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p-ROBUST EQUILIBRATED FLUX RECONSTRUCTION IN $\boldsymbol{H}$(curl) BASED ON LOCAL MINIMIZATIONS: APPLICATION TO A POSTERIORI ANALYSIS OF THE CURL-CURL PROBLEM*

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Abstract. We present a local construction of $\boldsymbol{H}$(curl)-conforming piecewise polynomials satisfying a prescribed curl constraint. We start from a piecewise polynomial not contained in the $\boldsymbol{H}$(curl) space but satisfying a suitable orthogonality property. The procedure employs minimizations in vertex patches, and the outcome is, up to a generic constant independent of the underlying polynomial degree, as accurate as the best approximations over the entire local versions of $\boldsymbol{H}$(curl). This allows to design guaranteed, fully computable, constant-free, and polynomial-degree-robust a posteriori error estimates of Prager–Synge type for Nédélec's finite element approximations of the curl-curl problem. A divergence-free decomposition of a divergence-free $\boldsymbol{H}$(div)-conforming piecewise polynomial, relying on overconstrained minimizations in Raviart–Thomas spaces, is the key ingredient. Numerical results illustrate the theoretical developments.

Key words. Sobolev space $\boldsymbol{H}$(curl), Sobolev space $\boldsymbol{H}$(div), equilibrated flux reconstruction, a posteriori error estimate, divergence-free decomposition, broken polynomial extension

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1. Introduction.

A posteriori error estimation by equilibrated flux reconstruction has achieved a great deal of attention for elliptic model problems like the Poisson problem. For an H^1-conforming discretization whose flux is not in $\boldsymbol{H}$(div), one has to reconstruct a flux in $\boldsymbol{H}$(div) satisfying a prescribed divergence constraint. To design high-performance algorithms, the procedure must furthermore be localized and cannot involve a solution of any supplementary global problem. Then a guaranteed, fully computable, and constant-free upper bound on the unknown discretization error follows from the equality of Prager and Synge [35]. There are several techniques of such an equilibrated flux reconstruction. Following Ladevèze and Leguillon [29] and Ainsworth and Oden [2], normal fluxes on mesh faces can first be constructed and then lifted elementwise as in Nicaise, Witowski, and Wohlmuth [34]; dual Voronoï-type grids can be employed for local nonoverlapping minimizations in $\boldsymbol{H}$(div) as in Luce and Wohlmuth [31] or Hannukainen, Stenberg, and Vohralík [27]; or a localization by the partition of unity via the finite element hat basis functions can be used for an overlapping combination of best-possible vertex-patch fluxes as in Destuynder and Métivet [15] or Braess and Schöberl [8]. This last approach is conceptual and, as established in Braess, Pillwein, and Schöberl [7] and Ern and Vohralík [21],

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gives estimates robust with respect to the polynomial degree p (henceforth termed p-robust).

In contrast, only a handful of results are available for the curl-curl problem, where, for an $\boldsymbol{H}(\mathrm{curl})$-conforming discretization whose curl is not in $\boldsymbol{H}(\mathrm{curl})$, one has to locally reconstruct a flux in $\boldsymbol{H}(\mathrm{curl})$ satisfying a prescribed curl constraint. An approach based on patchwise minimizations for the lowest-order case $p = 0$ has been designed in [8]. Its generalization for arbitrary $p \geq 1$, however, turns surprisingly difficult and, to the best of our knowledge, has not been presented yet. Several workarounds appeared in the literature recently, though. A conceptual discussion appears in Licht [30], whereas a construction following in spirit [29, 2] has been proposed and analyzed in Gedicke, Geevers, and Perugia [23]. This last approach has been recently modified in Gedicke et al. [24] in order to achieve p-robustness. A broken patchwise equilibration procedure that bypasses the Prager–Synge theorem is proposed and proved p-robust in Chaumont-Frelet, Ern, and Vohralík [10]; it relies on smaller edge patches, but the arising estimates are not constant-free.

The purpose of this contribution is to design an equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{curl})$ employing best-possible local fluxes. In doing so, we rely on localization by the partition of unity via the hat functions and overlapping flux combinations, in generalization of the concept of [8] to arbitrary $p \geq 0$. Consequently, we identify the equivalent in $\boldsymbol{H}(\mathrm{curl})$ of the concept of equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{div})$ from [15, 8, 7, 20, 21]. This is then used for a posteriori error estimation when the Nédélec (edge) finite elements of arbitrary degree $p \geq 0$ are used for approximation of the curl-curl problem. It leads to guaranteed, fully computable, and constant-free a posteriori error estimates that are locally efficient and robust with respect to the polynomial degree p; (higher-order) data oscillation terms are rigorously included in our analysis. Our p-robust efficiency proofs are based on the seminal volume and tangential trace p-robust extensions on a single tetrahedron of Costabel and McIntosh [13, Proposition 4.2] and Demkowicz, Gopalakrishnan, and Schöberl [14, Theorem 7.2]. These results were recently extended into a stable broken polynomial extension for a single tetrahedron in Chaumont-Frelet, Ern, and Vohralík [9, Theorem 2]; for an edge patch of tetrahedra in Chaumont-Frelet, Ern, and Vohralík [10, Theorem 3.1]; and for a vertex patch of tetrahedra in Chaumont-Frelet and Vohralík [11, Theorem 3.3, see also Corollary 4.3].

An important step in the construction of our estimators is to decompose the given divergence-free right-hand side into locally supported divergence-free contributions. Starting from the available (lowest-order Galerkin) orthogonality property, we propose a multistage procedure relying on two central technical results of independent interest: overconstrained minimization in Raviart–Thomas spaces leading to suitable elementwise orthogonality properties and a decomposition of a divergence-free piecewise polynomial with the above elementwise orthogonality properties into local divergence-free contributions. These issues are related to the developments on divergence-free decompositions in Scheichl [37], Alonso Rodríguez et al. [3, 4], and the references therein.

This contribution is organized as follows. Section 2 fixes the notation. Section 3 introduces the curl-curl problem, its Nédélec finite element discretization, and identifies therefrom two abstract assumptions under which our analysis is performed. In section 4, we motivate our approach at the continuous level. Section 5 then presents our main results: In section 5.1, we develop a divergence-free decomposition of the given target curl; in section 5.2, we present the equilibrated flux reconstruction based on local minimization in $\boldsymbol{H}(\mathrm{curl})$ as well as its p-robust stability; and finally, these abstract results are applied in section 5.3 to the Nédélec finite element discretization of the curl-curl problem. Section 6 is dedicated to a numerical illustration, whereas section 7 collects some technical details and proofs. Finally, in Appendices A and B, we present the two central technical results on overconstrained minimization and divergence-free decomposition.

2. Notation. The purpose of this section is to set the necessary notation. Let $\omega, \Omega \subset \mathbb{R}^3$ be open, Lipschitz polyhedra; Ω will be used to denote the computational domain, while we reserve the notation $\omega \subseteq \Omega$ for its simply connected subsets. Notice that we do not require Ω to be simply connected.

2.1. Sobolev spaces $\boldsymbol{H}^1$, $\boldsymbol{H}(\mathbf{curl})$, and $\boldsymbol{H}(\mathbf{div})$. We let $L^2(\omega)$ be the space of scalar-valued square-integrable functions defined on ω; we use the notation $\boldsymbol{L}^2(\omega) := [L^2(\omega)]^3$ for vector-valued functions with each component in $L^2(\omega)$. We denote by $\|\cdot\|_\omega$ the $L^2(\omega)$ or $\boldsymbol{L}^2(\omega)$ norm and by $(\cdot,\cdot)_\omega$ the corresponding scalar product; we drop the index when $\omega = \Omega$. We will extensively work with the following three Sobolev spaces: (1) $H^1(\omega)$, the space of scalar-valued $L^2(\omega)$ functions with weak gradients in $\boldsymbol{L}^2(\omega)$, $H^1(\omega) := \{v \in L^2(\omega);\ \nabla v \in \boldsymbol{L}^2(\omega)\}$; (2) $\boldsymbol{H}(\mathrm{curl},\omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak curls in $\boldsymbol{L}^2(\omega)$, $\boldsymbol{H}(\mathrm{curl},\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \nabla\times\boldsymbol{v} \in \boldsymbol{L}^2(\omega)\}$; and (3) $\boldsymbol{H}(\mathrm{div},\omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak divergences in $L^2(\omega)$, $\boldsymbol{H}(\mathrm{div},\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \nabla\cdot\boldsymbol{v} \in L^2(\omega)\}$. We refer the reader to Adams [1] and Girault and Raviart [25] for an in-depth description of these spaces. Moreover, componentwise $H^1(\omega)$ functions will be denoted by $\boldsymbol{H}^1(\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\ \boldsymbol{v}_i \in H^1(\omega),\ i = 1,\ldots,3\}$. We will employ the notation $\langle\cdot,\cdot\rangle_S$ for the integral product on boundary (sub)sets $S \subset \partial\omega$.

2.2. Sobolev spaces with partially vanishing traces on $\boldsymbol{\partial\Omega}$. Let Γ_{D}, Γ_{N} be two disjoint, relatively open, and possibly empty subsets of the computational domain boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$. We assume in addition that each boundary face of the mesh $\mathcal{T}_h$ defined below lies entirely either in $\overline{\Gamma_{\mathrm{D}}}$ or in $\overline{\Gamma_{\mathrm{N}}}$. Then $H^1_{0,\mathrm{D}}(\Omega)$ is the subspace of $H^1(\Omega)$ formed by functions vanishing on Γ_{D} in the sense of traces, $H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$. Let $\boldsymbol{n}_\Omega$ be the unit normal vector on $\partial\Omega$, outward to Ω. Let $\mathrm{T} = \mathrm{D}$ or N; then $\boldsymbol{H}_{0,\mathrm{T}}(\mathrm{curl},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{curl},\Omega)$ formed by functions with vanishing tangential trace on Γ_{T}, $\boldsymbol{H}_{0,\mathrm{T}}(\mathrm{curl},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}_\Omega = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{T}}\}$, where $\boldsymbol{v}\times\boldsymbol{n}_\Omega = \boldsymbol{0}$ on Γ_{T} means that $(\nabla\times\boldsymbol{v},\boldsymbol{\varphi}) - (\boldsymbol{v},\nabla\times\boldsymbol{\varphi}) = 0$ for all functions $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\varphi}\times\boldsymbol{n}_\Omega = \boldsymbol{0}$ on $\partial\Omega \setminus \Gamma_{\mathrm{T}}$. Finally, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{div},\Omega)$ formed by functions with vanishing normal trace on Γ_{N}, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_{\mathrm{N}}\}$, where $\boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0$ on Γ_{N} means that $(\boldsymbol{v},\nabla\varphi) + (\nabla\cdot\boldsymbol{v},\varphi) = 0$ for all functions $\varphi \in H^1_{0,\mathrm{D}}(\Omega)$. Fernandes and Gilardi [22] present a thorough characterization of tangential (resp., normal) traces of $\boldsymbol{H}(\mathrm{curl},\Omega)$ (resp., $\boldsymbol{H}(\mathrm{div},\Omega)$) on a part of the boundary $\partial\Omega$.

2.3. Cohomology space. The space $\boldsymbol{\mathcal{H}}(\Omega,\Gamma_{\mathrm{D}})$ of functions $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ such that $\nabla\times\boldsymbol{v} = \boldsymbol{0}$ and $\nabla\cdot\boldsymbol{v} = 0$ is the "cohomology" space associated with the domain Ω and the partition of its boundary $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$. When Ω is simply connected and Γ_{D} is connected, this space is trivial; then the conditions associated with it below can be disregarded. In the general case, $\boldsymbol{\mathcal{H}}(\Omega,\Gamma_{\mathrm{D}})$ is finite-dimensional, and its dimension depends on the topology of Ω and Γ_{D}; see [22, 26].

2.4. Tetrahedral mesh, patches of elements, and the hat functions. Let $\mathcal{T}_h$ be a simplicial mesh of the domain Ω, i.e., $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$, where any element $K \in \mathcal{T}_h$ is a closed tetrahedron with nonzero measure and where the intersection of two different tetrahedra is either empty or their common vertex, edge, or face. The shape-regularity parameter of the mesh $\mathcal{T}_h$ is the positive real number $\kappa_{\mathcal{T}_h} := \max_{K\in\mathcal{T}_h} h_K/\rho_K$, where h_K is the diameter of the tetrahedron K and ρ_K is the diameter of the largest ball contained in K. These assumptions are standard and allow for strongly graded meshes with local refinements. We will use the notation $a \lesssim b$ when there exists a positive constant C only depending on $\kappa_{\mathcal{T}_h}$ such that $a \leq Cb$.

We denote the set of vertices of the mesh $\mathcal{T}_h$ by $\mathcal{V}_h$; it is composed of interior vertices lying in Ω and of vertices lying on the boundary $\partial\Omega$. For an element $K \in \mathcal{T}_h$, $\mathcal{F}_K$ denotes the set of its faces and $\mathcal{V}_K$ the set of its vertices. Conversely, for a vertex $\boldsymbol{a} \in \mathcal{V}_h$, $\mathcal{T}_{\boldsymbol{a}}$ denotes the patch of the elements of $\mathcal{T}_h$ that share $\boldsymbol{a}$, and $\omega_{\boldsymbol{a}}$ is the corresponding open subdomain with diameter $h_{\omega_{\boldsymbol{a}}}$. A particular role below will be played by the continuous, piecewise affine "hat" function $\psi^{\boldsymbol{a}}$, which takes value 1 at the vertex $\boldsymbol{a}$ and zero at the other vertices. We note that $\omega_{\boldsymbol{a}}$ corresponds to the support of $\psi^{\boldsymbol{a}}$ and that the functions $\psi^{\boldsymbol{a}}$ form the partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1. \tag{2.1}$$

We will also need the patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ extended by one layer of neighbors and the associated subdomain $\widetilde{\omega}_{\boldsymbol{a}}$, corresponding to the supports of the hat functions $\psi^{\boldsymbol{b}}$ for all vertices $\boldsymbol{b}$ contained in the patch $\mathcal{T}_{\boldsymbol{a}}$.

2.5. Piecewise polynomial spaces. Let $q \geq 0$ be an integer. For a single tetrahedron $K \in \mathcal{T}_h$, denote by $\mathcal{P}_q(K)$ the space of scalar-valued polynomials on K of total degree at most q and by $[\mathcal{P}_q(K)]^3$ the space of vector-valued polynomials on K with each component in $\mathcal{P}_q(K)$. The Nédélec [6, 33] space of degree q on K is then given by

$$\boldsymbol{\mathcal{N}}_q(K) := [\mathcal{P}_q(K)]^3 + \boldsymbol{x}\times[\mathcal{P}_q(K)]^3. \tag{2.2}$$

Similarly, the Raviart–Thomas [6, 36] space of degree q on K is given by

$$\boldsymbol{\mathcal{RT}}_q(K) := [\mathcal{P}_q(K)]^3 + \mathcal{P}_q(K)\boldsymbol{x}. \tag{2.3}$$

We note that (2.2) and (2.3) are equivalent to the writing with a direct sum and only homogeneous polynomials in the second terms. The second term in (2.2) can also equivalently be given by homogeneous $(q+1)$-degree polynomials $\boldsymbol{v}_h$ such that $\boldsymbol{x}\cdot\boldsymbol{v}_h(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in K$.

