|_{0,K}^2 = \int_K g^2(\mathbf{x})\,\mathrm{d}\mathbf{x}$, by $|K|$ the d-dimensional Lebesgue measure of K, by $|\sigma|$ the $(d-1)$-dimensional Lebesgue measure of σ, a part of a hyperplane in $\mathbb{R}^d$, and in particular by $|\mathbf{s}|$ the length of a segment $\mathbf{s}$. Let $\mathbf{b}$ be a vector. We shall mean by the thickness of K in the direction of $\mathbf{b}$, denoted by $\mathrm{thick}_{\mathbf{b}}(K)$, the supremum of the lengths of segments $\mathbf{s}$ with the direction vector $\mathbf{b}$ such that $\mathbf{s} \subset K$. The thickness of K is then the supremum of the lengths of all the segments $\mathbf{s}$ such that $\mathbf{s} \subset K$. Recall that the diameter of K is the supremum of the distances between all pairs of points of K. For K convex, thickness and diameter coincide.

Triangulation

We suppose that $\mathcal{T}_h$ for all $h > 0$ consists of closed simplices such that $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and such that if $K, L \in \mathcal{T}_h$, $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of K and L. The parameter h is defined by $h := \max_{K \in \mathcal{T}_h} \mathrm{diam}(K)$. We denote by $\mathcal{E}_h$ the set of all sides, by $\mathcal{E}_h^{\mathrm{int}}$ the set of all interior sides, by $\mathcal{E}_h^{\mathrm{ext}}$ the set of all exterior sides, and by $\mathcal{E}_K$ the set of all the sides of an element $K \in \mathcal{T}_h$. We make the following shape regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$:

Assumption (A) (Shape regularity assumption)
There exists a constant $\kappa_\mathcal{T} > 0$ such that

$$\min_{K\in\mathcal{T}_h} \frac{|K|}{\mathrm{diam}(K)^d} \geq \kappa_\mathcal{T} \qquad \forall h > 0\,.$$

Assumption (A) is equivalent to the existence of a constant $\theta_\mathcal{T} > 0$ such that

$$\max_{K\in\mathcal{T}_h} \frac{\mathrm{diam}(K)}{\rho_K} \leq \theta_\mathcal{T} \qquad \forall h > 0\,, \tag{2.1}$$

where ρ_K is the diameter of the largest ball inscribed in the simplex K. Finally, Assumption (A) is equivalent to the existence of a constant $\phi_\mathcal{T} > 0$ such that

$$\min_{K\in\mathcal{T}_h} \phi_K \geq \phi_\mathcal{T} \qquad \forall h > 0\,. \tag{2.2}$$

Here ϕ_K is the smallest angle of the simplex K (plain angle in radians if $d = 2$ and spheric angle in steradians if $d = 3$).

In the sequel we shall consider apart triangulations that may not be locally refined, i.e. the case where the following assumption holds:

Assumption (B) (Inverse assumption)
There exists a constant $\zeta_\mathcal{T} > 0$ such that

$$\max_{K\in\mathcal{T}_h} \frac{h}{\mathrm{diam}(K)} \leq \zeta_\mathcal{T} \qquad \forall h > 0\,.$$

Assumptions (A) and (B) imply

$$\min_{K\in\mathcal{T}_h} \frac{|K|}{h^d} \geq \tilde{\kappa}_\mathcal{T} \qquad \forall h > 0\,, \tag{2.3}$$

where $\tilde{\kappa}_\mathcal{T} := \kappa_\mathcal{T}/\zeta_\mathcal{T}^d$.

Dual mesh

In the sequel we will use a dual mesh $\mathcal{D}_h$ to $\mathcal{T}_h$ such that $\overline{\Omega} = \bigcup_{D\in\mathcal{D}_h} D$. There is one dual element D associated with each side $\sigma_D \in \mathcal{E}_h$. We construct it by connecting the barycentres of every $K \in \mathcal{T}_h$ that contains σ_D through the vertices of σ_D. For $\sigma_D \in \mathcal{E}_h^{\mathrm{ext}}$, the contour of D is completed by the side σ_D itself. We refer to Fig. 1 for the two-dimensional case. We denote by $\mathcal{D}_h^{\mathrm{int}}$ the set of all interior and by $\mathcal{D}_h^{\mathrm{ext}}$ the set of all boundary dual elements. As for the primal mesh, we set $\mathcal{F}_h$, $\mathcal{F}_h^{\mathrm{int}}$, and $\mathcal{F}_h^{\mathrm{ext}}$ for the dual mesh sides. We denote by Q_D the barycentre of a side σ_D and for two adjacent elements $D, E \in \mathcal{D}_h$, we set $\sigma_{D,E} := \partial D \cap \partial E$, $d_{D,E} := |Q_E - Q_D|$, and $K_{D,E}$ the element of $\mathcal{T}_h$ such that $\sigma_{D,E} \subset K_{D,E}$. We remark that

$$|K \cap D| = \frac{|K|}{d+1} \tag{2.4}$$

for each $K \in \mathcal{T}_h$ and $D \in \mathcal{D}_h$ such that $\sigma_D \in \mathcal{E}_K$. Let us now consider $\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}$, $\sigma_{D,E} = \partial D \cap \partial E$ in the two-dimensional case. Let $K_{D,E} \cap D$ be in the clockwise direction from $K_{D,E} \cap E$. We then define $v_{D,E}$ as the height of the triangle $|K_{D,E} \cap D|$ with respect to its base $\sigma_{D,E}$ and have (see Fig. 1)

$$|K_{D,E} \cap D| = \frac{|\sigma_{D,E}| v_{D,E}}{2}\,. \tag{2.5}$$

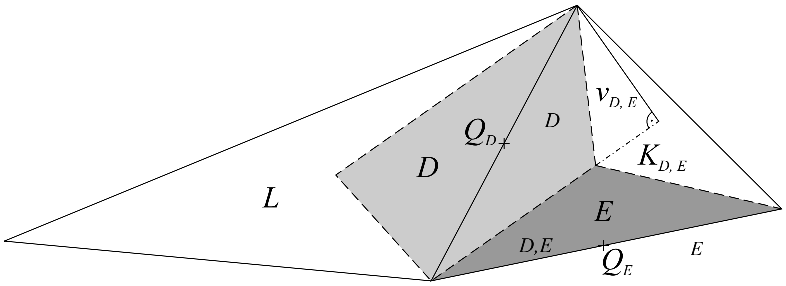

Figure 1: Triangles $K, L \in \mathcal{T}_h$ and dual elements $D, E \in \mathcal{D}_h$ with edges $\sigma_D, \sigma_E \in \mathcal{E}_h$

Function spaces

We define the space $W(\mathcal{T}_h)$ by

$$\begin{aligned} W(\mathcal{T}_h) \quad := \quad & \Big\{ g \in L^2(\Omega)\,;\, g|_K \in H^1(K) \qquad \forall K \in \mathcal{T}_h\,, \\ & \int_{\sigma_{K,L}} g|_K(\mathbf{x})\,\mathrm{d}\gamma(\mathbf{x}) = \int_{\sigma_{K,L}} g|_L(\mathbf{x})\,\mathrm{d}\gamma(\mathbf{x}) \\ & \qquad \forall \sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}\,,\ \sigma_{K,L} = \partial K \cap \partial L \Big\}\,. \end{aligned} \tag{2.6}$$

We keep the same notation for the function g and its trace and denote $\mathrm{d}\gamma(\mathbf{x})$ the integration symbol for the Lebesgue measure on a hyperplane of Ω. The space $W_0(\mathcal{T}_h)$ is defined by

$$W_0(\mathcal{T}_h) := \Big\{ g \in W(\mathcal{T}_h)\,;\, \int_\sigma g(\mathbf{x})\,\mathrm{d}\gamma(\mathbf{x}) = 0 \qquad \forall \sigma \in \mathcal{E}_h^{\mathrm{ext}} \Big\}\,. \tag{2.7}$$

We finally define

$$|g|_{1,\mathcal{T}} := \Big(\sum_{K \in \mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \Big)^{\frac{1}{2}}\,,$$

which is a seminorm on $W(\mathcal{T}_h)$ and a norm on $W_0(\mathcal{T}_h)$. The spaces $X(\mathcal{T}_h) \subset W(\mathcal{T}_h)$ and $X_0(\mathcal{T}_h) \subset W_0(\mathcal{T}_h)$ are defined by piecewise linear functions on $\mathcal{T}_h$. Note that the functions from $X(\mathcal{T}_h)$ are continuous in barycentres of interior sides and that the functions from $X_0(\mathcal{T}_h)$ are moreover equal to zero in barycentres of exterior sides.

The space $Y(\mathcal{D}_h)$ is the space of piecewise constant functions on $\mathcal{D}_h$,

$$Y(\mathcal{D}_h) := \big\{ c \in L^2(\Omega)\,;\, c|_D \text{ is constant } \forall D \in \mathcal{D}_h \big\}\ ,$$

and $Y_0(\mathcal{D}_h)$ is its subspace of functions equal to zero on all $D \in \mathcal{D}_h^{\mathrm{ext}}$,

$$Y_0(\mathcal{D}_h) := \big\{ c \in Y(\mathcal{D}_h)\,;\, c|_D = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{ext}} \big\}\ .$$

For $c \in Y(\mathcal{D}_h)$ given by the values c_D on $D \in \mathcal{D}_h$, we define

$$\begin{aligned} |c|_{1,\mathcal{T},*} \quad &:= \quad \Big(\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{v_{D,E}} (c_E - c_D)^2 \Big)^{\frac{1}{2}}\,, \\ |c|_{1,\mathcal{T},\dagger} \quad &:= \quad \Big(\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{\mathrm{diam}(K_{D,E})} (c_E - c_D)^2 \Big)^{\frac{1}{2}}\,, \\ |c|_{1,\mathcal{T},\ddagger} \quad &:= \quad \Big(\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{d_{D,E}} (c_E - c_D)^2 \Big)^{\frac{1}{2}}\,; \end{aligned}$$

$|\cdot|_{1,\mathcal{T},*}$, $|\cdot|_{1,\mathcal{T},\dagger}$, and $|\cdot|_{1,\mathcal{T},\dagger}$ are seminorms on $Y(\mathcal{D}_h)$ and norms on $Y_0(\mathcal{D}_h)$.