We will below extensively use the broken, piecewise polynomial spaces formed from the above element spaces

$$\begin{aligned}
\mathcal{P}_q(\mathcal{T}_h) &:= \{v_h \in L^2(\Omega);\, v_h|_K \in \mathcal{P}_q(K) \quad \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_q(K) \quad \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\, \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{RT}}_q(K) \quad \forall K \in \mathcal{T}_h\}.
\end{aligned}$$

To form the usual finite-dimensional Sobolev subspaces, we will write $\mathcal{P}_q(\mathcal{T}_h) \cap H^1(\Omega)$ (for $q \geq 1$), $\boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl},\Omega)$, $\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ (both for $q \geq 0$), and similarly for the subspaces reflecting the different boundary conditions. The same notation will also be used on the patches of mesh elements $\mathcal{T}_{\boldsymbol{a}}$.

2.6. L^2-orthogonal projectors and the Raviart–Thomas interpolator. For $q\geq 0$, let Π_q denote the $L^2(K)$-orthogonal projector onto $\mathcal{P}_q(K)$. Since this is an elementwise procedure, we keep the same notation for the $L^2(\Omega)$-orthogonal projector onto $\mathcal{P}_q(\mathcal{T}_h)$ given for $v\in L^2(\Omega)$ as $\Pi_q(v)\in\mathcal{P}_q(\mathcal{T}_h)$ such that $(\Pi_q(v),w_h)=(v,w_h)$ for all $w_h\in\mathcal{P}_q(\mathcal{T}_h)$. Then $\boldsymbol{\Pi}_q$ is given componentwise by Π_q.

Let $K\in\mathcal{T}_h$ be a mesh tetrahedron, and let $\boldsymbol{v}\in[C^1(K)]^3$ be given. Following [6, 36], the canonical q-degree Raviart–Thomas interpolant $\boldsymbol{I}^{\mathcal{RT}}_{K,q}(\boldsymbol{v})\in\boldsymbol{\mathcal{RT}}_q(K)$, $q\geq 0$, is given by

$$\langle \boldsymbol{I}^{\mathcal{RT}}_{K,q}(\boldsymbol{v})\cdot\boldsymbol{n}_K, r_h\rangle_F=\langle\boldsymbol{v}\cdot\boldsymbol{n}_K,r_h\rangle_F \qquad \forall r_h\in\mathcal{P}_q(F),\quad \forall F\in\mathcal{F}_K, \tag{2.4a}$$

$$(\boldsymbol{I}^{\mathcal{RT}}_{K,q}(\boldsymbol{v}),\boldsymbol{r}_h)_K=(\boldsymbol{v},\boldsymbol{r}_h)_K \qquad \forall\boldsymbol{r}_h\in[\mathcal{P}_{q-1}(K)]^3. \tag{2.4b}$$

Less regular functions can be used in (2.4), but $\boldsymbol{v}\in[C^1(K)]^3$ will be sufficient for our purposes; we will actually only employ polynomial $\boldsymbol{v}$ as arguments of $\boldsymbol{I}^{\mathcal{RT}}_{K,q}$. This interpolator crucially satisfies, on the tetrahedron K, the commuting property

$$\nabla\cdot\boldsymbol{I}^{\mathcal{RT}}_{K,q}(\boldsymbol{v})=\Pi_q(\nabla\cdot\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in[C^1(K)]^3. \tag{2.5}$$

2.7. Sobolev spaces on the patch subdomains $\boldsymbol{\omega_a}$. Let $\boldsymbol{a}\in\mathcal{V}_h$ be an interior vertex. Then we set (1) $H^1_*(\omega_{\boldsymbol{a}}):=\{v\in H^1(\omega_{\boldsymbol{a}});\,(v,1)_{\omega_{\boldsymbol{a}}}=0\}$, so that $H^1_*(\omega_{\boldsymbol{a}})$ is the subspace of those $H^1(\omega_{\boldsymbol{a}})$ functions whose mean value vanishes; (2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\,\boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=\boldsymbol{0}\text{ on }\partial\omega_{\boldsymbol{a}}\}$, where the tangential trace is understood by duality as above in section 2.2; and, similarly, (3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\,\boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=0\text{ on }\partial\omega_{\boldsymbol{a}}\}$. We will also need (4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}):=\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$ (the symbol † is used here for notational purposes). The situation is more subtle for boundary vertices. As a first possibility, if $\boldsymbol{a}\in\Gamma_{\mathrm{N}}$ (i.e., $\boldsymbol{a}\in\mathcal{V}_h$ is a boundary vertex such that all the faces sharing the vertex $\boldsymbol{a}$ lie in Γ_{N}), then the spaces $H^1_*(\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, and $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}})$ are defined as above. Second, when $\boldsymbol{a}\in\overline{\Gamma_{\mathrm{D}}}$, at least one of the faces sharing the vertex $\boldsymbol{a}$ lies in $\overline{\Gamma_{\mathrm{D}}}$; we denote by γ_{D} the subset of Γ_{D} formed by all mesh faces sharing the vertex $\boldsymbol{a}$ and lying in $\overline{\Gamma_{\mathrm{D}}}$. In this situation, we let (1) $H^1_*(\omega_{\boldsymbol{a}}):=\{v\in H^1(\omega_{\boldsymbol{a}});\,v=0\text{ on }\gamma_{\mathrm{D}}\}$; (2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\,\boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=\boldsymbol{0}\text{ on }\partial\omega_{\boldsymbol{a}}\setminus\gamma_{\mathrm{D}}\}$; (3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\,\boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=0\text{ on }\partial\omega_{\boldsymbol{a}}\setminus\gamma_{\mathrm{D}}\}$; and (4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}):=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\,\boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=\boldsymbol{0}\text{ on }\gamma_{\mathrm{D}}\}$.

2.8. Functional inequalities. To work with data oscillation terms, we will employ the following three functional inequalities. First, from [12, Theorems 3.4 and 3.5], [28, Theorem 2.1], and the discussion in [10, section 3.2.1], it follows that there exists a constant C_{L} such that for all $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, there exists $\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\cap\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega))$ such that $\nabla\times\boldsymbol{w}=\nabla\times\boldsymbol{v}$ and

$$\|\nabla\boldsymbol{w}\|\leq C_{\mathrm{L}}\|\nabla\times\boldsymbol{v}\|. \tag{2.6}$$

When either Γ_{D} or Γ_{N} has zero measure and if Ω is convex, one can take $C_{\mathrm{L}}=1$; see [12] together with [25, Theorem 3.7] for Dirichlet boundary conditions and [25, Theorem 3.9] for Neumann boundary conditions.

Second, for any mesh element $K\in\mathcal{T}_h$ and $\boldsymbol{v}\in\boldsymbol{H}^1(K)$, there holds the Poincaré inequality

$$\|\boldsymbol{v}-\boldsymbol{\Pi}_0(\boldsymbol{v})\|_K\leq\frac{h_K}{\pi}\|\nabla\boldsymbol{v}\|_K, \tag{2.7}$$

where $\boldsymbol{\Pi}_0(\boldsymbol{v})$ denotes the componentwise mean value of $\boldsymbol{v}$ on K.

Third, the Poincaré–Friedrichs–Weber inequality (see [22, Proposition 7.4] and more precisely [10, Theorem A.1] for the form of the constant) will be useful: For all vertices $\boldsymbol{a} \in \mathcal{V}_h$ and all vector-valued functions $\boldsymbol{v} \in \boldsymbol{H}^{\dagger}(\mathrm{curl}, \omega_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla\cdot\boldsymbol{v} = 0$, we have

$$\|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}. \tag{2.8}$$

Strictly speaking, the inequality is established in [10, Theorem A.1] for edge patches, but the proof can be easily extended to vertex patches.

3. Setting. The purpose of this section is to introduce the curl-curl problem and its Nédélec finite element approximation. We also identify, in a form of two self-standing assumptions, the kernel properties solely needed for our analysis.

3.1. Current density. The following assumption is central for us.

Assumption 3.1 (current density $\boldsymbol{j}$). Let $\boldsymbol{j}$ be $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$-conforming, divergence-free, and $L^2(\Omega)$-orthogonal to the cohomology space $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_{\mathrm{D}})$, i.e.,

$$\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega), \tag{3.1a}$$

$$\nabla\cdot\boldsymbol{j} = 0, \tag{3.1b}$$

$$(\boldsymbol{j}, \boldsymbol{\varphi}) = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{H}}(\Omega, \Gamma_{\mathrm{D}}). \tag{3.1c}$$

Let us recall from section 2.3 that when Ω is simply connected and Γ_{D} is connected, (3.1c) can be disregarded. Sometimes, to illustrate the main ideas, we will additionally suppose that $\boldsymbol{j}$ is a piecewise p-degree Raviart–Thomas polynomial, $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$. Assumption 3.1 equivalently means that $\boldsymbol{j}$ belongs to the range of the curl operator; i.e., there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla\times\boldsymbol{v} = \boldsymbol{j}$.

3.2. The curl-curl problem. The curl-curl problem we study here reads as follows: Find the magnetic vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ such that

$$\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}, \qquad \nabla\cdot\boldsymbol{A} = 0 \qquad \text{in } \Omega, \tag{3.2a}$$

$$\boldsymbol{A}\times\boldsymbol{n}_\Omega = \boldsymbol{0} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{3.2b}$$

$$(\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega = \boldsymbol{0}, \qquad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0 \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{3.2c}$$

with the additional requirement that $(\boldsymbol{A}, \boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi}$ from the cohomology space $\boldsymbol{\mathcal{H}}(\Omega, \Gamma_{\mathrm{D}})$ introduced in section 2.3 to ensure uniqueness. Introducing $\boldsymbol{K}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega);\ \nabla\times\boldsymbol{v} = \boldsymbol{0}\}$, the weak formulation of problem (3.2) (cf., e.g., [6]) consists in finding a pair $(\boldsymbol{A}, \boldsymbol{q}) \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega) \times \boldsymbol{K}(\Omega)$ such that

$$(\boldsymbol{A}, \boldsymbol{\varphi}) = 0 \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{K}(\Omega), \tag{3.3a}$$

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) + (\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega). \tag{3.3b}$$

Picking the test function $\boldsymbol{v} = \boldsymbol{q}$ in (3.3b), we see that $\boldsymbol{q} = \boldsymbol{0}$. Thus, $\boldsymbol{A}$ is such that

$$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega), \tag{3.4a}$$

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega). \tag{3.4b}$$

Remark 3.2 (characterization (3.4) of the magnetic vector potential $\boldsymbol{A}$). All the main developments below actually rely solely on (3.4), so that in particular, the vector field $\boldsymbol{A}$ can in our setting only be defined up to a curl-free component. Remark that the existence of $\boldsymbol{A}$ satisfying (3.4) is a direct consequence of Assumption 3.1 and that a direct consequence of (3.4) is that $\nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ with $\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}$.

3.3. Nédélec finite element approximation. For an integer $p \geq 0$ that we consider fixed henceforth, let the Nédélec finite element space be given by $\boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$. The subspace $\boldsymbol{K}_h := \{\boldsymbol{v}_h \in \boldsymbol{V}_h;\ \nabla\times\boldsymbol{v}_h = \boldsymbol{0}\}$ is simply $\nabla(\mathcal{P}_{p+1}(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega))$ when Ω is simply connected and Γ_D is connected and can be readily identified by introducing "cuts" in the mesh mimicking the construction of the cohomology space $\boldsymbol{\mathcal{H}}(\Omega,\Gamma_\mathrm{D})$; see [26, Chapter 6]. The finite element approximation of (3.3) is a pair $(\boldsymbol{A}_h, \boldsymbol{q}_h) \in \boldsymbol{V}_h \times \boldsymbol{K}_h$ such that

$$(\boldsymbol{A}_h, \boldsymbol{\varphi}_h) = 0 \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{K}_h, \tag{3.5a}$$

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) + (\boldsymbol{q}_h, \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.5b}$$

Observing that $\boldsymbol{K}_h \subset \boldsymbol{K}$, this actually leads to $\boldsymbol{A}_h \in \boldsymbol{V}_h$ such that

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.6}$$

In the developments below, we can actually still weaken (3.6) and rely solely on the following.

Assumption 3.3 (discrete magnetic vector potential $\boldsymbol{A}_h$). Let $\boldsymbol{A}_h$ be a piecewise p-degree Nédélec polynomial satisfying a lowest-order Nédélec orthogonality property:

$$\boldsymbol{A}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega), \tag{3.7a}$$

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega). \tag{3.7b}$$

4. Motivation. Let $\boldsymbol{j}$ satisfy Assumption 3.1. We motivate here our approach by showing how an equilibrated flux $\boldsymbol{h}$ may be constructed locally from any $\boldsymbol{A}$ satisfying (3.4) at the continuous level. These observations are the basis of the actual flux equilibration procedure involving $\boldsymbol{A}_h$ satisfying Assumption 3.3 at the discrete level that we develop in sections 5.1 and 5.2 below. We would in particular like to identify a patchwise construction such that

$$\boldsymbol{h}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}), \tag{4.1a}$$

$$\boldsymbol{h} := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}^{\boldsymbol{a}} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega), \tag{4.1b}$$

$$\nabla\times\boldsymbol{h} = \boldsymbol{j}. \tag{4.1c}$$

At the continuous level, the solution is trivially

$$\boldsymbol{h}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}), \tag{4.2}$$

where we implicitly extend by 0 or restrict to $\omega_{\boldsymbol{a}}$.

We now rewrite the above definition implicitly. The idea is to introduce

$$\boldsymbol{h}^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}^{\boldsymbol{a}}}} \|\boldsymbol{v} - \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})\|^2_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h \tag{4.3}$$

with a suitable curl constraint $\boldsymbol{j}^{\boldsymbol{a}}$. Since

$$\nabla\times(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})) = \psi^{\boldsymbol{a}}\,\underbrace{(\nabla\times(\nabla\times\boldsymbol{A}))}_{\boldsymbol{j}} + \underbrace{\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})}_{\boldsymbol{\theta}^{\boldsymbol{a}}}, \tag{4.4}$$

we have

$$\boldsymbol{j}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \boldsymbol{j} + \boldsymbol{\theta}^{\boldsymbol{a}}, \qquad \boldsymbol{\theta}^{\boldsymbol{a}} := \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}). \tag{4.5}$$

Importantly, it holds that

$$\boldsymbol{\theta}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \tag{4.6a}$$

$$\nabla \cdot \boldsymbol{\theta}^{\boldsymbol{a}} = \underbrace{\nabla \times \nabla \psi^{\boldsymbol{a}}}_{0} \cdot (\nabla \times \boldsymbol{A}) - \nabla \psi^{\boldsymbol{a}} \cdot \underbrace{\nabla \times (\nabla \times \boldsymbol{A})}_{\boldsymbol{j}} = -\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}, \tag{4.6b}$$

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\theta}^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}) = \boldsymbol{0}, \tag{4.6c}$$

where the last property follows by the partition of unity (2.1). Consequently,

$$\boldsymbol{j}^{\boldsymbol{a}} = \psi^{\boldsymbol{a}} \boldsymbol{j} + \boldsymbol{\theta}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \tag{4.7a}$$

$$\nabla \cdot \boldsymbol{j}^{\boldsymbol{a}} = \nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j} + \psi^{\boldsymbol{a}} \underbrace{\nabla \cdot \boldsymbol{j}}_{0} + \nabla \cdot \boldsymbol{\theta}^{\boldsymbol{a}} = 0, \tag{4.7b}$$

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}^{\boldsymbol{a}} = \boldsymbol{j}, \tag{4.7c}$$

which gives a decomposition of the divergence-free current density $\boldsymbol{j}$ into divergence-free contributions $\boldsymbol{j}^{\boldsymbol{a}}$ defined over the vertex patch subdomains $\omega_{\boldsymbol{a}}$. The above auxiliary fields $\boldsymbol{\theta}^{\boldsymbol{a}}$ can also be defined implicitly as the solution to the minimization problems:

$$\boldsymbol{\theta}^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = -\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}}} \|\boldsymbol{v} - \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A})\|_{\omega_{\boldsymbol{a}}}^2 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{4.8}$$

We shall now mimic (4.3), (4.7), and (4.8) at the discrete level.

5. Main results.

In this section, we summarize our main results.

5.1. Stable divergence-free patchwise decomposition of the given current density $\boldsymbol{j}$.

The central issue for our approach is a stable divergence-free patchwise decomposition of the current density $\boldsymbol{j}$ in the spirit of (4.7). For this purpose, we first design an appropriate discrete variant of (4.8), where we crucially rely on the patchwise orthogonality stemming from Assumption 3.3. We will initially be requested to work with the increased polynomial degree $p' := \min\{p, 1\}$,

$$p' := \min\{p, 1\}, \tag{5.1}$$

recalling that $p \geq 0$ is fixed in section 3.3. We start with the following.

Definition 5.1 (patchwise contributions $\boldsymbol{j}_h^{\boldsymbol{a}}$). *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy, respectively, Assumptions* 3.1 *and* 3.3. *Carry out the three following steps:*

1. *For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, consider the p'-degree Raviart–Thomas patchwise (seemingly overconstrained) minimizations*

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{p'}(-\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}) \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h), \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}^2, \tag{5.2}$$

where in addition to the usual normal trace and divergence, the constraints additionally also concern elementwise product with piecewise vector-valued constants.

2. *Extending $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$, set*

$$\boldsymbol{\delta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}}. \tag{5.3}$$

For all tetrahedra $K \in \mathcal{T}_h$, consider the $(p+1)$-degree Raviart–Thomas elementwise minimizations,

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_1(K)\\ \nabla\cdot\boldsymbol{v}_h=0\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=\boldsymbol{I}_{K,1}^{\boldsymbol{\mathcal{RT}}}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}_{K,1}^{\boldsymbol{\mathcal{RT}}}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K^2 \quad \forall \boldsymbol{a}\in\mathcal{V}_\mu \quad \textit{when } p=0, \tag{5.4a}$$

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{p+1}(K)\\ \nabla\cdot\boldsymbol{v}_h=0\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K^2 \quad \forall \boldsymbol{a}\in\mathcal{V}_K \quad \textit{when } p\geq 1, \tag{5.4b}$$

which yields the divergence-free decomposition

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}.$$

3. *For all vertices $\boldsymbol{a}\in\mathcal{V}_h$, define*

$$\boldsymbol{j}_h^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}, \tag{5.5}$$

with an implicit restriction of $\psi^{\boldsymbol{a}}\boldsymbol{j}$ to $\omega_{\boldsymbol{a}}$.

For a vertex $\boldsymbol{a}\in\mathcal{V}_h$ and the extended (second-order) patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, define the data oscillation term

$$\widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}} := \left\{ \sum_{K\in\widetilde{\mathcal{T}}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K \right)^2 \right\}^{\frac{1}{2}}. \tag{5.6}$$

We crucially have the following.

THEOREM 5.2 (stable divergence-free patchwise decomposition of $\boldsymbol{j}$). *Let $\boldsymbol{j}$ and $\boldsymbol{A}_h$ satisfy, respectively, Assumptions* 3.1 *and* 3.3. *Let $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition* 5.1 *for all vertices $\boldsymbol{a}\in\mathcal{V}_h$. Then*

$$\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}), \tag{5.7a}$$

$$\nabla\cdot\boldsymbol{j}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})), \tag{5.7b}$$

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}, \tag{5.7c}$$

where the extension of $\boldsymbol{j}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$ is understood in the last two properties. Moreover, when $\boldsymbol{j}\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ is piecewise polynomial, in strengthening of (5.7a)–(5.7b),

$$\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}), \tag{5.8a}$$

(5.8b) $$\nabla \cdot \boldsymbol{j}_h^{\boldsymbol{a}} = 0.$$

Let in addition $\boldsymbol{A}$ *satisfying* (3.4) *be arbitrary, and let, as in* (4.5),

$$\boldsymbol{j}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \boldsymbol{j} + \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}).$$

Then

(5.9) $$\|\boldsymbol{j}^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1} \left[\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}} \right].$$

Remarks. Several remarks about Definition 5.1 and Theorem 5.2 are in order:

1. At the discrete level, $\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h) \notin \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$, in contrast to $\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A})$; see (4.5)–(4.6). The auxiliary field $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ from (5.2) is the projection of $\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)$ to $\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ satisfying $\nabla \cdot \boldsymbol{\theta}_h^{\boldsymbol{a}} = \Pi_{p'}(-\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j})$. Step 1 of Definition 5.1 thus mimics (4.8) and achieves equivalents to (4.6a) and (4.6b). Unfortunately, $\boldsymbol{\delta}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}}$ given by (5.3) typically does not equal $\mathbf{0}$, which would mimic (4.6c).
2. Step 2 of Definition 5.1 yields the corrected fields $\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}$, which mimic (4.6) entirely in that (see Lemma 7.4 below for details)
$$\begin{aligned} \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}} &\in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \\ \nabla \cdot (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}) &= \Pi_{p'}(-\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j}), \\ \sum_{\boldsymbol{a} \in \mathcal{V}_h} (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}) &= \boldsymbol{\delta}_h - \boldsymbol{\delta}_h = \mathbf{0}. \end{aligned}$$
3. Step 3 of Definition 5.1, in view of Theorem 5.2, finally materializes (4.7) at the discrete level.
4. Property (5.9) from Theorem 5.2 shows that the local discrete decomposition (5.7) compares in a p-robust way to the continuous-level decomposition (4.7), up to data oscillation given by (5.6).
5. Minimization (5.2) contains an additional constraint on the elementwise product with piecewise vector-valued constants. Existence, uniqueness, and p-robust stability theory for such problems is developed in Appendix A. Section 7 shows that this applies to our setting under the orthogonality in Assumption 3.3.
6. The additional constraint in (5.2) also ensures the existence, uniqueness, and p-robust stability of the elementwise problems (5.4), where $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ form a divergence-free local decomposition of $\boldsymbol{\delta}_h$ following Appendix B; see Lemma 7.4 below.

5.2. Equilibrated flux reconstruction based on local patchwise minimizations in $\boldsymbol{H}(\mathbf{curl})$. We now identify an appropriate discrete variant of (4.1)–(4.3), giving a locally computable equilibrated flux $\boldsymbol{h}_h$. Let $\boldsymbol{j}_h^{\boldsymbol{a}}$ be given by Definition 5.1, and set

(5.10) $$\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{j}_h^{\boldsymbol{a}}, \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2.$$

When $\boldsymbol{j}$ is piecewise polynomial, $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$, (5.8) implies that $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} = \boldsymbol{j}_h^{\boldsymbol{a}}$, so that there is no need for (5.10). In general, the role of (5.10) is to prepare a piecewise polynomial datum for a discrete version of (4.3): It projects the nonpolynomial and non-divergence-free $\boldsymbol{j}_h^{\boldsymbol{a}}$ to $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot \bar{\boldsymbol{j}}_h^{\boldsymbol{a}} = 0$.

Problem (5.10) has the same form as problem (5.2) and is well-posed, following Appendix A below. With the Raviart–Thomas divergence-free $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}$, the following Nédélec local equilibration problem is well-posed by standard arguments; see, e.g., [6].

DEFINITION 5.3 (equilibrated flux reconstruction based on local minimization in $\boldsymbol{H}(\mathrm{curl})$). *Let* $\boldsymbol{j}$ *and* $\boldsymbol{A}_h$ *satisfy, respectively, Assumptions* 3.1 *and* 3.3*, and let, for all vertices* $\boldsymbol{a} \in \mathcal{V}_h$*,* $\boldsymbol{j}_h^{\boldsymbol{a}}$ *be given by Definition* 5.1 *and* $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}$ *by* (5.10)*. Consider the patchwise minimizations*

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \bar{\boldsymbol{j}}_h^{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}^2. \tag{5.11a}$$

Extending $\boldsymbol{h}_h^{\boldsymbol{a}}$ *by zero outside of* $\omega_{\boldsymbol{a}}$*, define*

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}. \tag{5.11b}$$

Recall $\widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}$ from (5.6), and define

$$\eta_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}}^{\boldsymbol{a}} := h_{\omega_{\boldsymbol{a}}} \|\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}. \tag{5.12}$$

Crucially, the construction of Definition 5.3 is a p-robustly stable equilibration.

THEOREM 5.4 (equilibrium property and p-robust stability of the flux reconstruction). *Let* $\boldsymbol{j}$ *and* $\boldsymbol{A}_h$ *satisfy, respectively, Assumptions* 3.1 *and* 3.3*. Then the equilibrated flux reconstruction* $\boldsymbol{h}_h$ *from Definition* 5.3 *satisfies*

$$\boldsymbol{h}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega), \tag{5.13a}$$

$$\nabla \times \boldsymbol{h}_h = \boldsymbol{j} \quad \text{when } \boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega). \tag{5.13b}$$

Let in addition $\boldsymbol{A}$ *satisfying* (3.4) *be arbitrary. Then*

$$\|\boldsymbol{h}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \eta_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}}^{\boldsymbol{a}} + \widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}.$$

5.3. Guaranteed, fully computable, constant-free, and p-robust a posteriori error estimates for the curl-curl problem. We apply here the results of sections 5.1 and 5.2 to a posteriori error analysis of the curl-curl problem (3.2).

THEOREM 5.5 (guaranteed, fully computable, and constant-free upper bound). *Let* $\boldsymbol{j}$ *satisfy Assumption* 3.1*, let* $\boldsymbol{A}$ *be the weak solution to the curl-curl problem given by* (3.3)*, and let* $\boldsymbol{A}_h$ *be its Nédélec finite element approximation given by* (3.5)*. Let* $\boldsymbol{j}_h^{\boldsymbol{a}}$ *be given by Definition* 5.1 *for all vertices* $\boldsymbol{a} \in \mathcal{V}_h$*, and let* $\boldsymbol{h}_h$ *be given by Definition* 5.3*. Then*

$$\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\| \le \eta_{\mathrm{tot}} := \underbrace{\|\boldsymbol{h}_h - \nabla \times \boldsymbol{A}_h\|}_{\eta} + C_{\mathrm{L}} \underbrace{\left\{ \sum_{K \in \mathcal{T}_h} \underbrace{\frac{h_K^2}{\pi^2} \|\boldsymbol{j} - \nabla \times \boldsymbol{h}_h\|_K^2}_{\eta_{\mathrm{osc},K}^2} \right\}^{\frac{1}{2}}}_{\eta_{\mathrm{osc}}}$$

and

$$\|\boldsymbol{h}_h - \nabla \times \boldsymbol{A}_h\|_K + \eta_{\mathrm{osc},K} \lesssim \sum_{\boldsymbol{a} \in \mathcal{V}_K} \left[\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \eta_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}}^{\boldsymbol{a}} + \widetilde{\eta}_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}} \right].$$

Remarks. Several remarks are in order:

1. On the discrete level, $\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h) \notin \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$, in contrast to $\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})$ on the continuous level; see (4.1)–(4.2). The equilibrated flux contribution $\boldsymbol{h}_h^{\boldsymbol{a}}$ from (5.11a) is its constrained projection to $\boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$. It mimics (4.3) at the discrete level.
2. When $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, all the data oscillation estimators in Theorem 5.5 vanish. Indeed, (5.13b) implies that $\eta_{\mathrm{osc}} = 0$, whereas $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} = \boldsymbol{j}_h^{\boldsymbol{a}}$ gives $\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}} = 0$; see (5.12). Similarly, $\widetilde{\eta}^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ from (5.6) vanishes as well (this is actually true up to $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ since $\nabla\cdot\boldsymbol{j} = 0$). Moreover, all these terms are higher-order with respect to $\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$ if $\boldsymbol{j}$ is piecewise smooth. We also note that using (5.11b), (5.11a), and (5.7c), the data oscillation term $\eta_{\mathrm{osc},K}$ can equivalently be rewritten with

$$\boldsymbol{j} - \nabla\times\boldsymbol{h}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\boldsymbol{j}_h^{\boldsymbol{a}} - \bar{\boldsymbol{j}}_h^{\boldsymbol{a}}). \tag{5.14}$$

3. The equilibration of Definition 5.3 is performed in local Nédélec spaces of order $p+1$. This is in agreement with p-robust flux equilibrations from [7, 20, 21, 11]. Similarly to [7, 19], it is also possible to design a downgrade of the orders of the local problems (5.11a) from $p+1$ to p.

   Let us first discuss the case $p \geq 1$. The first step is to replace (5.4b) by (5.4a) with $\boldsymbol{\mathcal{RT}}_1$ replaced by $\boldsymbol{\mathcal{RT}}_p$. Then, according to Theorem B.1 with $q' = p$, we obtain $\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ in place of (7.5a) below. Second, we employ (elementwise) the projector $\boldsymbol{I}_{K,p}^{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{j})$ in (5.5) in place of $\psi^{\boldsymbol{a}}\boldsymbol{j}$. Let $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Then $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}$ and $\nabla\cdot\boldsymbol{j}_h^{\boldsymbol{a}} = 0$ as in (5.7c), (5.8b), but $\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ in place of (5.8a). Consequently, (5.11a) can be brought down to

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \boldsymbol{I}_p^{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h))\|^2_{\omega_{\boldsymbol{a}}}, \tag{5.15}$$

   where $\boldsymbol{I}_p^{\boldsymbol{\mathcal{N}}}$ is the elementwise canonical p-degree Nédélec interpolate, analogue to (2.4). This leads to a cheaper procedure where the guaranteed estimate of Theorem 5.5 (with $\eta_{\mathrm{osc}} = 0$) still holds true when $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$; general $\boldsymbol{j}$ can be covered by data oscillation terms. Similarly, the local efficiency of Theorem 5.5 is also preserved, with, however, the p-robustness theoretically lost. In particular, from (B.6b), estimate (7.6) below still holds true, up to a possibly p-dependent constant.