Interpolation operator

The interpolation operator I associates to a function $g \in W(\mathcal{T}_h)$ a function $I(g) \in Y(\mathcal{D}_h)$ such that

$$I(g)|_D = g_D := \frac{1}{|\sigma_D|} \int_{\sigma_D} g|_K(\mathbf{x})\, \mathrm{d}\gamma(\mathbf{x}) \qquad \forall D \in \mathcal{D}_h\,,$$

where $K \in \mathcal{T}_h$ is such that $\sigma_D \in \mathcal{E}_K$. Note that by (2.6), if $\sigma_D \in \mathcal{E}_K$ and $\sigma_D \in \mathcal{E}_L$, $K \neq L$, the choice between K and L does not matter. We recall that $\sigma_D \in \mathcal{E}_h$ is the side associated with the dual element $D \in \mathcal{D}_h$. Note that for $g \in W_0(\mathcal{T}_h)$, $I(g) \in Y_0(\mathcal{D}_h)$.

3 DISCRETE FRIEDRICHS INEQUALITY FOR PIECEWISE CONSTANT FUNCTIONS

In finite volume methods (cf. [7]) one can prove the discrete Friedrichs inequality for piecewise constant functions for meshes that satisfy the following orthogonality property: there exists a point associated with each element of the mesh such that the straight line connecting these points for two neighboring elements is orthogonal to the common side of these two elements. The proofs in [6, 7] rely on this property of the meshes. We present in this section analogies of Lemma 9.5 and consequent Remark 9.13 and of Lemma 9.1 of [7] for the mesh $\mathcal{D}_h$, where the orthogonality property is not necessarily satisfied.

Theorem 3.1 (Discrete Friedrichs inequality for piecewise constant functions in two space dimensions). *Let $d = 2$. Then for all $c \in Y_0(\mathcal{D}_h)$,*

$$\|c\|^2_{0,\Omega} \leq \frac{|\Omega|}{2} |c|^2_{1,\mathcal{T},*}\,.$$

Proof. Let $\mathbf{b}_1 = (1,0)$ and $\mathbf{b}_2 = (0,1)$ be two fixed unit vectors in the axis directions. For all $\mathbf{x} \in \Omega$, let $\mathcal{B}^1_{\mathbf{x}}$ and $\mathcal{B}^2_{\mathbf{x}}$ be the straight lines going through $\mathbf{x}$ and defined by the vectors $\mathbf{b}_1$, $\mathbf{b}_2$ respectively. Let the functions $\chi^{(i)}_\sigma(\mathbf{x})$, $i = 1,2$, for each $\sigma \in \mathcal{F}^{\mathrm{int}}_h$ be defined by

$$\chi^{(i)}_\sigma(\mathbf{x}) := \begin{cases} 1 & \text{if } \sigma \cap \mathcal{B}^i_{\mathbf{x}} \neq \emptyset \\ 0 & \text{if } \sigma \cap \mathcal{B}^i_{\mathbf{x}} = \emptyset \end{cases}.$$

Let finally $D \in \mathcal{D}^{\mathrm{int}}_h$ be fixed. Then for a.e. $\mathbf{x} \in D$, $\mathcal{B}^i_{\mathbf{x}}$, $i = 1,2$, do not contain any vertex of the dual mesh and $\mathcal{B}^i_{\mathbf{x}} \cap \sigma$, $i = 1,2$, contain at most one point of all $\sigma \in \mathcal{F}_h$. This implies that for a.e. $\mathbf{x} \in D$, $\mathcal{B}^i_{\mathbf{x}}$, $i = 1,2$, always have to intersect the interior of some $E \in \mathcal{D}^{\mathrm{ext}}_h$ before "leaving" or after "entering" Ω (we recall that Ω may be nonconvex). Using this, the fact that $c_E = 0$ for all $E \in \mathcal{D}^{\mathrm{ext}}_h$, and the triangle inequality, we have

$$2|c_D| \leq \sum_{\sigma_{F,G} \in \mathcal{F}^{\mathrm{int}}_h} |c_G - c_F| \chi^{(i)}_{\sigma_{F,G}}(\mathbf{x}) \qquad \text{for a.e. } \mathbf{x} \in D\,,\ i = 1,2\,.$$

This gives

$$|c_D|^2 \leq \frac{1}{4} \sum_{\sigma_{F,G} \in \mathcal{F}^{\mathrm{int}}_h} |c_G - c_F| \chi^{(1)}_{\sigma_{F,G}}(\mathbf{x}) \sum_{\sigma_{F,G} \in \mathcal{F}^{\mathrm{int}}_h} |c_G - c_F| \chi^{(2)}_{\sigma_{F,G}}(\mathbf{x}) \qquad \text{for a.e. } \mathbf{x} \in D\,,$$

which is obviously valid also for $D \in \mathcal{D}^{\mathrm{ext}}_h$, considering that $c_D = 0$ on $D \in \mathcal{D}^{\mathrm{ext}}_h$. Integrating the above inequality over D and summing over $D \in \mathcal{D}_h$ yields

$$\sum_{D \in \mathcal{D}_h} c_D^2 |D| \leq \frac{1}{4} \int_\Omega \Big(\sum_{\sigma_{D,E} \in \mathcal{F}^{\mathrm{int}}_h} |c_E - c_D| \chi^{(1)}_{\sigma_{D,E}}(\mathbf{x}) \sum_{\sigma_{D,E} \in \mathcal{F}^{\mathrm{int}}_h} |c_E - c_D| \chi^{(2)}_{\sigma_{D,E}}(\mathbf{x}) \Big)\, \mathrm{d}\mathbf{x}\,.$$

Let $\alpha = \inf\{x_1; (x_1, x_2) \in \Omega\}$ and $\beta = \sup\{x_1; (x_1, x_2) \in \Omega\}$. For each $x_1 \in (\alpha, \beta)$, we denote by $J(x_1)$ the set of x_2 such that $\mathbf{x} = (x_1, x_2) \in \Omega$. We now notice that $\chi^{(1)}_\sigma(\mathbf{x})$ only depends on x_2 and that $\chi^{(2)}_\sigma(\mathbf{x})$ only depends on x_1. Thus

$$\begin{aligned}
&\int_\alpha^\beta \int_{J(x_1)} \Big(\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| \chi^{(1)}_{\sigma_{D,E}}(x_2) \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| \chi^{(2)}_{\sigma_{D,E}}(x_1) \Big) \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
= &\int_\alpha^\beta \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| \chi^{(2)}_{\sigma_{D,E}}(x_1) \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| \int_{J(x_1)} \chi^{(1)}_{\sigma_{D,E}}(x_2) \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
\le &\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| |\sigma_{D,E}| \int_\alpha^\beta \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |c_E - c_D| \chi^{(2)}_{\sigma_{D,E}}(x_1) \,\mathrm{d}x_1 \,,
\end{aligned}$$

where we have used $\int_{J(x_1)} \chi^{(1)}_{\sigma_{D,E}}(x_2) \,\mathrm{d}x_2 \le |\sigma_{D,E}|$. Using analogously $\int_\alpha^\beta \chi^{(2)}_{\sigma_{D,E}}(x_1) \,\mathrm{d}x_1 \le |\sigma_{D,E}|$, we come to

$$\sum_{D \in \mathcal{D}_h} c_D^2 |D| \le \frac{1}{4} \Big(\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |\sigma_{D,E}| |c_E - c_D| \Big)^2 .$$

Finally, using the Cauchy–Schwarz inequality, we have

$$\sum_{D \in \mathcal{D}_h} c_D^2 |D| \le \frac{1}{4} \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |\sigma_{D,E}| v_{D,E} \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{v_{D,E}} (c_E - c_D)^2 .$$

The equality $\sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |\sigma_{D,E}| v_{D,E} = 2|\Omega|$, which follows from (2.5), concludes the proof. $\square$

Remark 3.2 (Discrete Friedrichs inequality for piecewise constant functions on equilateral simplices). *Let $\mathbf{b} \in \mathbb{R}^d$ be a fixed vector and let $\mathcal{T}_h$ consist of equilateral simplices. Then for all $c \in Y_0(\mathcal{D}_h)$,*

$$\|c\|^2_{0,\Omega} \le [\mathrm{thick}_{\mathbf{b}}(\Omega) + 2h]^2 |c|^2_{1,\mathcal{T},\ddagger} .$$

This follows from [7, Lemma 9.1] (cf. alternatively [6, Lemma 1]), since the dual mesh $\mathcal{D}_h$ satisfies in this case the orthogonality property.

Our purpose now will be to extend this result to general triangulations.