   Alternatively, for $p = 0$ in particular, because of $p' = p+1$ employed in (5.2), we need to replace (5.5) by

$$\boldsymbol{j}_h^{\boldsymbol{a}}|_K := \boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}((\psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})|_K) \qquad \forall K \in \mathcal{T}_h.$$

   Let $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Then clearly $\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_0(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$. Moreover, from (2.5), $\nabla\cdot\boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K) = \Pi_0(\nabla\cdot(\boldsymbol{\delta}_h^{\boldsymbol{a}})|_K) = 0$, whereas $\nabla\cdot\boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}((\psi^{\boldsymbol{a}}\boldsymbol{j})|_K) = \Pi_0(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{j})|_K) = (\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j})|_K$, also using that $\boldsymbol{j} \in [\mathcal{P}_0(\mathcal{T}_{\boldsymbol{a}})]^3$ as above in point 2, and similarly $\nabla\cdot\boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\theta}_h^{\boldsymbol{a}}|_K) = \Pi_0(\nabla\cdot(\boldsymbol{\theta}_h^{\boldsymbol{a}})|_K) = (-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j})|_K$. Thus, $\nabla\cdot\boldsymbol{j}_h^{\boldsymbol{a}} = 0$. Finally, $(\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}})|_K = \boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}(\sum_{\boldsymbol{a}\in\mathcal{V}_h}(\psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})|_K) = \boldsymbol{I}_{K,0}^{\boldsymbol{\mathcal{RT}}}(\boldsymbol{j}|_K) = \boldsymbol{j}|_K$ by the linearity of the Raviart–Thomas projector $\boldsymbol{\mathcal{RT}}_0$. Then the above discussion for $p \geq 1$ applies.

4. The approach of [23, 24] includes solutions of local, a priori overdetermined problems on vertex patches in a multistage procedure. The present (again a priori overconstrained) problems (5.2) and consecutive steps in Definitions 5.1 and 5.3 share this spirit, though the minimizations directly determine the best-possible local energy error estimator contributions.

6. Numerical illustration. This section presents some numerical examples illustrating the key features of the estimator of Theorem 5.5. We impose the Dirichlet boundary condition on the whole boundary, i.e., $\Gamma_{\mathrm{D}} := \partial\Omega$. We consider both structured meshes and unstructured meshes. When we speak about a "structured" mesh, we mean a Cartesian partition of Ω into $N \times N \times N$ cubes where each cube is first subdivided into 6 pyramids (with the basis a face and the apex the barycenter of the cube) and then each pyramid into 4 tetrahedra. The corresponding mesh size is $h = \sqrt{3}/(2N)$. On the other hand, the "unstructured" meshes are generated with the software pacakge `MMG3D` [16], where we simply require a maximum element size. These are typically quasi-uniform, but do not have any particular repeating structure (every vertex patch is different). For both types of meshes, we consider the Nédélec finite element approximation (3.5) with different degrees $p \geq 1$.

6.1. $\boldsymbol{H}^3(\Omega)$ solution with a polynomial right-hand side. We first consider the unit cube $\Omega := (0,1)^3$ and a polynomial right-hand side $\boldsymbol{j} := (0,0,1)$, so that the data oscillation estimator η_{osc} vanishes. We can show that the solution is given by $\boldsymbol{A} = (0,0,A_3)$ with

$$A_3(\boldsymbol{x}) := \frac{16}{\pi^4} \sum_{n,m\geq 1} \frac{1}{nm(n^2+m^2)} \sin(n\pi\boldsymbol{x}_1)\sin(m\pi\boldsymbol{x}_2). \tag{6.1}$$

This function belongs to $H^3(\Omega)$ but not to $H^4(\Omega)$. In practice, we cut the series at $n = m = 100$ and obtain $\nabla\times\boldsymbol{A}$ by analytically differentiating (6.1).

We first fix the polynomial degree and consider a sequence of meshes. We use $p = 1$ and structured meshes with $N = 1, 2, 4, 8$ and then $p = 2$ and a sequence of unstructured meshes. Figure 1 presents the corresponding errors, estimates, and effectivity indices. We observe the expected convergence rate h^2 (recall that $\boldsymbol{A} \in \boldsymbol{H}^3(\Omega)$ merely). The estimator $\eta = \eta_{\mathrm{tot}}$ ($\eta_{\mathrm{osc}} = 0$ here) closely follows the error $\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|$, and the effectivity index given by the ratio $\eta/\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|$ is above but close to the optimal value 1; we actually numerically observe asymptotic exactness.

We then fix a mesh and increase the polynomial degree p from 1 to 6. We consider two configurations: a structured mesh where the unit cube is split into 24 tetrahedra as described above and an unstructured mesh consisting of 20 tetrahedra. Figure 2 reports the results. The convergence is not exponential, which is expected because of the solution's finite regularity. Also in this setting, the estimator closely follows the actual error, and the effectivity index always remains above but close to 1. In particular, the effectivity index does not increase with p, which illustrates the p-robustness of the estimator.

Although this is not reported in the figures, we also numerically check that the reconstructed flux $\boldsymbol{h}_h$ is indeed equilibrated, i.e., $\|\boldsymbol{j} - \nabla\times\boldsymbol{h}_h\| = 0$. Because of finite precision arithmetics, this value is not exactly zero but ranges between 10^{-15} and 10^{-11}, which is perfectly reasonable compared to the actual error levels.

6.2. Analytical solution with a general right-hand side. We consider again the unit cube $\Omega := (0,1)^3$, this time with a nonpolynomial right-hand side $\boldsymbol{j} := 8\pi^2(\sin(2\pi\boldsymbol{x}_2)\sin(2\pi\boldsymbol{x}_3), 0, 0)$. The associated solution is analytic:

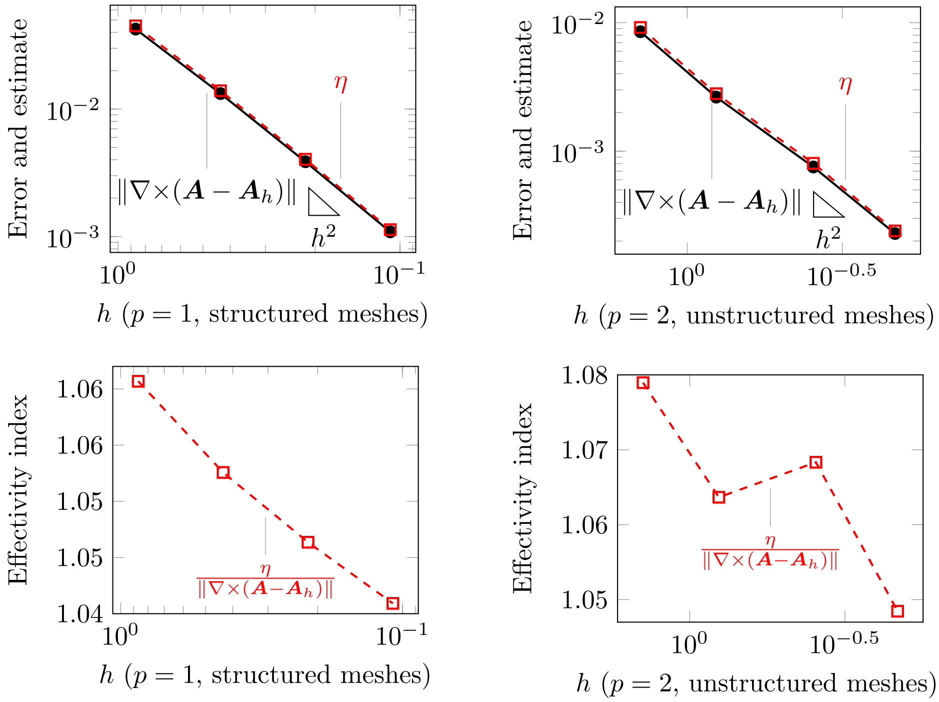

FIG. 1. *[Smooth solution with limited regularity* (6.1)*] Uniform mesh refinement.*

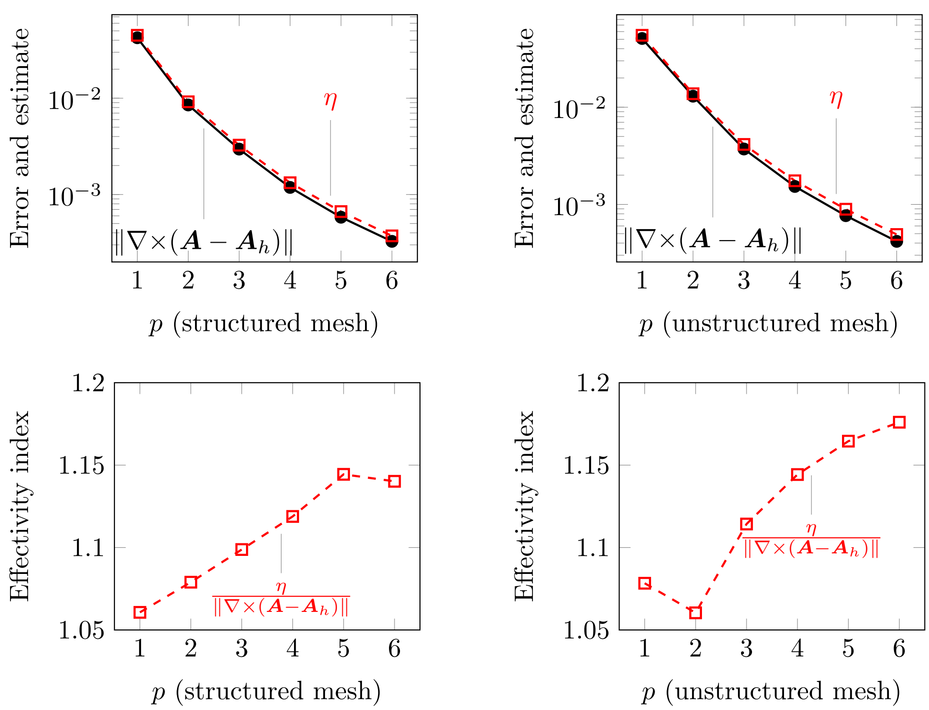

FIG. 2. *[Smooth solution with limited regularity* (6.1)*] Uniform polynomial degree refinement.*

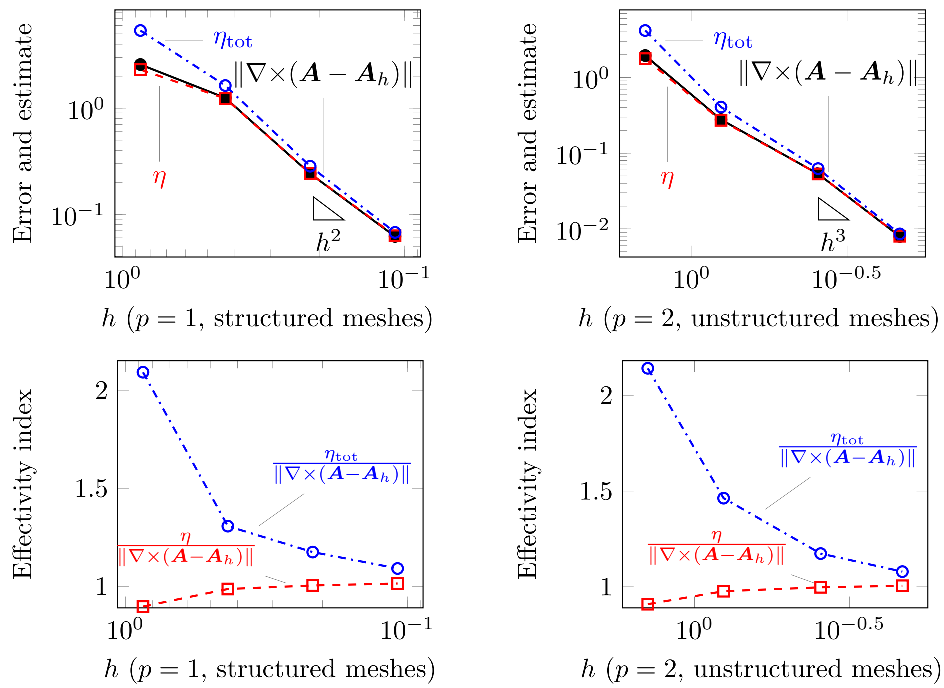

FIG. 3. *[Analytical solution* (6.2)*] Uniform mesh refinement.*

$$\boldsymbol{A} := (\sin(2\pi\boldsymbol{x}_2)\sin(2\pi\boldsymbol{x}_3), 0, 0). \tag{6.2}$$

Figure 3 presents an h convergence experiment with the same settings as above. The optimal convergence rate h^{p+1} is observed for $\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$. The oscillation-free estimator η closely follows the actual error, with a possible slight underestimation, whereas the total estimator including data oscillation $\eta_{\text{tot}} = \eta + C_{\text{L}}\eta_{\text{osc}}$ from Theorem 5.5 gives a guaranteed upper bound with a slight overestimation; as discussed in section 2.8, we can take here $C_{\text{L}} = 1$. In agreement with the theory, the influence of η_{osc} diminishes with mesh refinement, and we again numerically observe asymptotic exactness. We then consider a p convergence test. In Figure 4, we now observe the expected exponential convergence rate of the error and a perfect behavior of the effectivity indices. More precisely, as the mesh is not refined here, η_{osc} does not necessarily go faster to zero than the error; this would be the case if the hp-version of (2.7), with $\boldsymbol{\Pi}_p$ and $Ch_K/(p+1)$ in place of, respectively, $\boldsymbol{\Pi}_0$ and h_K/π, were used. We put forward here (2.7), where the is no unknown constant C, leading to a fully computable η_{osc}.

6.3. Adaptivity with a singular solution. Our last experiment features a singular solution in a nonconvex domain, following [10, 23]. Specifically, we consider an L-shape example where $\Omega := L \times (0,1)$, with

$$L := \{\boldsymbol{x} = (r\cos\theta, r\sin\theta);\ |\boldsymbol{x}_1|, |\boldsymbol{x}_2| \leq 1, \quad 0 \leq \theta \leq 3\pi/2\}.$$

The right-hand side $\boldsymbol{j}$ is nonpolynomial and chosen such that

$$\boldsymbol{A}(\boldsymbol{x}) = \big(0, 0, \chi(r)r^{\alpha}\sin(\alpha\theta)\big), \tag{6.3}$$

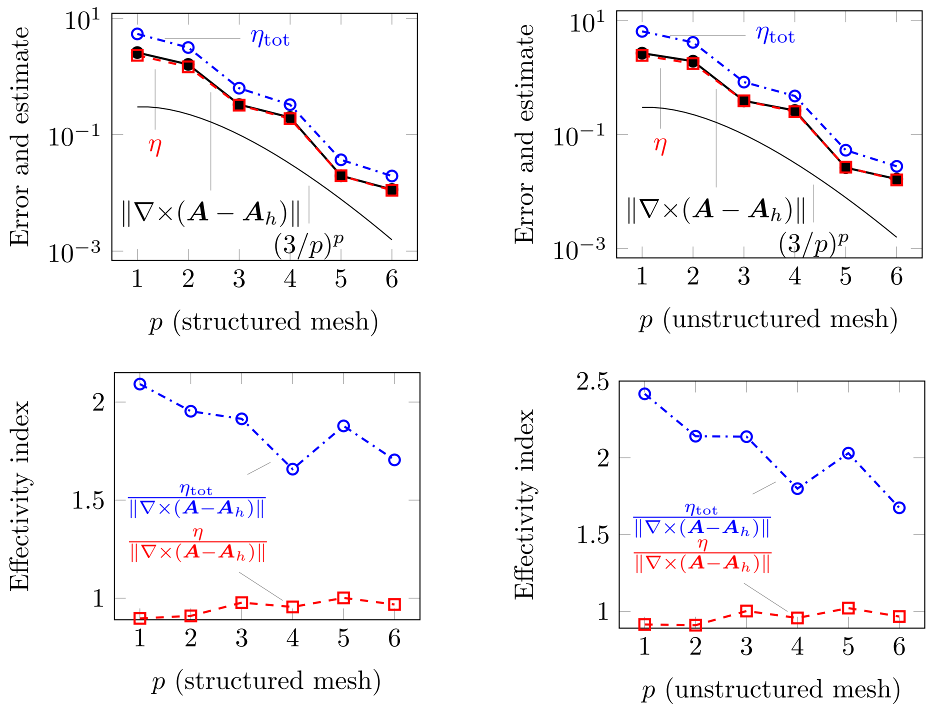

FIG. 4. *[Analytical solution* (6.2)*] Uniform polynomial degree refinement.*

where $\alpha := 3/2$, $r^2 := |\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2$, $(\boldsymbol{x}_1, \boldsymbol{x}_2) = r(\cos\theta, \sin\theta)$, and $\chi : (0,1) \to \mathbb{R}$ is a smooth cutoff function such that $\chi = 0$ in a neighborhood of 1. One easily checks that $\nabla\cdot\boldsymbol{A} = 0$. Besides, since $\Delta(r^\alpha \sin(\alpha\theta)) = 0$ near the origin, the right-hand side is nonsingular (i.e., $\boldsymbol{j} \in \boldsymbol{L}^2(\Omega)$), and the singularity appearing in the solution is solely due to the reentrant edge.

We couple our estimator with an adaptive strategy based on Dörfler's marking [17] for $\eta_K := \|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K$ and MMG3D [16] to build a series of adaptively refined meshes. We select $p = 2$ and an initial mesh made of 415 elements.

The behaviors of the error and of the estimators η and $\eta_{\rm tot}$ with respect to the number of degrees of freedom $N_{\rm dofs}$ are presented in Figure 5. Here we still take $C_{\rm L} = 1$ in front of $\eta_{\rm osc}$, though we do not anymore have a theoretical support for this. The effectivity indices stay close to one, even on unstructured and locally refined meshes, with $\eta_{\rm tot}/\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|$ always above one. Besides, the optimal convergence rate is observed (it is limited to $-2/3$ when using isotropic elements in the presence of an edge singularity; see [5, section 4.2.3]). This seems to indicate that the estimator is perfectly suited to drive adaptive mesh refinement and illustrates the local efficiency of Theorem 5.5.

Finally, Figures 6 and 7 present the meshes generated by the adaptive algorithm, the estimators $\eta_K = \|\boldsymbol{h}_h - \nabla\times\boldsymbol{A}_h\|_K$, and the elementwise errors $\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|_K$ (the top face and the faces sharing the reentrant edge). The meshes are refined close to the reentrant edge, as expected. The estimated error distribution closely matches the actual one, illustrating again the local efficiency of the estimator.

7. Technical details and proofs. This section collects some technical details and the proofs of all the claims above.

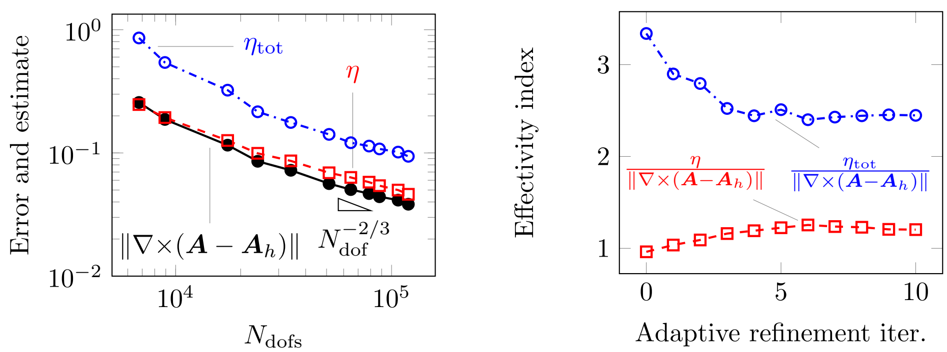

FIG. 5. *[Singular solution* (6.3)*] Adaptive mesh refinement.*

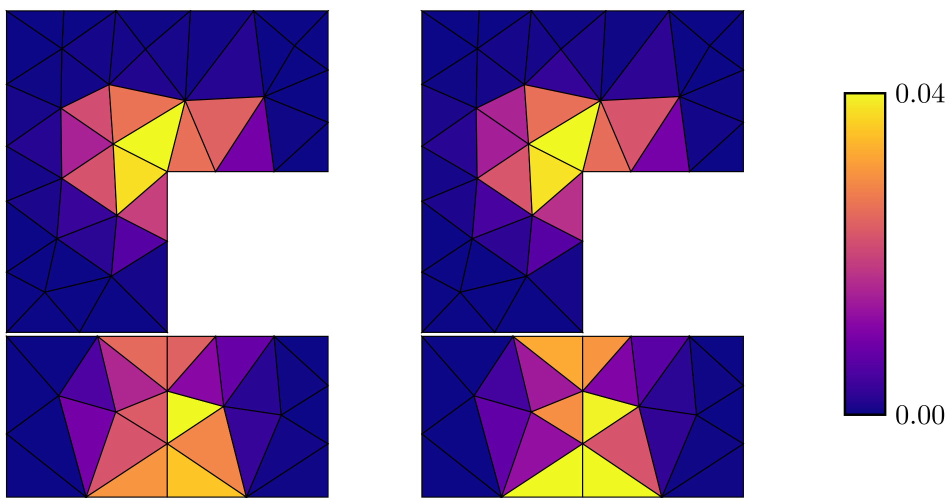

FIG. 6. *[Singular solution* (6.3)*]. Estimated (left) and actual (right) error distributions on the initial mesh. Top view (top) and side view (bottom).*

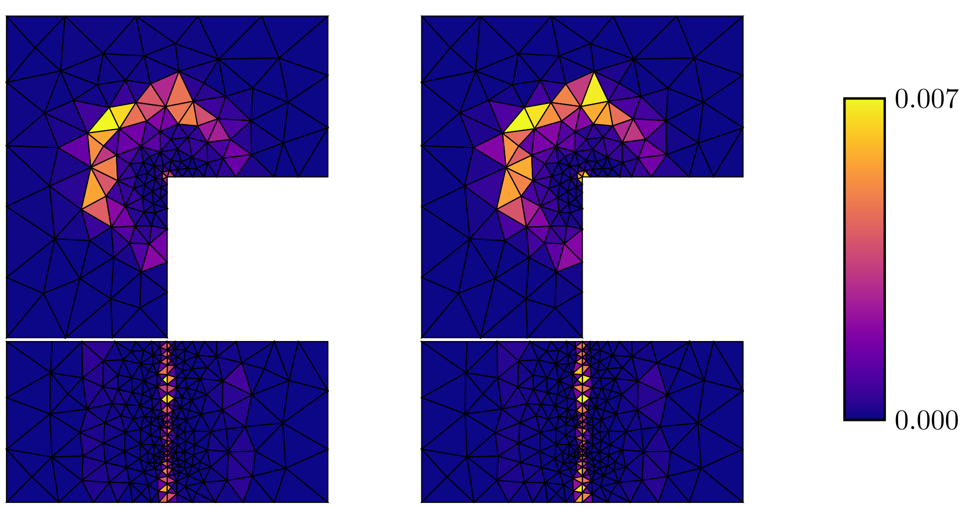

FIG. 7. *[Singular solution* (6.3)*] Estimated (left) and actual (right) error distributions at adaptive mesh refinement iteration #10. Top view (top) and side view (bottom).*

7.1. Equivalent form of Assumption 3.3. Recall from section 2.4 the piecewise affine "hat" function $\psi^{\boldsymbol{a}}$ associated with the vertex $\boldsymbol{a} \in \mathcal{V}_h$, as well as the notation $H^1_*(\omega_{\boldsymbol{a}})$ from section 2.7. The following technical result holds true.

LEMMA 7.1 (equivalence of images by the curl operator). *There holds*

$$\nabla\times \left[\operatorname*{span}_{\boldsymbol{a}\in\mathcal{V}_h} \left(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}} \nabla(\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}))\right)\right] = \nabla\times[\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)]. \tag{7.1}$$

Proof. Let $\boldsymbol{a} \in \mathcal{V}_h$. For any $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, clearly $\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h$, extended by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, lies in $\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ (though in general not in $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h)$). Moreover, $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\times\nabla q_h$, which is a piecewise constant vector-valued polynomial on the patch $\mathcal{T}_{\boldsymbol{a}}$ whose extension by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$ has a continuous normal trace on interfaces and zero normal trace on Γ_D. Thus, this extension belongs to the lowest-order divergence-free Raviart–Thomas space, which implies $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\times\boldsymbol{w}_h$ on $\omega_{\boldsymbol{a}}$ for $\boldsymbol{w}_h$ which belongs to $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$. Thus, in (7.1), there holds the inclusion $\subseteq$.

Conversely, following, e.g., Monk [32, section 5.5.1] or Ern and Guermond [18, section 15.1], the space $\boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ is spanned by the set of the "edge functions" $\{\boldsymbol{\psi}^e\}_{e\in\mathcal{E}_h^\mathrm{D}}$, where $\mathcal{E}_h^\mathrm{D}$ denotes the mesh edges not lying in $\overline{\Gamma_\mathrm{D}}$. If e is the edge between vertices $\boldsymbol{a},\boldsymbol{b} \in \mathcal{V}_h$, then $\boldsymbol{\psi}^e = \psi^{\boldsymbol{a}}\nabla\psi^{\boldsymbol{b}} - \psi^{\boldsymbol{b}}\nabla\psi^{\boldsymbol{a}}$. Moreover, if one of the vertices of e lies in $\overline{\Gamma_\mathrm{D}}$, we choose the convention that $\boldsymbol{a} \in \overline{\Gamma_\mathrm{D}}$, so that we have $(\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}})|_{\omega_{\boldsymbol{a}}} \in H^1_*(\omega_{\boldsymbol{a}})$ for some constant $c_{\boldsymbol{b}}$ in all cases. Now, since $\nabla\times\boldsymbol{\psi}^e = 2\nabla\psi^{\boldsymbol{a}}\times\nabla\psi^{\boldsymbol{b}} = 2\nabla\times(\psi^{\boldsymbol{a}}\nabla\psi^{\boldsymbol{b}}) = 2\nabla\times(\psi^{\boldsymbol{a}}\nabla(\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}}))$, we have found $q_h := (\psi^{\boldsymbol{b}} - c_{\boldsymbol{b}})|_{\omega_{\boldsymbol{a}}}/2 \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$ such that, after zero extension, $\nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h) = \nabla\times\boldsymbol{\psi}^e$, and the inclusion $\supseteq$ in (7.1) holds. ∎

The following alternative formulation of Assumption 3.3 is crucial.

LEMMA 7.2 (patchwise orthogonality). *Let $\boldsymbol{j}$ satisfy Assumption* 3.1*. Then $\boldsymbol{A}_h$ satisfies Assumption* 3.3 *if and only if $\boldsymbol{A}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ and*

$$(\psi^{\boldsymbol{a}}\boldsymbol{j},\nabla q_h)_{\omega_{\boldsymbol{a}}} + (\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\nabla q_h)_{\omega_{\boldsymbol{a}}} = 0 \quad \forall q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}), \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{7.2}$$

Proof. Since $\nabla\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\times\nabla q_h = \nabla\times(\psi^{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}\nabla q_h)$,

$$(\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\nabla q_h)_{\omega_{\boldsymbol{a}}} = -(\nabla\times\boldsymbol{A}_h,\nabla\psi^{\boldsymbol{a}}\times\nabla q_h)_{\omega_{\boldsymbol{a}}} = -(\nabla\times\boldsymbol{A}_h,\nabla\times(\psi^{\boldsymbol{a}}\nabla q_h))_{\omega_{\boldsymbol{a}}}.$$

For any $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ such that $\boldsymbol{j} = \nabla\times\boldsymbol{v}$, the Green theorem in turn gives

$$(\psi^{\boldsymbol{a}}\boldsymbol{j},\nabla q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{j},\psi^{\boldsymbol{a}}\nabla q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{v},\nabla\times(\psi^{\boldsymbol{a}}\nabla q_h))_{\omega_{\boldsymbol{a}}}.$$

Finally, again by the Green theorem, for any $\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_0(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$,

$$(\boldsymbol{j},\boldsymbol{v}_h) = (\nabla\times\boldsymbol{v},\boldsymbol{v}_h) = (\boldsymbol{v},\nabla\times\boldsymbol{v}_h).$$

Applying these identities, respectively, in (7.2) and (3.7b), the assertion follows from Lemma 7.1. ∎

7.2. Properties of the auxiliary fields $\boldsymbol{\theta}_h^{\boldsymbol{a}}$, $\boldsymbol{\delta}_h$, and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ from Definition 5.1. We collect here some important results on $\boldsymbol{\theta}_h^{\boldsymbol{a}}$, $\boldsymbol{\delta}_h$, and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ from (5.2)–(5.4). We start with the following application of the self-standing result on overconstrained minimization in the Raviart–Thomas spaces that we present in Appendix A below. Let $\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ be defined as $\widetilde{\eta}^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}}$ in (5.6) but on the patch $\mathcal{T}_{\boldsymbol{a}}$ only.

LEMMA 7.3 (existence, uniqueness, and stability of $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ from (5.2)). *There exists a unique solution $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ to problem* (5.2) *for all $\boldsymbol{a} \in \mathcal{V}_h$. Moreover, it satisfies the stability estimate*

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}}} \|\boldsymbol{v} - \nabla\psi^{\boldsymbol{a}} \times (\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}.$$

Proof. We choose $g^{\boldsymbol{a}} := (-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j})|_{\omega_{\boldsymbol{a}}}$, $\boldsymbol{\tau}_h^{\boldsymbol{a}} := (\nabla\psi^{\boldsymbol{a}} \times (\nabla\times\boldsymbol{A}_h))|_{\omega_{\boldsymbol{a}}}$, and $q := p$ and verify the assumptions of Theorem A.2 in three steps. Note that $\Pi_{p'}(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\Pi}_{p'}(\boldsymbol{j})$ and that $\|\nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j}))\|_K \leq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}\|\boldsymbol{j} - \boldsymbol{\Pi}_{p'}(\boldsymbol{j})\|_K$, where $\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}$ follows from the shape regularity of the mesh, which gives rise to $h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}$ from the data oscillation term in Theorem A.2.

Step 1. Assumption (A.1a). From (3.1a), $g^{\boldsymbol{a}} \in L^2(\omega_{\boldsymbol{a}})$, so that the first condition in (A.1a) is satisfied. From (3.7a), in turn, on $\omega_{\boldsymbol{a}}$, it follows that $\nabla\times\boldsymbol{A}_h \in [\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})]^3$ (see, e.g., [6, Corollary 2.3.2]), so that $\boldsymbol{\tau}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}} \times (\nabla\times\boldsymbol{A}_h) \in [\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})]^3 \subset \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \subset \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}})$. Thus, the second (polynomial) condition in (A.1a) is also satisfied.