Lemma 3.3. *Let Assumption (B) be satisfied and let $\mathbf{b} \subset \Omega$ be a segment that does not contain any vertex of the dual mesh $\mathcal{D}_h$. Then*

$$A := \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}},\, \sigma_{D,E} \cap \mathbf{b} \neq \emptyset} \mathrm{diam}(K_{D,E}) \le C_{d,\mathcal{T}} \, \mathrm{thick}_{\mathbf{b}}(\Omega) \,,$$

where

$$C_{d,\mathcal{T}} = \frac{2^d (d-1)}{\tilde{\kappa}_{\mathcal{T}}} (1 + 2\theta_{\mathcal{T}}) . \tag{3.1}$$

Proof. The number of nonzero terms of A is equal to the number of interior dual sides intersected by $\mathbf{b}$. In view of the fact that $\mathbf{b}$ does not contain any vertex of the dual mesh, this number is bounded by $2(d-1)$-times the number of simplices $K \in \mathcal{T}_h$ whose interior is intersected by $\mathbf{b}$. All intersected simplices have to be entirely in the rectangle/rectangular parallelepiped constructed around $\mathbf{b}$, with the distance between $\mathbf{b}$ and its boundary equal to h. Considering the consequence (2.3) of Assumptions (A) and (B), we can estimate the number of intersected elements by

$$\frac{(2h)^{d-1}(|\mathbf{b}| + 2h)}{\tilde{\kappa}_{\mathcal{T}} h^d}.$$

Using in addition $\operatorname{diam}(K_{D,E}) \leq h$ and $|\mathbf{b}| \leq \operatorname{thick}_{\mathbf{b}}(\Omega)$, we have

$$A \leq \frac{2^d(d-1)}{\tilde{\kappa}_{\mathcal{T}}}(\operatorname{thick}_{\mathbf{b}}(\Omega) + 2h).$$

Noticing that

$$h \leq \theta_{\mathcal{T}} \operatorname{thick}_{\mathbf{b}}(\Omega) \tag{3.2}$$

by the consequence (2.1) of Assumption (A) concludes the proof. □

Lemma 3.4. *Let $\mathbf{b} \subset \Omega$ be a segment that does not contain any vertex of the dual mesh $\mathcal{D}_h$. Then*

$$A := \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}},\, \sigma_{D,E} \cap \mathbf{b} \neq \emptyset} \operatorname{diam}(K_{D,E}) \leq C_{d,\mathcal{T}} \operatorname{thick}_{\mathbf{b}}(\Omega),$$

where

$$C_{d,\mathcal{T}} = 4(d-1)N\theta_{\mathcal{T}}^N, \quad N = \frac{2^{d-2}\pi}{\phi_{\mathcal{T}}}. \tag{3.3}$$

Proof. The number of nonzero terms of A is equal to the number of interior dual sides intersected by $\mathbf{b}$. In view of the fact that $\mathbf{b}$ does not contain any vertex of the dual mesh, this number is bounded by $2(d-1)$-times the number of simplices $K \in \mathcal{T}_h$ whose interior is intersected by $\mathbf{b}$. We next follow the ideas of [15, Lemma V.4.3] rather than those originally used in [16, Lemma 2.3.4], yielding a slightly better value of the constant $C_{d,\mathcal{T}}$.

Let us group the intersected simplices by N, defining a system of nonoverlapping segments $\{\mathbf{b}_k\}_{k=1}^M$ lying on the intersection of the straight line given by $\mathbf{b}$ and Ω, such that each $\mathbf{b}_k$ intersects exactly N simplices and such that the intersection always contains two points from ∂K (stretches over the whole K). At most $N-1$ simplices whose interior is intersected by $\mathbf{b}$ remain. Using that

$$\operatorname{diam}(K) \leq \theta_{\mathcal{T}} \rho_K \leq \theta_{\mathcal{T}} \operatorname{thick}_{\mathbf{b}}(\Omega) \qquad \forall K \in \mathcal{T}_h,$$

following from the consequence (2.1) of Assumption (A), we have

$$A \leq 2(d-1) \sum_{k=1}^{M} \sum_{K \in \mathcal{T}_h;\, K^\circ \cap \mathbf{b}_k \neq \emptyset} \operatorname{diam}(K) + 2(d-1)N\theta_{\mathcal{T}}^N \operatorname{thick}_{\mathbf{b}}(\Omega), \tag{3.4}$$

noticing as well that $\theta_{\mathcal{T}} > 1$ and $N > 1$. We next estimate the first term of the above expression. It follows from the consequence (2.1) of Assumption (A) that $\rho_K \leq \theta_{\mathcal{T}} \rho_L$ if

$K, L \in \mathcal{T}_h$ are neighboring elements. Recall that ρ_K is the diameter of the largest ball inscribed in the simplex K. Thus we come to

$$\frac{\max\limits_{K\in\mathcal{T}_h;\, K^\circ\cap\mathbf{b}_k\neq\emptyset} \rho_K}{\min\limits_{K\in\mathcal{T}_h;\, K^\circ\cap\mathbf{b}_k\neq\emptyset} \rho_L} \leq \theta_{\mathcal{T}}^{N-1} \qquad \forall k = 1, \dots, M\,.$$

We further claim that

$$\min_{K\in\mathcal{T}_h;\, K^\circ\cap\mathbf{b}_k\neq\emptyset} \rho_K \leq |\mathbf{b}_k| \qquad \forall k = 1, \dots, M\,,$$

i.e. if we take N simplices, where N is given by (3.3), intersected by a straight line, then the length of the intersection is at least equal to the smallest diameter of the inscribed balls of the simplices. Let V be a vertex of a simplex K and let us consider the hyperplane joining the midpoints of the edges sharing V. Clearly, as soon as $\mathbf{b}_k$ intersects this hyperplane or as soon as the intersection lies entirely in the part of K bounded by this hyperplane and not containing V, the intersection is longer than ρ_K. Hence, in order not to exceed ρ_K for some K, all the intersected simplices would have to share the same vertex and the intersection would only have to lie in the part of K bounded by the above defined hyperplane and containing V for each K. However, with each intersected simplex, we would in this case add an angle greater or equal to $\phi_{\mathcal{T}}$ by the consequence (2.2) of Assumption (A). Since we have N simplices, their angles would fill the whole semi-circle (π, $d = 2$) or semi-sphere (2π, $d = 3$), which shows that this is not possible.

Using the last two estimates and once more the consequence (2.1) of Assumption (A), we have

$$\sum_{k=1}^{M} \sum_{K\in\mathcal{T}_h;\, K^\circ\cap\mathbf{b}_k\neq\emptyset} \operatorname{diam}(K) \leq \sum_{k=1}^{M} \theta_{\mathcal{T}} \sum_{K\in\mathcal{T}_h;\, K^\circ\cap\mathbf{b}_k\neq\emptyset}^{N} \rho_K \leq \sum_{k=1}^{M} N\theta_{\mathcal{T}}^N |\mathbf{b}_k| \leq N\theta_{\mathcal{T}}^N \operatorname{thick}_{\mathbf{b}}(\Omega)\,.$$

In combination with (3.4), this concludes the proof of the lemma. □

Theorem 3.5 (Discrete Friedrichs inequality for piecewise constant functions). *Let $\mathbf{b} \in \mathbb{R}^d$ be a fixed vector. Then for all $c \in Y_0(\mathcal{D}_h)$,*

$$\|c\|_{0,\Omega}^2 \leq C_{d,\mathcal{T}} [\operatorname{thick}_{\mathbf{b}}(\Omega)]^2 |c|_{1,\mathcal{T},\dagger}^2\,,$$

where $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case.

Proof. For all $\mathbf{x} \in \Omega$, we denote by $\mathcal{B}_{\mathbf{x}}$ the straight semi-line defined by the origin $\mathbf{x}$ and the vector $\mathbf{b}$. Let $\mathbf{y}(\mathbf{x}) \in \partial\Omega \cap \mathcal{B}_{\mathbf{x}}$ be the point where $\mathcal{B}_{\mathbf{x}}$ intersects $\partial\Omega$ for the first time. Then $[\mathbf{x}, \mathbf{y}(\mathbf{x})] \subset \overline{\Omega}$. We finally define a function $\chi_\sigma(\mathbf{x})$ for each $\sigma \in \mathcal{F}_h^{\text{int}}$ by

$$\chi_\sigma(\mathbf{x}) := \begin{cases} 1 & \text{if } \sigma \cap [\mathbf{x}, \mathbf{y}(\mathbf{x})] \neq \emptyset \\ 0 & \text{if } \sigma \cap [\mathbf{x}, \mathbf{y}(\mathbf{x})] = \emptyset \end{cases}.$$

Let $D \in \mathcal{D}_h^{\text{int}}$ be fixed. Then for a.e. $\mathbf{x} \in D$, $\mathcal{B}_{\mathbf{x}}$ does not contain any vertex of the dual mesh and $\mathcal{B}_{\mathbf{x}} \cap \sigma$ contains at most one point of all $\sigma \in \mathcal{F}_h$. This implies that for a.e. $\mathbf{x} \in D$, $\mathcal{B}_{\mathbf{x}}$ always has to intersect the interior of some $E \in \mathcal{D}_h^{\text{ext}}$ before "leaving" Ω. Using this, the fact that $c_E = 0$ for all $E \in \mathcal{D}_h^{\text{ext}}$, and the triangle inequality, we have

$$|c_D| \leq \sum_{\sigma_{F,G}\in\mathcal{F}_h^{\text{int}}} |c_G - c_F| \chi_{\sigma_{F,G}}(\mathbf{x}) \quad \text{for a.e. } \mathbf{x} \in D\,.$$