Step 2. *Assumption* (A.1b). For vertices $\boldsymbol{a} \in \mathcal{V}_h$ such that $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$, the Green theorem and $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ from (3.1a) together with $\nabla\cdot\boldsymbol{j} = 0$ from (3.1b) imply

$$-(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, 1)_{\omega_{\boldsymbol{a}}} = -(\nabla\psi^{\boldsymbol{a}}, \boldsymbol{j})_{\omega_{\boldsymbol{a}}} = (\psi^{\boldsymbol{a}}, \nabla\cdot\boldsymbol{j})_{\omega_{\boldsymbol{a}}} = 0.$$

Step 3. *Assumption* (A.1c). For any $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H_*^1(\omega_{\boldsymbol{a}})$, again the Green theorem yields

$$-(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, q_h)_{\omega_{\boldsymbol{a}}} \overset{(3.1\mathrm{b})}{=} -(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{j}), q_h)_{\omega_{\boldsymbol{a}}} = (\psi^{\boldsymbol{a}}\boldsymbol{j}, \nabla q_h)_{\omega_{\boldsymbol{a}}},$$

so that the patchwise orthogonality property (7.2) implies

$$(\nabla\psi^{\boldsymbol{a}} \times (\nabla\times\boldsymbol{A}_h), \nabla q_h)_{\omega_{\boldsymbol{a}}} - (\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}, q_h)_{\omega_{\boldsymbol{a}}} = 0. \tag{7.3}$$

∎

Similarly, an important part of the results of the following lemma are consequences of Appendix B below.

LEMMA 7.4 (auxiliary correction fields $\boldsymbol{\delta}_h$ and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$). *For $\boldsymbol{\delta}_h$ given by* (5.3)*, there holds*

$$\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \textit{and} \quad \nabla\cdot\boldsymbol{\delta}_h = 0. \tag{7.4}$$

In addition, there exists a unique solution $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ to problems (5.4) *for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, yielding the local divergence-free decomposition*

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \quad \textit{and} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h, \tag{7.5a}$$

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{7.5b}$$

Moreover, for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, there holds the local stability estimate

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim \|\boldsymbol{\delta}_h\|_K. \tag{7.6}$$

Proof. The patchwise contributions $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ extended by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$ belong to $\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, so that the first property in (7.4) is immediate. The second property in (7.4) then follows by the divergence constraint in (5.2), the linearity of the projector $\Pi_{p'}$, and the partition of unity (2.1) since

$$\nabla\cdot\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = \Pi_{p'}\left[\sum_{\boldsymbol{a}\in\mathcal{V}_h} -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}\right] = \Pi_{p'}(0) = 0.$$

Let $K\in\mathcal{T}_h$ and $\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^3$ be fixed. Then definition (5.3), which gives $\boldsymbol{\delta}_h|_K = \sum_{\boldsymbol{b}\in\mathcal{V}_K}\boldsymbol{\theta}_h^{\boldsymbol{b}}$; the partition of unity (2.1), which implies $\sum_{\boldsymbol{b}\in\mathcal{V}_K}(\nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h))|_K = \boldsymbol{0}$; and the elementwise orthogonality constraint in (5.2) lead to

$$(\boldsymbol{\delta}_h,\boldsymbol{r}_h)_K = \sum_{\boldsymbol{b}\in\mathcal{V}_K}(\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h),\boldsymbol{r}_h)_K = 0.$$

This is condition (B.2). Thus, Theorem B.1 can be employed, where we choose $q := p'$ together with $q' := p'$ for $p = 0$ and $q' := p'+1$ for $p\geq 1$. This implies the existence and uniqueness of solutions $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ to problems (5.4), the properties (7.5a), the decomposition (7.5b), and the stability bound (7.6). Note in particular that we only employ (B.6b) with $q' = q$ in the lowest-order case with $q = 1$, whereas in other cases, we employ (B.6b) with $q' = q+1$, so there is indeed no polynomial degree dependence in (7.6). ∎

7.3. Decomposition of the current density $\boldsymbol{j}$ and its stability from Theorem 5.2. We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2 *(decomposition).* Property (5.7a) is immediate since $\psi^{\boldsymbol{a}}\boldsymbol{j}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ in view of assumption (3.1a), from (5.2), which gives $\boldsymbol{\theta}_h^{\boldsymbol{a}}\in\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, and from the first property in (7.5a). Property (5.7b) follows since $\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{j}) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}$ in view of assumption (3.1b) and using $\nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \Pi_{p'}(-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}) = -\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\Pi}_{p'}(\boldsymbol{j})$ from (5.2) and $\nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0$, which is the second property in (7.5a). Finally, (5.7c) follows from the partition of unity (2.1), which gives $\sum_{\boldsymbol{a}\in\mathcal{V}_h}\psi^{\boldsymbol{a}}\boldsymbol{j} = \boldsymbol{j}$ together with (5.3) and (7.5b). When $\boldsymbol{j}\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, (5.8) immediately follows from (5.7) and the fact that $\psi^{\boldsymbol{a}}\boldsymbol{j}\in\boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$. ∎

Proof of Theorem 5.2 *(stability).* From (4.5) and (5.5), we develop

$$\begin{aligned}\boldsymbol{j}^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}) - \boldsymbol{\theta}_h^{\boldsymbol{a}} + \boldsymbol{\delta}_h^{\boldsymbol{a}} &= \nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h))\\ &\quad - (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)) + \boldsymbol{\delta}_h^{\boldsymbol{a}}.\end{aligned}$$

For the first term above, we immediately see

$$\|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.$$

For the second term above, we employ Lemma 7.3 with $\boldsymbol{v} = \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})$, which lies in $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with divergence equal to $-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}$ by virtue of (4.6), which leads to

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\eta_{\mathrm{osc},\boldsymbol{j}}^{\boldsymbol{a}}. \tag{7.7}$$

For the last term, we first recall (7.6), i.e., $\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim \|\boldsymbol{\delta}_h\|_K$ for every $K\in\mathcal{T}_{\boldsymbol{a}}$. Now definition (5.3), the partition of unity (2.1), and the triangle inequality imply

$$\|\boldsymbol{\delta}_h\|_K = \left\|\sum_{\boldsymbol{b}\in\mathcal{V}_K}(\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h))\right\|_K \leq \sum_{\boldsymbol{b}\in\mathcal{V}_K}\|\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{b}}},$$

which extends by one layer beyond the patch $\omega_{\boldsymbol{a}}$ and can be bounded by (7.7). The shape regularity of the mesh ensures that $\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1}$ and $\|\nabla\psi^{\boldsymbol{b}}\|_{\infty,\omega_{\boldsymbol{b}}} \simeq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}$ for all vertices $\boldsymbol{b}$ in the patch $\mathcal{T}_{\boldsymbol{a}}$. Hence, (5.9) follows on combining the above developments. ∎

7.4. Equilibrated flux reconstruction from Section 5.2 and its stability. To prove Theorem 5.4, we rely on the following crucial result.

THEOREM 7.5 (p-robust $\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$ stability). *For a vertex $\boldsymbol{a}\in\mathcal{V}_h$, let $\boldsymbol{A}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$ and $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}\in\boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}=0$ be given. Then*

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}_h=\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}}}\|\boldsymbol{v}_h-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}\lesssim\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}}}\|\boldsymbol{v}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}. \tag{7.8}$$

On a single tetrahedron K in place of the vertex patch $\mathcal{T}_{\boldsymbol{a}}$, Theorem 7.5 follows by the seminal contributions of Costabel and McIntosh [13, Proposition 4.2] and Demkowicz, Gopalakrishnan, and Schöberl [14, Theorem 7.2]; see [9, Theorem 2]. On an edge patch, such a result has been established in [10, Theorem 3.1]. The further extension to a vertex patch has recently been established in [11, Theorem 3.3; see also Corollary 4.3].

Proof of Theorem 5.4 *(equilibration).* Property (5.13a) follows immediately from $\boldsymbol{h}_h^{\boldsymbol{a}}\in\boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$ of (5.11a) and (5.11b). For piecewise polynomial $\boldsymbol{j}\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, $\nabla\times\boldsymbol{h}_h^{\boldsymbol{a}}=\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}=\boldsymbol{j}_h^{\boldsymbol{a}}$ from (5.11a) and (5.10). Property (5.13b) is then a direct consequence of (5.7c) and (5.11b). ∎

Proof of Theorem 5.4 *(stability).* Fix a vertex $\boldsymbol{a}\in\mathcal{V}_h$, and use $\boldsymbol{j}^{\boldsymbol{a}}=\psi^{\boldsymbol{a}}\boldsymbol{j}+\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})=\nabla\times(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}))$ as in property (4.4). This implies $(\boldsymbol{j}^{\boldsymbol{a}},\boldsymbol{v})_{\omega_{\boldsymbol{a}}}=(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}),\nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}}$ for any $\boldsymbol{v}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})$. Then Theorem 7.5 and a primal-dual equivalence as in, e.g., [10, Lemma 5.5] imply

$$\begin{aligned}\|\boldsymbol{h}_h^{\boldsymbol{a}}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}&\lesssim\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}}}\|\boldsymbol{v}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}\\ &=\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}}\left\{(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}},\boldsymbol{v})_{\omega_{\boldsymbol{a}}}-(\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h),\nabla\times\boldsymbol{v})_{\omega_{\boldsymbol{a}}}\right\}\\ &\le\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}}(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}-\boldsymbol{j}^{\boldsymbol{a}},\boldsymbol{v})_{\omega_{\boldsymbol{a}}}+\|\psi^{\boldsymbol{a}}(\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h))\|_{\omega_{\boldsymbol{a}}}\\ &\le\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}}(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}-\boldsymbol{j}^{\boldsymbol{a}},\boldsymbol{v})_{\omega_{\boldsymbol{a}}}+\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.\end{aligned}$$

We are thus left to treat the first term above.

Fix $\boldsymbol{v}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})$ with $\|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1$. Consider $q\in H^1_*(\omega_{\boldsymbol{a}})$ such that

$$(\nabla q,\nabla w)_{\omega_{\boldsymbol{a}}}=(\boldsymbol{v},\nabla w)_{\omega_{\boldsymbol{a}}}\qquad\forall w\in H^1_*(\omega_{\boldsymbol{a}}).$$

Then $\tilde{\boldsymbol{v}}:=\boldsymbol{v}-\nabla q$ lies in both $\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})$ and $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ and is divergence-free, $\nabla\cdot\tilde{\boldsymbol{v}}=0$. Thus, the Poincaré–Friedrichs–Weber inequality (2.8) implies

$$\|\tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}}\lesssim h_{\omega_{\boldsymbol{a}}}\|\nabla\times\tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}}=h_{\omega_{\boldsymbol{a}}}\|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=h_{\omega_{\boldsymbol{a}}}. \tag{7.9}$$

Note that $\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}-\boldsymbol{j}^{\boldsymbol{a}}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}-\boldsymbol{j}^{\boldsymbol{a}})=0$; indeed, this follows from (4.7a)–(4.7b) together with (5.10). Thus, the Green theorem gives

$$(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}}-\boldsymbol{j}^{\boldsymbol{a}},\nabla q)_{\omega_{\boldsymbol{a}}}=0. \tag{7.10}$$

Thanks to this crucial property, we can play in $\tilde{\boldsymbol{v}}$ and use (7.9): Employing additionally the Cauchy–Schwarz inequality and the triangle inequality, we have

$$
\begin{aligned}
(\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}, \boldsymbol{v})_{\omega_{\boldsymbol{a}}} &= (\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}, \tilde{\boldsymbol{v}})_{\omega_{\boldsymbol{a}}} \leq \|\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \|\tilde{\boldsymbol{v}}\|_{\omega_{\boldsymbol{a}}} \\
&\lesssim h_{\omega_{\boldsymbol{a}}} \left[\|\bar{\boldsymbol{j}}_h^{\boldsymbol{a}} - \boldsymbol{j}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\boldsymbol{j}_h^{\boldsymbol{a}} - \boldsymbol{j}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \right],
\end{aligned} \tag{7.11}
$$

and we conclude by (5.9) from Theorem 5.2. ∎

7.5. A posteriori error estimates from Section 5.3. We can now finally prove Theorem 5.5.

Proof of Theorem 5.5 *(reliability).* For a piecewise polynomial current density, $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$, Theorem 5.4 implies $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ with $\nabla \times \boldsymbol{h}_h = \boldsymbol{j}$. Thus, in this case, the claim follows with $\eta_{\mathrm{osc}} = 0$ by the Prager–Synge theorem [35] in the $\boldsymbol{H}(\mathrm{curl})$-context; see, e.g., [8, Theorem 10] or [23, Theorem 3.1].

In general, we proceed as follows. Since $\boldsymbol{A}, \boldsymbol{A}_h \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$,

$$
\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\| = \max_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega) \\ \|\nabla \times \boldsymbol{v}\| = 1}} (\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h), \nabla \times \boldsymbol{v}).
$$

Fix $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ with $\|\nabla \times \boldsymbol{v}\| = 1$, and consider $\boldsymbol{w}$ from (2.6). Note that since $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ from Theorem 5.4, the Green theorem and $\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{v}$ give

$$
(\nabla \times \boldsymbol{h}_h, \boldsymbol{w}) = (\boldsymbol{h}_h, \nabla \times \boldsymbol{w}) = (\boldsymbol{h}_h, \nabla \times \boldsymbol{v}).
$$

Similarly, $\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{v}$ and the weak solution characterization (3.4) lead to

$$
(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v}) = (\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{w}) = (\boldsymbol{j}, \boldsymbol{w}).
$$

Thus,

$$
(\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h), \nabla \times \boldsymbol{v}) = (\boldsymbol{j} - \nabla \times \boldsymbol{h}_h, \boldsymbol{w}) + (\boldsymbol{h}_h - \nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}).
$$

The second term is trivially bounded by the estimator η via the Cauchy–Schwarz inequality, so that we are left with bounding the first one.

Property (5.14) and the additional orthogonality constraint in (5.10) lead to

$$
(\boldsymbol{j} - \nabla \times \boldsymbol{h}_h, \boldsymbol{w}) = \sum_{K \in \mathcal{T}_h} \left(\sum_{\boldsymbol{a} \in \mathcal{V}_K} (\boldsymbol{j}_h^{\boldsymbol{a}} - \bar{\boldsymbol{j}}_h^{\boldsymbol{a}}), \boldsymbol{w} \right)_K = \sum_{K \in \mathcal{T}_h} \big(\boldsymbol{j} - \nabla \times \boldsymbol{h}_h, \boldsymbol{w} - \boldsymbol{\Pi}_0(\boldsymbol{w})\big)_K.
$$

Consequently, the Poincaré inequality (2.7), (2.6), and $\|\nabla \times \boldsymbol{v}\| = 1$ give

$$
(\boldsymbol{j} - \nabla \times \boldsymbol{h}_h, \boldsymbol{w}) \leq \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{osc},K} \|\nabla \boldsymbol{w}\|_K \leq \eta_{\mathrm{osc}} \|\nabla \boldsymbol{w}\| \leq C_{\mathrm{L}} \eta_{\mathrm{osc}} \|\nabla \times \boldsymbol{v}\| = C_{\mathrm{L}} \eta_{\mathrm{osc}}.
$$
∎

Remark 7.6 (comparison with (7.11)). Above, we could also write

$$
(\boldsymbol{j} - \nabla \times \boldsymbol{h}_h, \boldsymbol{w}) = \sum_{\boldsymbol{a} \in \mathcal{V}_h} (\boldsymbol{j}_h^{\boldsymbol{a}} - \bar{\boldsymbol{j}}_h^{\boldsymbol{a}}, \boldsymbol{w})_{\omega_{\boldsymbol{a}}},
$$

where the terms in the sum are similar to (7.11) from section 7.4. In contrast to (7.11), it seems that we cannot pass through the Poincaré–Friedrichs–Weber inequality (2.8) in the absence of an exactly divergence-free field (recall from (5.7b) that $\nabla \cdot \boldsymbol{j}_h^{\boldsymbol{a}} =$

$\nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{j}-\boldsymbol{\Pi}_{p'}(\boldsymbol{j}))$ only in general) and rather need to resort to the switch from $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ to $\boldsymbol{w}$ of (2.6) and to make use of the Poincaré inequality (2.7).

Proof of Theorem 5.5 *(efficiency).* Property (5.14), the triangle inequality, and definition (5.12) immediately lead to $\eta_{\mathrm{osc},K}\le\sum_{\boldsymbol{a}\in\mathcal{V}_K}\eta^{\boldsymbol{a}}_{\mathrm{osc},\boldsymbol{j}_h^{\boldsymbol{a}}}$. Moreover, definition (5.11b) together with the partition of unity (2.1) imply

$$\|\boldsymbol{h}_h-\nabla\times\boldsymbol{A}_h\|_K=\left\|\sum_{\boldsymbol{a}\in\mathcal{V}_K}(\boldsymbol{h}_h^{\boldsymbol{a}}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h))\right\|_K\le\sum_{\boldsymbol{a}\in\mathcal{V}_K}\|\boldsymbol{h}_h^{\boldsymbol{a}}-\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}.$$

Thus, employing Theorem 5.4 concludes the proof. ∎

Appendix A. Overconstrained minimization in Raviart–Thomas spaces.

A.1. Assumption and statement of the overconstrained minimization.

In this appendix, we consider a fixed mesh vertex $\boldsymbol{a}\in\mathcal{V}_h$. Let an integer $q\ge 0$ be fixed, and set $q':=\min\{q,1\}$. We employ the notation of section 2 and in particular recall that $\lesssim$ means smaller or equal to up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$. We also assume a polynomial form, mean value zero, and patchwise orthogonality conditions on the two data $g^{\boldsymbol{a}}$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}}$:

Assumption A.1 (data $g^{\boldsymbol{a}}$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}}$). The data $g^{\boldsymbol{a}}$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}}$ satisfy

$$g^{\boldsymbol{a}}\in L^2(\omega_{\boldsymbol{a}})\quad\text{and}\quad\boldsymbol{\tau}_h^{\boldsymbol{a}}\in\boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}),\tag{A.1a}$$

$$(g^{\boldsymbol{a}},1)_{\omega_{\boldsymbol{a}}}=0\quad\text{when }\boldsymbol{a}\notin\overline{\Gamma_{\mathrm{D}}},\tag{A.1b}$$

$$(\boldsymbol{\tau}_h^{\boldsymbol{a}},\nabla q_h)_{\omega_{\boldsymbol{a}}}+(g^{\boldsymbol{a}},q_h)_{\omega_{\boldsymbol{a}}}=0\qquad\forall q_h\in\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})\cap H^1_*(\omega_{\boldsymbol{a}}).\tag{A.1c}$$

We consider the following (seemingly overconstrained) minimization problem in the Raviart–Thomas space $\boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$:

$$\boldsymbol{\theta}_h^{\boldsymbol{a}}:=\arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=\Pi_{q'}(g^{\boldsymbol{a}})\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\boldsymbol{\tau}_h^{\boldsymbol{a}},\boldsymbol{r}_h)_K\quad\forall\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^3,\forall K\in\mathcal{T}_{\boldsymbol{a}}}}\|\boldsymbol{v}_h-\boldsymbol{\tau}_h^{\boldsymbol{a}}\|^2_{\omega_{\boldsymbol{a}}}.\tag{A.2}$$

The following result is of independent interest.

THEOREM A.2 (overconstrained minimization in the Raviart–Thomas spaces). *Let Assumption* A.1 *hold. Then there exists a unique solution $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ to problem* (A.2), *satisfying the stability estimate*

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}}-\boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}\lesssim\min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}=g^{\boldsymbol{a}}}}\|\boldsymbol{v}-\boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}+\left\{\sum_{K\in\mathcal{T}_{\boldsymbol{a}}}\left(\frac{h_K}{\pi}\|\Pi_{q'}(g^{\boldsymbol{a}})-g^{\boldsymbol{a}}\|_K\right)^2\right\}^{\frac{1}{2}}.$$

A.2. Auxiliary conventional minimization.

In addition to (A.2), it will be useful to also consider

$$\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}:=\arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=\Pi_{q'}(g^{\boldsymbol{a}})}}\|\boldsymbol{v}_h-\boldsymbol{\tau}_h^{\boldsymbol{a}}\|^2_{\omega_{\boldsymbol{a}}}.\tag{A.3}$$

Minimizations (A.3) are in a conventional format in that the constraints only concern normal trace and divergence. Moreover, they fulfill the following important property.

Lemma A.3 (existence, uniqueness, and stability of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ from (A.3)). *Let Assumption* A.1 *hold. Then there exists a unique solution* $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ *to problem* (A.3)*, satisfying the stability estimate*

(A.4)
$$\|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = g^{\boldsymbol{a}}}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}\|_K \right)^2 \right\}^{\frac{1}{2}}.$$

Proof. Existence and uniqueness of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ from (A.3) are classical following, e.g., [6], thanks to the Neumann boundary compatibility condition (A.1b); note that this implies $(\Pi_{q'}(g^{\boldsymbol{a}}), 1)_{\omega_{\boldsymbol{a}}} = 0$ when $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$. Moreover, since $\Pi_{q'}(g^{\boldsymbol{a}}) \in \mathcal{P}_{q'}(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}})$, taking $p = q'$, $\boldsymbol{\tau}_p = -\boldsymbol{\tau}_h^{\boldsymbol{a}}$, and $r_K = (\Pi_{q'}(g^{\boldsymbol{a}}) - \nabla\cdot\boldsymbol{\tau}_h^{\boldsymbol{a}})|_K$ in [21, Corollaries 3.3 and 3.6] for an interior vertex $\boldsymbol{a}$ and [21, Corollary 3.8] and [11, Proposition 3.1] for a boundary vertex $\boldsymbol{a}$ leads to

$$\|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = \Pi_{q'}(g^{\boldsymbol{a}})}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} = \|\nabla \tilde{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}.$$

The equality above is a classical primal-dual equivalence, with $\tilde{r}^{\boldsymbol{a}} \in H^1_*(\omega_{\boldsymbol{a}})$ given by

$$(\nabla \tilde{r}^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}), v)_{\omega_{\boldsymbol{a}}} \qquad \forall v \in H^1_*(\omega_{\boldsymbol{a}}).$$

Thus, as in, e.g., [20, Theorem 3.17],

$$\begin{aligned} \|\nabla \tilde{r}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} &= \max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{(\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}), v)_{\omega_{\boldsymbol{a}}}\} \\ &= \max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{(\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}}\}. \end{aligned}$$

The projection orthogonality and the elementwise Poincaré inequality then lead to

$$\begin{aligned} |(\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}}| &= \left| \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}, v - \Pi_0 v)_K \right| \\ &\leq \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{\pi} \|\Pi_{q'}(g^{\boldsymbol{a}}) - g^{\boldsymbol{a}}\|_K \right)^2 \right\}^{\frac{1}{2}} \|\nabla v\|_{\omega_{\boldsymbol{a}}}, \end{aligned}$$

and (A.4) follows since

$$\max_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \{(\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}}\} = \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = g^{\boldsymbol{a}}}} \|\boldsymbol{v} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

by the same primal-dual equivalence argument. ∎

A.3. Auxiliary first-order overconstrained minimization and proof of Theorem A.2. Let, in addition to (A.2) and (A.3), the *first-order* Raviart–Thomas piecewise polynomial $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ be given by

(A.5)
$$\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \boldsymbol{\tau}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}\|^2_{\omega_{\boldsymbol{a}}}.$$

The field $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ can be seen as the correction of $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ from (A.3) necessary to fulfill the constraints on the elementwise product with piecewise vector-valued constants in (A.2). As one might expect, the patchwise orthogonality assumption (A.1c) turns to be the key for the following crucial technical result.

LEMMA A.4 (existence, uniqueness, and stability of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ from (A.5)). *Let Assumption* A.1 *hold. Then there exists a unique solution* $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ *to problem* (A.5)*, and the following stability estimate holds true:*

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}. \tag{A.6}$$

We postpone the proof Lemma A.4 to the sections below; let us now first show that Lemma A.4 implies the results announced in Theorem A.2.

Proof of Theorem A.2. It follows straightforwardly from (A.3) and (A.5) that $\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ lies in the minimization set of (A.2). Consequently, the existence and uniqueness of (A.2) follows since the minimized functional in (A.2) is convex. Moreover, the triangle inequality together with Lemma A.4 implies

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} + \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \boldsymbol{\tau}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}},$$

and we conclude by Lemma A.3. ∎

A.4. Piola mappings. In order to prove the technical results below, we will rely on Piola mappings. An extensive description may be found in, e.g., [18, Chapters 7.2 and 9.2], and we only list here the essential properties we need.

If $U, V \subset \mathbb{R}^3$ are open sets with Lipschitz boundaries and $\phi : U \to V$ is a bi-Lipschitz mapping, the gradient-preserving, curl-preserving, diverence-preserving, and broken Piola mappings are the applications $\phi^{\mathrm{g}} : L^2(U) \to L^2(V)$, $\phi^{\mathrm{c}}, \phi^{\mathrm{d}} : \boldsymbol{L}^2(U) \to \boldsymbol{L}^2(V)$, and $\phi^{\mathrm{b}} : L^2(U) \to L^2(V)$, respectively, defined by

$$\phi^{\mathrm{g}}(v) = v \circ \phi^{-1},\ \phi^{\mathrm{c}}(\boldsymbol{w}) = (\mathbb{J}^{-T}\boldsymbol{w}) \circ \phi^{-1},\ \phi^{\mathrm{d}}(\boldsymbol{w}) = \left(\frac{\mathbb{J}}{|\mathbb{J}|}\boldsymbol{w}\right) \circ \phi^{-1},\ \phi^{\mathrm{b}}(v) = \left(\frac{v}{|\mathbb{J}|}\right) \circ \phi^{-1} \tag{A.7}$$

for all $v \in L^2(U)$ and $\boldsymbol{w} \in \boldsymbol{L}^2(V)$, where $\mathbb{J}$ is the Jacobian matrix of ϕ and $|\mathbb{J}|$ its determinant. These mappings are invertible. In addition, if $\gamma_U \subset \partial U$ and $\gamma_V := \phi(\gamma_U)$, with a similar notation as in section 2.2, then ϕ^{g}, ϕ^{c}, and ϕ^{d}

$$H^1_{0,\gamma_U}(U) \to H^1_{0,\gamma_V}(V), \boldsymbol{H}_{0,\gamma_U}(\mathrm{curl}, U) \to \boldsymbol{H}_{0,\gamma_V}(\mathrm{curl}, V), \boldsymbol{H}_{0,\gamma_U}(\mathrm{div}, U) \to \boldsymbol{H}_{0,\gamma_V}(\mathrm{div}, V) \tag{A.8}$$

are bijections, and more generally, the full, tangential, and normal traces on γ_U are, respectively, transported by ϕ^{g}, ϕ^{c}, and ϕ^{d} on γ_V. The commutativity properties

$$\phi^{\mathrm{c}}(\nabla v) = \nabla(\phi^{\mathrm{g}}(v)) \quad \nabla\cdot(\phi^{\mathrm{d}}(\boldsymbol{w})) = \phi^{\mathrm{b}}(\nabla\cdot\boldsymbol{w}) \tag{A.9}$$

for $v \in H^1(U)$ and $\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}, U)$ will be useful. We will also need the formula

$$((\phi^{\mathrm{d}})^{-1}(\boldsymbol{v}), \boldsymbol{w})_U = \varepsilon(\boldsymbol{v}, \phi^{\mathrm{c}}(\boldsymbol{w}))_V \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(V),\ \forall \boldsymbol{w} \in \boldsymbol{L}^2(U), \tag{A.10}$$

where ε is the (constant) sign of the determinant of $\mathbb{J}$. Finally, if U is a polyhedron covered by a tetrahedral mesh $\mathcal{T}_U$ and $\phi|_K$ is affine for each $K \in \mathcal{T}_U$, denoting by $\mathcal{T}_V$ the tetrahedral mesh of V induced by ϕ, then we have the bijections

$$\phi^{\mathrm{g}} : \mathcal{P}_q(\mathcal{T}_U) \to \mathcal{P}_q(\mathcal{T}_V),\ \phi^{\mathrm{c}} : \boldsymbol{\mathcal{N}}_q(\mathcal{T}_U) \to \boldsymbol{\mathcal{N}}_q(\mathcal{T}_V),\ \phi^{\mathrm{d}} : \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_U) \to \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_V) \tag{A.11}$$

for all integers $q \geq 0$. In addition, in this case, we have

$$\|\phi^{\rm d}\|\|(\phi^{\rm d})^{-1}\| \leq C(\kappa_{\mathcal{T}_U}, \kappa_{\mathcal{T}_V}) \tag{A.12}$$

with $\|\phi^{\rm d}\|$ denoting the norm operator of $\phi^{\rm d} : \boldsymbol{L}^2(U) \to \boldsymbol{L}^2(V)$ (and vice versa for $\|(\phi^{\rm d})^{-1}\|$) and $\kappa_{\mathcal{T}_U}$, $\kappa_{\mathcal{T}_V}$ the shape-regularity constants of $\mathcal{T}_U$ and $\mathcal{T}_V$ as in section 2.4.

A.5. A preliminary result. Before proving Lemma A.4, we establish the following preliminary result.

LEMMA A.5 (orthogonalities). *Let $\boldsymbol{\mu}^{\boldsymbol{a}} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$ satisfy $(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} = 0$ for all $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$. Then the following set is nonempty:*

$$W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}}) := \left\{ \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \;\middle|\; \begin{array}{l} \nabla\cdot \boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\mu}^{\boldsymbol{a}}, \boldsymbol{r}_h)_K \\ \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \, \forall K \in \mathcal{T}_{\boldsymbol{a}} \end{array} \right\}.$$

Proof. Step 1: *Interior patches.* We start with the case where the vertex $\boldsymbol{a} \in \mathcal{V}_h$ does not lie on the boundary $\partial\Omega$; cf. Figure 8 (left). We will construct a particular element $\boldsymbol{w}_h \in W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$ by an explicit run through the patch $\mathcal{T}_{\boldsymbol{a}}$ of tetrahedra sharing the vertex $\boldsymbol{a} \in \mathcal{V}_h$, similarly as in [7, 21]. Specifically, following the concept of shelling of a polytopal complex (see [38, Theorem 8.12] and [21, Lemma B.1]), there exists an enumeration K_i, $1 \leq i \leq |\mathcal{T}_{\boldsymbol{a}}|$, of the tetrahedra in the patch $\mathcal{T}_{\boldsymbol{a}}$ such that, except for the first tetrahedron in the enumeration K_1, (i) if there are at least two faces corresponding to the neighbors of K_i which have been already enumerated, then all the tetrahedra of $\mathcal{T}_{\boldsymbol{a}}$ sharing this edge come sooner in the enumeration, and (ii) except for the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$, there are one or two neighbors of K_i which have been already enumerated and correspondingly two or one neighbors of K_i which have not been enumerated yet.