The Cauchy–Schwarz inequality yields

$$|c_D|^2 \le \sum_{\sigma_{F,G}\in\mathcal{F}_h^{\rm int}} \chi_{\sigma_{F,G}}(\mathbf{x})\,{\rm diam}(K_{F,G}) \sum_{\sigma_{F,G}\in\mathcal{F}_h^{\rm int}} \frac{(c_G-c_F)^2}{{\rm diam}(K_{F,G})}\chi_{\sigma_{F,G}}(\mathbf{x}) \quad \text{for a.e. } \mathbf{x}\in D\,,$$

which is obviously valid also for $D \in \mathcal{D}_h^{\rm ext}$, considering that $c_D = 0$ on $D \in \mathcal{D}_h^{\rm ext}$. Integrating the above inequality over D, summing over $D \in \mathcal{D}_h$, and using Lemma 3.3 when Assumption (B) is satisfied and Lemma 3.4 in the general case yields

$$\sum_{D\in\mathcal{D}_h} |c_D|^2|D| \le C_{d,\mathcal{T}}{\rm thick}_{\mathbf{b}}(\Omega) \sum_{\sigma_{D,E}\in\mathcal{F}_h^{\rm int}} \frac{(c_E-c_D)^2}{{\rm diam}(K_{D,E})}\int_\Omega \chi_{\sigma_{D,E}}(\mathbf{x})\,{\rm d}\mathbf{x}\,.$$

Now the value $\int_\Omega \chi_{\sigma_{D,E}}(\mathbf{x})\,{\rm d}\mathbf{x}$ is the measure of the set of points of Ω located inside a cylinder whose basis is $\sigma_{D,E}$ and generator vector is $-\mathbf{b}$. Thus

$$\int_\Omega \chi_{\sigma_{D,E}}(\mathbf{x})\,{\rm d}\mathbf{x} \le |\sigma_{D,E}|\,{\rm thick}_{\mathbf{b}}(\Omega)\,,$$

which leads to the assertion of the lemma. □

4 INTERPOLATION ESTIMATES ON FUNCTIONS FROM $H^1(K)$

We give some interpolation estimates for a simplex K in this section.

Lemma 4.1. *Let K be a simplex, σ its side, and $g \in H^1(K)$. We set*

$$\begin{aligned} g_K &:= \frac{1}{|K|}\int_K g(\mathbf{x})\,{\rm d}\mathbf{x}\,,\\ g_\sigma &:= \frac{1}{|\sigma|}\int_\sigma g(\mathbf{x})\,{\rm d}\gamma(\mathbf{x})\,. \end{aligned}$$

Then

$$(g_K-g_\sigma)^2 \le c_d\frac{{\rm diam}(K)^2}{|K|}\int_K |\nabla g(\mathbf{x})|^2\,{\rm d}\mathbf{x}\,, \tag{4.1}$$

$$\int_K [g(\mathbf{x})-g_\sigma]^2\,{\rm d}\mathbf{x} \le c_d\,{\rm diam}(K)^2\int_K |\nabla g(\mathbf{x})|^2\,{\rm d}\mathbf{x}\,, \tag{4.2}$$

where

$$c_d = 6 \ \text{for}\ d=2\,, \quad c_d = 9 \ \text{for}\ d=3\,. \tag{4.3}$$

Proof. The inequality (4.1) is proved as a part of [7, Lemma 9.4] or [6, Lemma 2] for $d = 2$. In these references a general convex polygonal element K is considered; the fact that $c_d = 6$ follows by considering a triangular element. The inequality (4.2) also follows from these proofs, using the Cauchy–Schwarz inequality. We now give the proof for the three-dimensional case, following the ideas of the proof for $d = 2$.

Let us consider a tetrahedron K and its face σ. Let us denote the space coordinates by x_1, x_2, x_3. We assume, without loss of generality, that $\sigma \subset \{0\}\times\mathbb{R}\times\mathbb{R}^+$, that one vertex of σ lies in the origin, that the longest edge of σ lies on x_2^+, and that $K \subset \mathbb{R}^+\times\mathbb{R}\times\mathbb{R}$. Let

$\mathbf{a} = (\alpha, \beta, \gamma)$ be the vertex that does not lie on σ. For all $x_1 \in [0, \alpha]$, we set $J(x_1) = \{x_2 \in \mathbb{R}$ such that $(x_1, x_2, x_3) \in K$ for some $x_3 \in \mathbb{R}\}$. For all $x_2 \in J(x_1)$ with $x_1 \in [0, \alpha]$ given, we set $J(x_1, x_2) = \{x_3 \in \mathbb{R}$ such that $(x_1, x_2, x_3) \in K\}$. For a.e. $\mathbf{x} = (x_1, x_2, x_3) \in K$ and a.e. $\mathbf{y} = (0, y_2, y_3) \in \sigma$, we set $\mathbf{z}(\mathbf{x}, \mathbf{y}) = t\mathbf{a} + (1-t)\mathbf{y}$ with $t = \frac{x_1}{\alpha}$. Since K is convex, $\mathbf{z}(\mathbf{x}, \mathbf{y}) \in K$ and we have $\mathbf{z}(\mathbf{x}, \mathbf{y}) = (x_1, z_2(x_1, y_2), z_3(x_1, y_3))$ with $z_2(x_1, y_2) = \frac{x_1}{\alpha}\beta + (1 - \frac{x_1}{\alpha})y_2$ and $z_3(x_1, y_3) = \frac{x_1}{\alpha}\gamma + (1 - \frac{x_1}{\alpha})y_3$.

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\int_K [g(\mathbf{x}) - g_\sigma]^2 \,\mathrm{d}\mathbf{x} = \int_K \Big[\frac{1}{|\sigma|} \int_\sigma g(\mathbf{x}) \,\mathrm{d}\gamma(\mathbf{y}) - \frac{1}{|\sigma|} \int_\sigma g(\mathbf{y}) \,\mathrm{d}\gamma(\mathbf{y}) \\
&\pm \frac{1}{|\sigma|} \int_\sigma g(\mathbf{z}(\mathbf{x}, \mathbf{y})) \,\mathrm{d}\gamma(\mathbf{y})\Big]^2 \mathrm{d}\mathbf{x} \le \frac{2}{|\sigma|^2} \int_K \Big[\int_\sigma \Big(g(\mathbf{x}) - g(\mathbf{z}(\mathbf{x}, \mathbf{y}))\Big) \,\mathrm{d}\gamma(\mathbf{y})\Big]^2 \mathrm{d}\mathbf{x} \\
&+ \frac{2}{|\sigma|^2} \int_K \Big[\int_\sigma \Big(g(\mathbf{z}(\mathbf{x}, \mathbf{y})) - g(\mathbf{y})\Big) \,\mathrm{d}\gamma(\mathbf{y})\Big]^2 \mathrm{d}\mathbf{x} \le \frac{2}{|\sigma|}(A + B),
\end{aligned}$$

where

$$\begin{aligned}
A &:= \int_K \int_\sigma \Big(g(\mathbf{x}) - g(\mathbf{z}(\mathbf{x}, \mathbf{y}))\Big)^2 \,\mathrm{d}\gamma(\mathbf{y}) \,\mathrm{d}\mathbf{x}, \\
B &:= \int_K \int_\sigma \Big(g(\mathbf{z}(\mathbf{x}, \mathbf{y})) - g(\mathbf{y})\Big)^2 \,\mathrm{d}\gamma(\mathbf{y}) \,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Similarly,

$$(g_K - g_\sigma)^2 \le \frac{2}{|K||\sigma|}(A + B).$$

We denote by $D_i g$ the partial derivative of g with respect to x_i, $i \in \{1, 2, 3\}$, and estimate A and B separately. For this purpose, we suppose that $g \in C^1(K)$ and use the density of $C^1(K)$ in $H^1(K)$ to extend the estimates to $g \in H^1(K)$.

We first estimate A. We have

$$\begin{aligned}
A = \int_0^\alpha \int_{J(x_1)} \int_{J(x_1, x_2)} \int_{J(0)} \int_{J(0, y_2)} &\Big(g(x_1, x_2, x_3) \\
&- g(x_1, z_2(x_1, y_2), z_3(x_1, y_3))\Big)^2 \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1.
\end{aligned}$$

Let us suppose that $x_3 \ge z_3$. This implies that $[x_1, x_2, z_3(x_1, y_3)] \in K$, since the cross-section of K and the plane $x_1 = const$ is a triangle whose bottom edge is horizontal and the longest of its three edges. We deduce the inequality

$$\begin{aligned}
&\Big(g(x_1, x_2, x_3) - g(x_1, z_2(x_1, y_2), z_3(x_1, y_3))\Big)^2 = \Big(g(x_1, x_2, x_3) - g(x_1, x_2, z_3(x_1, y_3)) \\
&+ g(x_1, x_2, z_3(x_1, y_3)) - g(x_1, z_2(x_1, y_2), z_3(x_1, y_3))\Big)^2 = \Big(\int_{z_3(x_1, y_3)}^{x_3} D_3 g(x_1, x_2, s) \,\mathrm{d}s \\
&+ \int_{z_2(x_1, y_2)}^{x_2} D_2 g(x_1, s, z_3(x_1, y_3)) \,\mathrm{d}s\Big)^2 \le 2\mathrm{diam}(K) \int_{J(x_1, x_2)} [D_3 g(x_1, x_2, s)]^2 \,\mathrm{d}s \\
&+ 2\mathrm{diam}(K)\Big(1 - \frac{x_1}{\alpha}\Big) \int_{z_2(x_1, y_2)}^{x_2} [D_2 g(x_1, s, z_3(x_1, y_3))]^2 \,\mathrm{d}s,
\end{aligned}$$

where we have used the Newton integration formula and the Cauchy–Schwarz inequality. Defining $D_i g$, $i \in \{1, 2, 3\}$, by 0 outside of K and considering also $x_3 < z_3$, we come to

$$A \le 2\mathrm{diam}(K)(A_1 + A_2 + A_3 + A_4)$$

with

$$A_1 := \int_0^\alpha \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0)} \int_{J(0,y_2)} \int_{J(x_1,x_2)} [D_3 g(x_1,x_2,s)]^2 \,\mathrm{d}s \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,,$$

$$A_2 := \int_0^\alpha \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0)} \int_{J(0,y_2)} \Big(1 - \frac{x_1}{\alpha}\Big) \int_{z_2(x_1,y_2)}^{x_2} [D_2 g(x_1,s,z_3(x_1,y_3))]^2 \,\mathrm{d}s \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,,$$

$$A_3 := \int_0^\alpha \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0)} \int_{J(0,y_2)} \int_{z_2(x_1,y_2)}^{x_2} [D_2 g(x_1,s,x_3)]^2 \,\mathrm{d}s \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,,$$

$$A_4 := \int_0^\alpha \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0)} \int_{J(0,y_2)} \Big(1 - \frac{x_1}{\alpha}\Big) \int_{z_3(x_1,y_3)}^{x_3} [D_3 g(x_1,z_2(x_1,y_2),s)]^2 \,\mathrm{d}s \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,.$$

We easily see that

$$A_1 \le \operatorname{diam}(K)|\sigma| \int_K [D_3 g(\mathbf{x})]^2 \,\mathrm{d}\mathbf{x} \,.$$

Next, we estimate A_2. Using the Fubini theorem and the change of variables $z_3 = z_3(x_1, y_3)$, we have

$$\begin{aligned}
&\int_{J(0,y_2)} \Big(1 - \frac{x_1}{\alpha}\Big) \int_{z_2(x_1,y_2)}^{x_2} [D_2 g(x_1,s,z_3(x_1,y_3))]^2 \,\mathrm{d}s \,\mathrm{d}y_3 \\
=& \int_{z_2(x_1,y_2)}^{x_2} \int_{J(x_1,z_2(x_1,y_2))} [D_2 g(x_1,s,z_3)]^2 \,\mathrm{d}z_3 \,\mathrm{d}s \\
\le& \int_{J(x_1)} \int_{J(x_1,s)} [D_2 g(x_1,s,z_3)]^2 \,\mathrm{d}z_3 \,\mathrm{d}s \,,
\end{aligned}$$

where the estimate follows by extending the integration region. Hence

$$A_2 \le \operatorname{diam}(K)|\sigma| \int_K [D_2 g(\mathbf{x})]^2 \,\mathrm{d}\mathbf{x} \,.$$

Using the Fubini theorem, we similarly estimate A_3 and A_4,

$$A_3 \le \operatorname{diam}(K)|\sigma| \int_K [D_2 g(\mathbf{x})]^2 \,\mathrm{d}\mathbf{x} \,,$$
$$A_4 \le \operatorname{diam}(K)|\sigma| \int_K [D_3 g(\mathbf{x})]^2 \,\mathrm{d}\mathbf{x} \,,$$

which finally yields

$$A \le 4\operatorname{diam}(K)^2|\sigma| \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \,. \tag{4.4}$$

We now turn to the study of B. We write it as

$$B = \int_0^\alpha \int_{J(x_1)} \int_{J(x_1,x_2)} \int_{J(0)} \int_{J(0,y_2)} \Big(g(x_1,z_2(x_1,y_2),z_3(x_1,y_3)) - g(0,y_2,y_3)\Big)^2 \,\mathrm{d}y_3 \,\mathrm{d}y_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \,.$$

Using the Newton integration formula and the Cauchy–Schwarz and Hölder inequalities, we have

$$\begin{aligned}\Big(g(x_1,z_2(x_1,y_2),z_3(x_1,y_3))-g(0,y_2,y_3)\Big)^2 &= \Big(\int_0^{x_1}\Big[D_1g(s,z_2(s,y_2),z_3(s,y_3))\\ &+D_2g(s,z_2(s,y_2),z_3(s,y_3))\frac{\beta-y_2}{\alpha}+D_3g(s,z_2(s,y_2),z_3(s,y_3))\frac{\gamma-y_3}{\alpha}\Big]\,\mathrm{d}s\Big)^2\\ &\le \alpha\Big(1+\Big(\frac{\beta-y_2}{\alpha}\Big)^2+\Big(\frac{\gamma-y_3}{\alpha}\Big)^2\Big)\int_0^{x_1}\sum_{i=1}^3[D_ig(s,z_2(s,y_2),z_3(s,y_3))]^2\,\mathrm{d}s\,.\end{aligned}$$

Hence

$$B\le\alpha\Big(1+\Big(\frac{\beta-y_2}{\alpha}\Big)^2+\Big(\frac{\gamma-y_3}{\alpha}\Big)^2\Big)\sum_{i=1}^3 B_i$$

with

$$B_i=\int_0^\alpha\int_{J(x_1)}\int_{J(x_1,x_2)}\int_{J(0)}\int_{J(0,y_2)}\int_0^{x_1}[D_ig(s,z_2(s,y_2),z_3(s,y_3))]^2\,\mathrm{d}s\,\mathrm{d}y_3\,\mathrm{d}y_2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,,$$

$i\in\{1,2,3\}$. Using the Fubini theorem, we have

$$B_i=\int_{J(0)}\int_{J(0,y_2)}\int_0^\alpha[D_ig(s,z_2(s,y_2),z_3(s,y_3))]^2\int_s^\alpha\int_{J(x_1)}\int_{J(x_1,x_2)}\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}s\,\mathrm{d}y_3\,\mathrm{d}y_2\,.$$

Hence

$$B_i\le\frac{|\sigma|}{2\alpha}\int_0^\alpha\int_{J(0)}\int_{J(0,y_2)}[D_ig(s,z_2(s,y_2),z_3(s,y_3))]^2(\alpha-s)^2\,\mathrm{d}y_3\,\mathrm{d}y_2\,\mathrm{d}s\,,$$

where we have used the estimate

$$\int_{J(x_1)}\int_{J(x_1,x_2)}\mathrm{d}x_3\,\mathrm{d}x_2\le|\sigma|\Big(1-\frac{x_1}{\alpha}\Big)$$

on the area of the cross-section of K and the plane $x_1=const$. Now using the change of variables $z_3=z_3(s,y_3)$ and $z_2=z_2(s,y_2)$ gives

$$\begin{aligned}&\int_{J(0)}\int_{J(0,y_2)}[D_ig(s,z_2(s,y_2),z_3(s,y_3))]^2(\alpha-s)^2\,\mathrm{d}y_3\,\mathrm{d}y_2\\ =\ &\alpha^2\int_{J(s)}\int_{J(s,z_2)}[D_ig(s,z_2,z_3)]^2\,\mathrm{d}z_3\,\mathrm{d}z_2\end{aligned}$$

and thus

$$B_i\le\frac{|\sigma|\alpha}{2}\int_K[D_ig(\mathbf{x})]^2\,\mathrm{d}\mathbf{x}\,,$$

which finally yields, noticing that $\alpha^2+(\beta-y_2)^2+(\gamma-y_3)^2=|\mathbf{a}-\mathbf{y}|^2\le\operatorname{diam}(K)^2$,

$$B\le\frac{|\sigma|}{2}\operatorname{diam}(K)^2\int_K|\nabla g(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,. \tag{4.5}$$

Now combining (4.4) and (4.5) leads to the assertion of the lemma for $d=3$. □

5 DISCRETE FRIEDRICHS INEQUALITY

We prove in this section the discrete Friedrichs inequality, using the results of the previous sections. We first give several auxiliary lemmas.

Lemma 5.1. *Let $d = 2$. Then*

$$|I(g)|^2_{1,\mathcal{T},*} \leq \frac{C_d}{\kappa^2_{\mathcal{T}}} |g|^2_{1,\mathcal{T}} \qquad \forall g \in W(\mathcal{T}_h)\,,$$

where C_d is given by (5.2) below.

Proof. Let $K \in \mathcal{T}_h$ and $\sigma_D, \sigma_E \in \mathcal{E}_K$. We define g_K as the mean value of g over K and deduce from the inequality $(a-b)^2 \leq 2a^2 + 2b^2$ and from (4.1) that

$$(g_E - g_D)^2 \leq 2(g_E - g_K)^2 + 2(g_D - g_K)^2 \leq 4c_d \frac{\operatorname{diam}(K)^2}{|K|} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,. \tag{5.1}$$

Using this, the definition of $|\cdot|_{1,\mathcal{T},*}$, $|\sigma_{D,E}| \leq 2/3 \operatorname{diam}(K)$, (2.5) and (2.4), the fact that each $K \in \mathcal{T}_h$ contains exactly three dual edges, and Assumption (A), we have

$$\begin{aligned} |I(g)|^2_{1,\mathcal{T},*} &= \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{v_{D,E}} (g_E - g_D)^2 \\ &\leq 4c_d \sum_{K \in \mathcal{T}_h} \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}, \sigma_{D,E} \subset K} \frac{|\sigma_{D,E}|^2}{v_{D,E}|\sigma_{D,E}|} \frac{\operatorname{diam}(K)^2}{|K|} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\ &\leq 8c_d \sum_{K \in \mathcal{T}_h} \Big[\frac{\operatorname{diam}(K)^2}{|K|}\Big]^2 \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leq \frac{8c_d}{\kappa^2_{\mathcal{T}}} \sum_{K \in \mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,. \end{aligned}$$

□

Lemma 5.2. *There holds*

$$|I(g)|^2_{1,\mathcal{T},\dagger} \leq \frac{C_d}{\kappa_{\mathcal{T}}} |g|^2_{1,\mathcal{T}} \qquad \forall g \in W(\mathcal{T}_h)\,,$$

where

$$C_d = 8c_d \ \text{ for } \ d = 2\,, \quad C_d = \frac{27}{4} c_d \ \text{ for } \ d = 3\,, \tag{5.2}$$

and c_d is given by (4.3).

Proof. Using the definition of $|\cdot|_{1,\mathcal{T},\dagger}$, (5.1), $|\sigma_{D,E}| \leq C^*_d \operatorname{diam}(K_{D,E})^{d-1}$ with $C^*_d = 2/3$ if $d = 2$ and $C^*_d = 9/32$ if $d = 3$, the fact that each $K \in \mathcal{T}_h$ contains $(d+1)d/2$ dual sides, and Assumption (A), we have

$$\begin{aligned} |I(g)|^2_{1,\mathcal{T},\dagger} &= \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \frac{|\sigma_{D,E}|}{\operatorname{diam}(K_{D,E})} (g_E - g_D)^2 \\ &\leq 4c_d \sum_{K \in \mathcal{T}_h} \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}, \sigma_{D,E} \subset K} \frac{|\sigma_{D,E}| \operatorname{diam}(K)}{|K|} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\ &\leq 2c_d(d+1)dC^*_d \sum_{K \in \mathcal{T}_h} \frac{\operatorname{diam}(K)^d}{|K|} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leq \frac{C_d}{\kappa_{\mathcal{T}}} \sum_{K \in \mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,. \end{aligned}$$

□

Lemma 5.3 (Interpolation estimate). *There holds*

$$\|g - I(g)\|_{0,\Omega}^2 \le c_d h^2 |g|_{1,\mathcal{T}}^2 \qquad \forall g \in W(\mathcal{T}_h)\,,$$

where c_d is given by (4.3).

Proof. We have

$$\begin{aligned} \|g - I(g)\|_{0,\Omega}^2 &= \sum_{K\in\mathcal{T}_h} \sum_{\sigma_D\in\mathcal{E}_K} \int_{K\cap D} [g(\mathbf{x}) - g_D]^2 \,\mathrm{d}\mathbf{x} \\ &\le c_d \sum_{K\in\mathcal{T}_h} \sum_{\sigma_D\in\mathcal{E}_K} [\mathrm{diam}(K\cap D)]^2 \int_{K\cap D} |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\ &\le c_d h^2 \sum_{K\in\mathcal{T}_h} \int_K |\nabla g(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,, \end{aligned}$$

using the estimate (4.2) for the simplex $K \cap D$ and $\mathrm{diam}(K \cap D) \le h$. □

We are now ready to state the first of the two main results of this paper.

Theorem 5.4 (Discrete Friedrichs inequality). *There holds*

$$\|g\|_{0,\Omega}^2 \le C_F |g|_{1,\mathcal{T}}^2 \qquad \forall g \in W_0(\mathcal{T}_h)\,,\ \forall h > 0$$

with

$$C_F = \frac{C_d}{\kappa_{\mathcal{T}}^2}|\Omega| + 2c_d h^2 \ \ \textit{for}\ \ d = 2\,, \quad C_F = 2C_d \frac{C_{d,\mathcal{T}}}{\kappa_{\mathcal{T}}} \Big[\inf_{\mathbf{b}\in\mathbb{R}^d} \mathrm{thick}_{\mathbf{b}}(\Omega)\Big]^2 + 2c_d h^2 \ \ \textit{for}\ \ d = 2, 3\,,$$

where $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case, c_d is given by (4.3), and C_d is given by (5.2).

Proof. One has

$$\|g\|_{0,\Omega}^2 \le 2\|g - I(g)\|_{0,\Omega}^2 + 2\|I(g)\|_{0,\Omega}^2\,.$$

The error $\|g - I(g)\|_{0,\Omega}^2$ of the approximation follows from Lemma 5.3. Note that $I(g) \in Y_0(\mathcal{D}_h)$ and hence the discrete Friedrichs inequality for piecewise constant functions given by Theorem 3.1 together with Lemma 5.1 yield

$$\|I(g)\|_{0,\Omega}^2 \le \frac{C_d}{2\kappa_{\mathcal{T}}^2}|\Omega||g|_{1,\mathcal{T}}^2$$

for the case where $d = 2$. Similarly, using the discrete Friedrichs inequality for piecewise constant functions given by Theorem 3.5 together with Lemma 5.2, one has

$$\|I(g)\|_{0,\Omega}^2 \le C_d \frac{C_{d,\mathcal{T}}}{\kappa_{\mathcal{T}}} [\mathrm{thick}_{\mathbf{b}}(\Omega)]^2 |g|_{1,\mathcal{T}}^2$$

for an arbitrary vector $\mathbf{b} \in \mathbb{R}^d$ for the case where $d = 2, 3$. □

Remark 5.5 (Dependence of C_F on Ω). *We have $h^2 \le |\Omega|/\kappa_{\mathcal{T}}$ by Assumption (A) and $h \le \theta_{\mathcal{T}} \mathrm{thick}_{\mathbf{b}}(\Omega)$ by the consequence (2.1) of Assumption (A). Hence the constant in the discrete Friedrichs inequality only depends on the area of Ω if $d = 2$ and on the square of the infimum over the thickness of Ω in one direction if $d = 2, 3$. This dependence is optimal: Nečas [9, Theorem 1.1] gives the same dependence for the Friedrichs inequality and $H_0^1(\Omega) \subset W_0(\mathcal{T}_h)$. Note however that the constant itself can still be better in the continuous case, see e.g. Rektorys [11, Chapters 18 and 30].*

Remark 5.6 (Dependence of C_F on the shape regularity parameter). *One can see that C_F depends on $1/\kappa_{\mathcal{T}}^2$ if $d = 2$ and when it is expressed using $|\Omega|$. We are able to establish the same result also when C_F is expressed using $\inf_{\mathbf{b}\in\mathbb{R}^d} \mathrm{thick}_{\mathbf{b}}(\Omega)$ only when the meshes are not locally refined (when Assumption (B) is satisfied). Indeed, C_F in this case depends on $C_{d,\mathcal{T}}/\kappa_{\mathcal{T}}$ and the constant $C_{d,\mathcal{T}}$ given by (3.1) is of the form $[2^d(d-1)\zeta_{\mathcal{T}}^d(2C+1)]/\kappa_{\mathcal{T}}$; this follows by replacing the inequality (3.2) by $h \le C\mathrm{thick}_{\mathbf{b}}(\Omega)$ for some suitable constant C. Example 6.3 below shows that this dependence is optimal. In the case where the meshes are only shape-regular, we only have (3.3). Note however that this dependence carries over to the case where the functions are only fixed to zero on a part of the boundary, cf. Remarks 5.8 and 5.9 below.*

Remark 5.7 (Discrete Friedrichs inequality for domains only bounded in one direction). *We see that the constant C_F only depends on the infimum over the thickness of Ω in one direction. Thus the discrete Friedrichs inequality may be extended onto domains only bounded in one direction, as it is the case for the Friedrichs inequality (cf. Nečas [9, Remark 1.1]).*

Remark 5.8 (Discrete Friedrichs inequality for functions only fixed to zero on a particular part of the boundary). *Let $\Gamma \subset \partial\Omega$ (given by a set of boundary sides) be such that there exists a vector $\mathbf{b} \in \mathbb{R}^d$ such that the first intersection of $\mathcal{B}_{\mathbf{x}}$ and $\partial\Omega$ lies in Γ for all $\mathbf{x} \in \Omega$, where $\mathcal{B}_{\mathbf{x}}$ is the straight semi-line defined by the origin $\mathbf{x}$ and the vector $\mathbf{b}$. We notice that the discrete Friedrichs inequality can immediately be extended onto functions only fixed to zero on Γ. This follows easily from the proof of Theorem 3.5 (the zero condition is only used on boundary sides lying in Γ). The dependence of C_F on the shape regularity parameter is thus given by $C_{d,\mathcal{T}}/\kappa_{\mathcal{T}}$, cf. Remark 5.6. The constant C_F in this case depends on the square of the infimum of* $\mathrm{thick}_{\mathbf{b}}(\Omega)$ *over suitable vectors* $\mathbf{b}$ *(compare with the general case treated in the next remark).*

Remark 5.9 (Discrete Friedrichs inequality for functions only fixed to zero on a general part of the boundary). *The discrete Friedrichs inequality can also be extended onto functions only fixed to zero on an arbitrary set of boundary sides, cf. Lemma 7.2 below. Then, for convex domains, C_F depends on the square of the diameter of Ω, on the ratio* $[\mathrm{diam}(\Omega)]^{d-1}/|\Gamma|$ *where Γ is the part of the boundary with the zero condition, and possibly additionally on the geometry of Ω, see Lemma 7.2. For nonconvex domains, the dependence of C_F on Ω is more complicated. The dependence of C_F on the shape regularity parameter again reveals given by $C_{d,\mathcal{T}}/\kappa_{\mathcal{T}}$, cf. Remark 5.6.*

6 DISCRETE FRIEDRICHS INEQUALITY FOR CROUZEIX–RAVIART FINITE ELEMENTS IN TWO SPACE DIMENSIONS

We show in this section how the proofs from the previous sections simplify for the case of Crouzeix–Raviart finite elements in two space dimensions. Let us consider the space $X(\mathcal{T}_h)$ introduced in Section 2. The basis of this space is spanned by the shape functions φ_D, $D \in \mathcal{D}_h$, such that $\varphi_D(Q_E) = \delta_{DE}$, $E \in \mathcal{D}_h$, δ being the Kronecker delta.

Lemma 6.1. *Let $d = 2$. Then for all $c \in X(\mathcal{T}_h)$,*

$$\|c\|_{0,\Omega} = \|I(c)\|_{0,\Omega}\,.$$

Proof. Let us write $c = \sum_{D\in\mathcal{D}_h} c_D\varphi_D$. Using that the quadrature formula $\int_K \psi\,\mathrm{d}\mathbf{x} \approx |K|/3\sum_{\sigma_D\in\mathcal{E}_K}\psi(Q_D)$ is exact for quadratic functions on triangles and (2.4), we have

$$\int_\Omega c^2(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{K\in\mathcal{T}_h}\int_K c^2(\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{K\in\mathcal{T}_h}\frac{1}{3}|K|\sum_{\sigma_D\in\mathcal{E}_K} c^2(Q_D) = \sum_{D\in\mathcal{D}_h} c_D^2|D|\,.$$

□

Lemma 6.2 (Discrete Friedrichs inequality for Crouzeix–Raviart finite elements in two space dimensions). *Let $d=2$. Then*

$$\|c\|_{0,\Omega}^2 \le C_F|c|_{1,\mathcal{T}}^2 \qquad \forall c\in X_0(\mathcal{T}_h)\,,\ \forall h>0$$

with

$$C_F = \frac{1}{4\kappa_\mathcal{T}^2}|\Omega| \ \ \text{or}\ \ C_F = \frac{C_{d,\mathcal{T}}}{2\kappa_\mathcal{T}}\Big[\inf_{\mathbf{b}\in\mathbb{R}^d}\mathrm{thick}_\mathbf{b}(\Omega)\Big]^2,$$

where $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case.

Proof. Let $c\in X_0(\mathcal{T}_h)$, $c=\sum_{D\in\mathcal{D}_h} c_D\varphi_D$. Note that by the definition of $X_0(\mathcal{T}_h)$, $c_D=0$ for all $D\in\mathcal{D}_h^{\mathrm{ext}}$. Using respectively Lemma 6.1 and Theorem 3.1 or Theorem 3.5, we get

$$\|c\|_{0,\Omega}^2 \le \frac{|\Omega|}{2}|I(c)|_{1,\mathcal{T},*}^2\,,\quad \|c\|_{0,\Omega}^2 \le C_{d,\mathcal{T}}[\mathrm{thick}_\mathbf{b}(\Omega)]^2|I(c)|_{1,\mathcal{T},\dagger}^2$$

for an arbitrary vector $\mathbf{b}\in\mathbb{R}^2$. Finally, we deduce that

$$\begin{aligned}
|I(c)|_{1,\mathcal{T},*}^2 &= \sum_{\sigma_{D,E}\in\mathcal{F}_h^{\mathrm{int}}}\frac{|\sigma_{D,E}|^2}{v_{D,E}|\sigma_{D,E}|}\Big(\nabla c|_{K_{D,E}}\cdot(Q_E-Q_D)\Big)^2\\
&\le \frac{2}{3}\sum_{\sigma_{D,E}\in\mathcal{F}_h^{\mathrm{int}}}\frac{\mathrm{diam}(K_{D,E})^2}{|K_{D,E}|}\Big|\nabla c|_{K_{D,E}}\Big|^2 d_{D,E}^2\\
&\le \frac{1}{2\kappa_\mathcal{T}^2}\sum_{K\in\mathcal{T}_h}\Big|\nabla c|_K\Big|^2|K| = \frac{1}{2\kappa_\mathcal{T}^2}|c|_{1,\mathcal{T}}^2\,,
\end{aligned}$$

using (2.5) and (2.4), $|\sigma_{D,E}|\le 2/3\ \mathrm{diam}(K_{D,E})$, the fact that the gradient of c is elementwise constant and that each $K\in\mathcal{T}_h$ contains exactly three dual edges, $d_{D,E}\le\mathrm{diam}(K_{D,E})/2$, and Assumption (A). Similarly, $|I(c)|_{1,\mathcal{T},\dagger}^2 \le 1/(2\kappa_\mathcal{T})\,|c|_{1,\mathcal{T}}^2$. □

Example 6.3 (Optimality of the dependence of C_F on the shape regularity parameter). *Let us consider a domain Ω, its triangulation $\mathcal{T}_h$, a vector $\mathbf{b}$, and a function $c\in X(\mathcal{T}_h)$ given by the values $0,1,-1$ as depicted in Figure 2. Using Lemma 6.1, we immediately have*

$$\|c\|_{0,\Omega}^2 = \sum_{K\in\mathcal{T}_h}\frac{1}{3}|K|(0+1+1) = \frac{2}{3}|\Omega|\,.$$

On each $K\in\mathcal{T}_h$, $|\nabla c|_K| = 4/h$, hence $|c|_{1,\mathcal{T}}^2 = 16/h^2\,|\Omega|$. Using Remark 5.8, the discrete Friedrichs inequality given by Lemma 6.2 holds true. The term occurring on its right hand side, independent of the shape regularity parameter, is $1/2[\mathrm{thick}_\mathbf{b}(\Omega)]^2|c|_{1,\mathcal{T}}^2 = 8v^2/h^2\,|\Omega|$. This term can be arbitrarily smaller than $\|c\|_{0,\Omega}^2$, letting $h\to+\infty$ or $v\to 0$. Next, $\kappa_\mathcal{T} =$

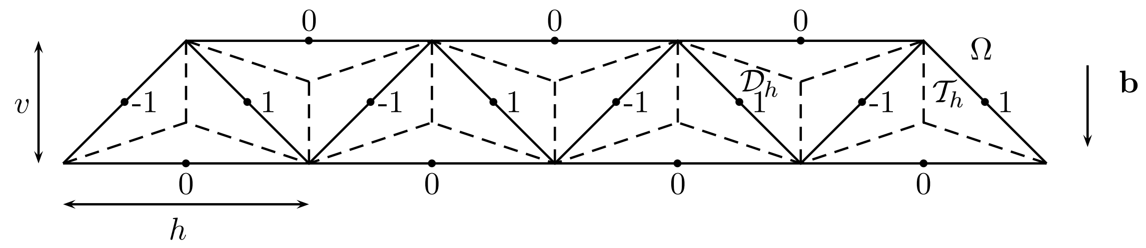

Figure 2: Domain Ω, triangulation $\mathcal{T}_h$, dual mesh $\mathcal{D}_h$, and values of a function $c \in X(\mathcal{T}_h)$ for the optimality example

$v/(2h)$. Note that $\mathcal{T}_h$ satisfies Assumption (B) and hence $C_{d,\mathcal{T}} \approx 1/\kappa_{\mathcal{T}}$. In fact, by a simple estimation of the term A from Lemma 3.3, one has $C_{d,\mathcal{T}} = 1/\kappa_{\mathcal{T}}$ in this case and thus $C_{d,\mathcal{T}}/\kappa_{\mathcal{T}} = 1/\kappa_{\mathcal{T}}^2 = 4h^2/v^2$. One immediately sees that the multiplication by this term is necessary.

Corollary 6.4 (Discrete Friedrichs inequality for Crouzeix–Raviart finite elements on equilateral triangles). *Let $d = 2$ and let $\mathcal{T}_h$ consist of equilateral triangles. Then*

$$\|c\|_{0,\Omega}^2 \leq C_F |c|_{1,\mathcal{T}}^2 \qquad \forall c \in X_0(\mathcal{T}_h)\,,\ \forall h > 0\,,$$

where

$$C_F = \frac{|\Omega|}{2} \ \text{ or } \ C_F = \Big[\inf_{\mathbf{b}\in\mathbb{R}^d} \operatorname{thick}_{\mathbf{b}}(\Omega) + 2h\Big]^2 .$$

Proof. Let c be as in the previous lemma. For equilateral triangles, one has $d_{D,E} = v_{D,E}$ and thus the norms $|\cdot|_{1,\mathcal{T},*}$ and $|\cdot|_{1,\mathcal{T},\ddagger}$ coincide. By (2.5) and (2.4), $|\sigma_{D,E}| v_{D,E} = 2/3\,|K|$, $\cos^2(\alpha) + \cos^2(\alpha + \pi/3) + \cos^2(\alpha + 2\pi/3) = 3/2$, so that

$$\sum_{K\in\mathcal{T}_h} \sum_{\sigma_{D,E}\in\mathcal{F}_h^{\text{int}},\, \sigma_{D,E}\subset K} \frac{|\sigma_{D,E}|}{d_{D,E}} \Big|\nabla c|_K\Big|^2 d_{D,E}^2 \cos^2(\nabla c|_K, Q_E - Q_D) = \sum_{K\in\mathcal{T}_h} \Big|\nabla c|_K\Big|^2 |K|\,.$$

Now using respectively Lemma 6.1 and Theorem 3.1 or Remark 3.2 yields the assertion. □

Remark 6.5 (C_F for Crouzeix–Raviart finite elements on equilateral triangles). *Let $d = 2$. Then the constant in the Friedrichs inequality may be expressed as $c_F = |\Omega|/2$ or $c_F = [\inf_{\mathbf{b}\in\mathbb{R}^d} \operatorname{thick}_{\mathbf{b}}(\Omega)]^2$, cf. Nečas [9, Theorem 1.1]. Corollary 6.4 shows that for Crouzeix–Raviart finite elements and equilateral triangles, we are able to achieve the same result (up to h) also for the constant C_F from the discrete Friedrichs inequality. We however remark that there exist sharper estimates in the continuous case, see e.g. Rektorys [11, Chapters 18 and 30].*

7 DISCRETE POINCARÉ INEQUALITY FOR PIECEWISE CONSTANT FUNCTIONS

As in the case of the discrete Friedrichs inequality, we start with the discrete Poincaré inequality for piecewise constant functions. [7, Lemma 10.2] states the discrete Poincaré inequality for piecewise constant functions on meshes satisfying the orthogonality property. We present in this section an analogy of this lemma for the mesh $\mathcal{D}_h$, where the orthogonality property is not necessarily satisfied.

Lemma 7.1. *Let ω be an open convex subset of Ω, $\omega \neq \emptyset$, and let*

$$m_\omega(c) := \frac{1}{|\omega|}\int_\omega c(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Then for all $c \in Y(\mathcal{D}_h)$,

$$\|c - m_\omega(c)\|_{0,\omega}^2 \le \frac{|B_\Omega|}{|\omega|} C_{d,\mathcal{T}} [\operatorname{diam}(\Omega)]^2 |c|_{1,\mathcal{T},\dagger}^2\,,$$

where B_Ω is the ball of $\mathbb{R}^d$ with center 0 and radius $\operatorname{diam}(\Omega)$ and $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case.

The proof of this lemma follows the proof of the first step of [7, Lemma 10.2], using the techniques introduced in Section 3 for meshes where the orthogonality property is not satisfied. Similarly, following the proof of the second step of [7, Lemma 10.2], we have:

Lemma 7.2. *Let ω be a polygonal open convex subset of Ω and let Γ be a subset of a hyperplane of $\mathbb{R}^d$ such that $\Gamma \subset \partial\omega$ and $|\Gamma| > 0$. Let*

$$m_\Gamma(c) := \frac{1}{|\Gamma|}\int_\Gamma c(\mathbf{x})\,\mathrm{d}\gamma(\mathbf{x})\,.$$

Then for all $c \in Y(\mathcal{D}_h)$,

$$\|c - m_\Gamma(c)\|_{0,\omega}^2 \le c_{\omega,\Gamma} C_{d,\mathcal{T}} \operatorname{diam}(\Omega)\operatorname{diam}(\omega)\frac{[\operatorname{diam}(\omega)]^{d-1}}{|\Gamma|}|c|_{1,\mathcal{T},\dagger}^2\,,$$

where $c_{\omega,\Gamma}$ only depends on Γ and the geometry of ω and $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case.

Remark 7.3 (The constant $c_{\omega,\Gamma}$ in two space dimensions). *Evaluating the constants from the proof of the second step of [7, Lemma 10.2], one has $c_{\omega,\Gamma} = 2 + 2/(\inf_{\mathbf{y}\in\Gamma} \mathbf{n}_\Gamma \cdot (\mathbf{a} - \mathbf{y})/|\mathbf{a}-\mathbf{y}|)$ in two space dimensions, where $\mathbf{a} \in \partial\omega$ is the most distant point from the straight line given by Γ.*

Theorem 7.4 (Discrete Poincaré inequality for piecewise constant functions). *Let*

$$m_\Omega(c) := \frac{1}{|\Omega|}\int_\Omega c(\mathbf{x})\,\mathrm{d}\mathbf{x}\,.$$

Then for all $c \in Y(\mathcal{D}_h)$,

$$\|c - m_\Omega(c)\|_{0,\Omega}^2 \le C_\Omega C_{d,\mathcal{T}} [\operatorname{diam}(\Omega)]^2 |c|_{1,\mathcal{T},\dagger}^2\,,$$

where

$$C_\Omega = \frac{|B_\Omega|}{|\Omega|} \tag{7.1}$$

when Ω is convex and

$$C_\Omega = 2\sum_{i=1}^n \frac{|B_\Omega|}{|\Omega_i|} + 16(n-1)^2\frac{|\Omega|}{|\Omega_i|_{\min}}\Big(\frac{|B_\Omega|}{|\Omega_i|_{\min}} + c_\Omega\Big) \tag{7.2}$$

when Ω is not convex but there exists a finite number of disjoint open convex polygonal sets Ω_i such that $\overline{\Omega} = \cup_{i=1}^n \overline{\Omega_i}$. Here, $|\Omega_i|_{\min} = \min_{i=1,\dots,n}\{|\Omega_i|\}$, B_Ω is the ball of $\mathbb{R}^d$ with center 0 and radius $\operatorname{diam}(\Omega)$, $c_\Omega = \max_{i=1,\dots,n}\max_{\Gamma=\partial\Omega_i\cap\partial\Omega_j \text{ for some } j,\,|\Gamma|>0} c_{\Omega_i,\Gamma}[\operatorname{diam}(\Omega_i)]^{d-1}/|\Gamma|$, and $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case.

Proof. When Ω is convex, the assertion of this theorem coincides with that of Lemma 7.1 for $\omega = \Omega$. When Ω is not convex, we have Lemmas 7.1 and 7.2 for each Ω_i. Then the third step of the proof of [7, Lemma 10.2] yields the assertion of the theorem. □

Remark 7.5. *One has*

$$\|c\|_{0,\Omega}^2 \le 2\|c - m_\Omega(c)\|_{0,\Omega}^2 + 2\|m_\Omega(c)\|_{0,\Omega}^2 .$$

Hence Theorem 7.4 implies the discrete Poincaré inequality for piecewise constant functions in the more common form

$$\|c\|_{0,\Omega}^2 \le 2C_\Omega C_{d,\mathcal{T}}[\mathrm{diam}(\Omega)]^2 |c|_{1,\mathcal{T},\dagger}^2 + \frac{2}{|\Omega|}\Big(\int_\Omega c(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)^2 \qquad \forall c \in Y(\mathcal{D}_h)\,,\ \forall h > 0\,.$$

8 DISCRETE POINCARÉ INEQUALITY

We state below the second of the two main results of this paper.

Theorem 8.1 (Discrete Poincaré inequality). *There holds*

$$\|g\|_{0,\Omega}^2 \le C_P |g|_{1,\mathcal{T}}^2 + \frac{4}{|\Omega|}\Big(\int_\Omega g(\mathbf{x})\,\mathrm{d}\mathbf{x}\Big)^2 \qquad \forall g \in W(\mathcal{T}_h)\,,\ \forall h > 0$$

with

$$C_P = 4C_d C_\Omega \frac{C_{d,\mathcal{T}}}{\kappa_{\mathcal{T}}}[\mathrm{diam}(\Omega)]^2 + 8c_d h^2\,,$$

where C_Ω is given by (7.1) when Ω is convex and by (7.2) otherwise, $C_{d,\mathcal{T}}$ is given by (3.1) when Assumption (B) is satisfied and by (3.3) in the general case, c_d is given by (4.3), and C_d is given by (5.2).

Proof. One has

$$\|g\|_{0,\Omega}^2 \le 4\|g - I(g)\|_{0,\Omega}^2 + 4\|I(g) - m_\Omega[I(g)]\|_{0,\Omega}^2 + 4\|m_\Omega[I(g)] - m_\Omega(g)\|_{0,\Omega}^2 + 4\|m_\Omega(g)\|_{0,\Omega}^2\,,$$

where $m_\Omega(f) = 1/|\Omega| \int_\Omega f(\mathbf{x})\,\mathrm{d}\mathbf{x}$. The discrete Poincaré inequality for piecewise constant functions given by Theorem 7.4 and Lemma 5.2 imply

$$\|I(g) - m_\Omega[I(g)]\|_{0,\Omega}^2 \le C_d C_\Omega \frac{C_{d,\mathcal{T}}}{\kappa_{\mathcal{T}}}[\mathrm{diam}(\Omega)]^2 |g|_{1,\mathcal{T}}^2\,.$$

We have

$$\|m_\Omega[I(g)] - m_\Omega(g)\|_{0,\Omega}^2 \le \|g - I(g)\|_{0,\Omega}^2$$

by the Cauchy–Schwarz inequality. Finally, the error $\|g - I(g)\|_{0,\Omega}^2$ of the approximation follows from Lemma 5.3. □

Remark 8.2 (Dependence of C_P on Ω). *Let Ω be a cube. We then have $h \le \mathrm{diam}(\Omega)$ and $C_\Omega \le \pi\sqrt{3}/2$ and hence the constant in the discrete Poincaré inequality in this case only depends on the square of the diameter of Ω. This dependence is optimal: Nečas [9, Theorem 1.3] gives the same dependence for the Poincaré inequality and $H^1(\Omega) \subset W(\mathcal{T}_h)$. Note however that the constant itself can still be better in the continuous case, see e.g. Payne and Weinberger [10] and Bebendorf [2] for convex domains.*

Remark 8.3 (Dependence of C_P on the shape regularity parameter). *Our results indicate that the dependence of C_P on the shape regularity parameter is given by $C_{d,\mathcal{T}}/\kappa_{\mathcal{T}}$, cf. Remark 5.6.*

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