Consider a pass through the patch $\mathcal{T}_{\boldsymbol{a}}$ in the sense of the above enumeration. For the tetrahedron K_i, $1 \leq i \leq |\mathcal{T}_{\boldsymbol{a}}|$, let us denote by $\mathcal{F}_i^\sharp$ the faces of K_i corresponding to the neighbors of K_i which have been already passed through and $F^j = \partial K_i \cap \partial K_j \in \mathcal{F}_i^\sharp$ the face corresponding to the neighbor K_j. Also, let $F_i^{\rm ext}$ be the face of K_i lying on the patch boundary $\partial\omega_{\boldsymbol{a}}$. Consider the problem

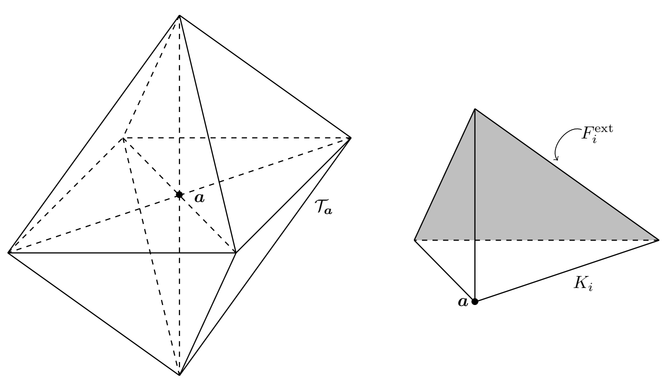

FIG. 8. *Interior vertex patch (left) and the element K_1 with the face $F_1^{\rm ext}$ (right).*

$$
\boldsymbol{w}_h^i := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(K_i) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_i}=0 \text{ on } F_i^{\mathrm{ext}} \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_i}=\boldsymbol{w}_h^j\cdot\boldsymbol{n}_{K_i} \text{ on all } F^j\in\mathcal{F}_i^\sharp \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_{K_i}=(\boldsymbol{\mu}^{\boldsymbol{a}},\boldsymbol{r}_h)_{K_i} \quad \forall \boldsymbol{r}_h\in[\mathcal{P}_0(K_i)]^3}} \|\boldsymbol{v}_h - \boldsymbol{\mu}^{\boldsymbol{a}}\|_{K_i}^2, \tag{A.13}
$$

similar to (A.5) but reduced to the single tetrahedron K_i. If a solution to (A.13) exists, for all $1 \le i \le |\mathcal{T}_{\boldsymbol{a}}|$, $\boldsymbol{w}_h$ defined as $\boldsymbol{w}_h|_{K_i} := \boldsymbol{w}_h^i$ belongs to $W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$. We are thus left to establish the existence (and uniqueness) of (A.13).

Step 1a: *The first element* K_1. Let us start with the first element K_1; cf. Figure 8 (right). Then the set $\mathcal{F}_1^\sharp$ is empty, and the constraints in (A.13) lead us to ask whether in the first-order Raviart–Thomas space $\boldsymbol{\mathcal{RT}}_1(K_1)$ one can impose simultaneously the divergence, the normal flux through one face, and moments against constant functions. We will reason by the canonical degrees of freedom (see, e.g., [6, Proposition 2.3.4 and Figure 2.14.c] or (2.4)) and find a suitable $\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_1(K_1)$ satisfying the constraints in (A.13). First, we see from (2.4) that in $\boldsymbol{\mathcal{RT}}_1(K_1)$, the normal flux $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}$ on F_1^{ext} can be fixed to zero and that the moments against constants $(\boldsymbol{v}_h, \boldsymbol{r}_h)_{K_1}$ can be fixed as in (A.13). We still have the freedom to choose the normal fluxes $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}$ on the faces of K_1 different from F_1^{ext}, and the question is whether this can be done so as to fix the divergence of $\boldsymbol{v}_h$ to zero. By [6, Proposition 2.3.3], there holds

$$
\nabla\cdot\boldsymbol{v}_h = 0 \quad \Leftrightarrow \quad (\nabla\cdot\boldsymbol{v}_h, q_h)_{K_1} = 0 \qquad \forall q_h \in \mathcal{P}_1(K_1).
$$

Employing the Green theorem and the fact that $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1} = 0$ on F_1^{ext},

$$
(\nabla\cdot\boldsymbol{v}_h, q_h)_{K_1} = \langle \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}, q_h\rangle_{\partial K_1 \setminus F_1^{\mathrm{ext}}} - (\boldsymbol{v}_h, \nabla q_h)_{K_1}.
$$

Now, since $\nabla q_h \in [\mathcal{P}_0(K_1)]^3$, the last term above is fixed from the last constraint in (A.13), so the question becomes whether can one choose $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}$ on $\partial K_1 \setminus F_1^{\mathrm{ext}}$ such that

$$
\langle \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}, q_h\rangle_{\partial K_1 \setminus F_1^{\mathrm{ext}}} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{K_1} \qquad \forall q_h \in \mathcal{P}_1(K_1), \tag{A.14}
$$

which gives 4 conditions for the 9 remaining degrees of freedom (there are 4 degrees of freedom in $\mathcal{P}_1(K_1)$ and 3 degrees of freedom per face in $\boldsymbol{\mathcal{RT}}_1(K_1)$ following [6, Proposition 2.3.4]).

We proceed as follows. Out of the three faces of K_1 different from F_1^{ext}, choose one, and impose $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1} = 0$ therein. Then we are left to set $\boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}$ on two faces, say, F and $\widetilde{F}$. For F, consider the three hat basis functions ψ_F^k, $1 \le k \le 3$, as in section 2.4, corresponding to its three vertices. Restricted to $\widetilde{F}$, which is necessary a face neighboring with F, they belong to $\mathcal{P}_1(\widetilde{F})$, and one of the restrictions, say, ψ_F^3, is zero on $\widetilde{F}$. Thus, there holds

$$
\langle \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}, \psi_F^3\rangle_{\widetilde{F}} = 0,
$$

and, following [6, Proposition 2.3.4], we can prescribe

$$
\langle \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}, \psi_F^k\rangle_{\widetilde{F}} := 0 \qquad 1 \le k \le 2.
$$

Moreover, restricted to F, ψ_F^k create a basis of $\mathcal{P}_1(F)$, whereas restricted to K_1, they belong to $\mathcal{P}_1(K_1)$. Thus, we can also set

$$
\langle \boldsymbol{v}_h\cdot\boldsymbol{n}_{K_1}, \psi_F^k\rangle_F := (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_1} \qquad 1 \le k \le 3.
$$

With the choices made so far, we see that (A.14) holds for the three hat functions ψ_F^k, $1 \leq k \leq 3$. Finally, consider ψ_F^4, the hat basis function corresponding to the vertex opposite to the face F. Restricted to F, it is zero, so that

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_1}, \psi_F^4 \rangle_F = 0.$$

Moreover, restricted to $\widetilde{F}$, it completes ψ_F^1 and ψ_F^2 (restricted to $\widetilde{F}$) to create a basis of $\mathcal{P}_1(\widetilde{F})$, and restricted to K_1, it belongs to $\mathcal{P}_1(K_1)$, so that we can prescribe

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_1}, \psi_F^4 \rangle_{\widetilde{F}} := (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_1}.$$

Thus, (A.14) also holds for ψ_F^4, and since ψ_F^k, $1 \leq k \leq 4$, restricted to K_1 create a basis of $\mathcal{P}_1(K_1)$, (A.14) holds true, and a unique $\boldsymbol{w}_h^1$ from (A.13) exists.

Step 1b: *Any element* K_i *with* $|\mathcal{F}_i^\sharp| = 1$. We now investigate those consecutive elements K_i which are such that two neighbors of K_i have not been passed through yet. This means that exactly one neighbor of K_i, say, K_j, has already been passed through, so there is one face F^j in the set $\mathcal{F}_i^\sharp$. Since $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i} = \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_i}$ on F^j is requested in (A.13), (A.14) asks if can one choose $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}$ on $\partial K_i \setminus \{F_i^{\text{ext}}, F^j\}$ such that

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{\partial K_i \setminus \{F_i^{\text{ext}}, F^j\}} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{K_i} - \langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{F^j} \tag{A.15}$$

for all $q_h \in \mathcal{P}_1(K_i)$, which is still undetermined, giving 4 conditions for the 6 remaining degrees of freedom. The reasoning is similar as for K_1. Still denoting F and $\widetilde{F}$ the two remaining faces and ψ_F^k, $1 \leq k \leq 4$, the hat basis functions, we again have

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^3 \rangle_{\widetilde{F}} = 0, \qquad \langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_F = 0.$$

Moreover, imposing

$$\begin{aligned}
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{\widetilde{F}} := 0 \qquad 1 \leq k \leq 2, \\
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{\widetilde{F}} := (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i} - \langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^j}, \\
&\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_F := (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_i} - \langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^j} \qquad 1 \leq k \leq 3
\end{aligned}$$

yields (A.15), and $\boldsymbol{w}_h^i$ exists.

Step 1c*: Any element* K_i *with* $|\mathcal{F}_i^\sharp| = 2$. We now investigate those consecutive elements K_i which are such that only one neighbor of K_i has not been passed through yet, with K_{j_1} and K_{j_2} already passed through and faces F^{j_1}, F^{j_2} in the set $\mathcal{F}_i^\sharp$. Denote F the only remaining face, so that F_i^{ext}, F^{j_1}, F^{j_2}, and F are the four faces of the tetrahedron K_i. As in (A.14) and (A.15), we need to ensure that

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, q_h \rangle_F = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{K_i} - \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{F^{j_1}} - \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, q_h \rangle_{F^{j_2}} \tag{A.16}$$

for all $q_h \in \mathcal{P}_1(K_1)$. This time, the system is overdetermined in that we request 4 conditions for the 3 remaining degrees of freedom of the normal components $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}$ on the face F. As above, we can impose

$$\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_F := (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_i} - \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^{j_1}} - \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^k \rangle_{F^{j_2}} \quad 1 \leq k \leq 3,$$

which fixes $\boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}$ on the face F. Now, noting that $\langle \boldsymbol{v}_h \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_F = 0$, it follows that to prove (A.16), we need to show that

$$(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i} - \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_1}} - \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_2}} = 0. \tag{A.17}$$

To prove (A.17), recall from property (i) of the enumeration (given that all other elements sharing the edge e common to F^{j_1} and F^{j_2} have been already passed through) and the previous steps (see (A.14) and (A.15)) that

$$\langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_j}, \psi_F^4 \rangle_{\partial K_j} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_j} \tag{A.18}$$

for all the tetrahedra K_j sharing the edge e, different from K_i. Recalling from the assumptions of Lemma A.5, we have

$$0 = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_e}, \tag{A.19}$$

where ω_e is the part of $\omega_{\boldsymbol{a}}$ corresponding to the elements sharing the edge e; the last equality holds since in the vertex patch subdomain $\omega_{\boldsymbol{a}}$, ψ_F^4 is only supported on the edge patch subdomain ω_e. Denote by $\widetilde{\omega}_e$ the part of ω_e without the element K_i. Then the normal traces orientation, the zero normal trace boundary conditions $\boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_j} = 0$ on the faces F_j^{ext} together with the zero values of ψ_F^4 on $\partial\omega_e \setminus \partial\omega_{\boldsymbol{a}}$, the Green theorem first applied on $\widetilde{\omega}_e$ and later individually on K_j, and the notation $\boldsymbol{w}_h|_{K_j} = \boldsymbol{w}_h^j$ for the previous $K_j^\circ \subset \widetilde{\omega}_e$ give

$$\begin{aligned}
&- \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_1}} - \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{K_i}, \psi_F^4 \rangle_{F^{j_2}} = \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{\widetilde{\omega}_e}, \psi_F^4 \rangle_{F^{j_1}} + \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{\widetilde{\omega}_e}, \psi_F^4 \rangle_{F^{j_2}} \\
&\quad = \langle \boldsymbol{w}_h \cdot \boldsymbol{n}_{\widetilde{\omega}_e}, \psi_F^4 \rangle_{\partial \widetilde{\omega}_e} = (\boldsymbol{w}_h, \nabla \psi_F^4)_{\widetilde{\omega}_e} + (\nabla \cdot \boldsymbol{w}_h, \psi_F^4)_{\widetilde{\omega}_e} \\
&\quad = \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} \left\{ (\boldsymbol{w}_h^j, \nabla \psi_F^4)_{K_j} + (\nabla \cdot \boldsymbol{w}_h^j, \psi_F^4)_{K_j} \right\} = \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} \langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_j}, \psi_F^4 \rangle_{\partial K_j} \\
&\quad \overset{(A.18)}{=} \sum_{j;\, K_j^\circ \subset \widetilde{\omega}_e} (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_j} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{\omega_e} - (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i} \overset{(A.19)}{=} -(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^4)_{K_i},
\end{aligned} \tag{A.20}$$

which is (A.17). Thus, there exists a unique $\boldsymbol{w}_h^i$ from (A.13) also on this K_i.

Step 1d: *The last element* $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. According to property (ii) of the enumeration, the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$ is such that $|\mathcal{F}^\sharp_{|\mathcal{T}_{\boldsymbol{a}}|}| = 3$, so that all the neighbors have been already passed through. Consequently, all the degrees of freedom of $\boldsymbol{v}_h$ are fixed from the last three constraints in (A.13), and we need to show that $\nabla \cdot \boldsymbol{v}_h = 0$, i.e., that

$$(\nabla \cdot \boldsymbol{v}_h, q_h)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} = 0 \qquad \forall q_h \in \mathcal{P}_1(K_{|\mathcal{T}_{\boldsymbol{a}}|}),$$

since $\nabla \cdot \boldsymbol{v}_h \in \mathcal{P}_1(K_{|\mathcal{T}_{\boldsymbol{a}}|})$. Using the Green theorem and the constraints in (A.13) as above, this is equivalent to verifying that

$$\begin{aligned}
0 = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} &- \langle \boldsymbol{w}_h^{j_1} \cdot \boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k \rangle_{F^{j_1}} \\
&- \langle \boldsymbol{w}_h^{j_2} \cdot \boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k \rangle_{F^{j_2}} - \langle \boldsymbol{w}_h^{j_3} \cdot \boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k \rangle_{F^{j_3}}
\end{aligned} \tag{A.21}$$

for all $1 \le k \le 4$, where F^{j_1}, F^{j_2}, F^{j_3} are the three faces in $\mathcal{F}^\sharp_{|\mathcal{T}_{\boldsymbol{a}}|}$ and ψ_F^k are the hat basis functions associated with the four vertices of $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. As in (A.19), the assumptions of Lemma A.5 imply

$$0 = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^k)_{\omega_{\boldsymbol{a}}} \qquad 1 \le k \le 4. \tag{A.22}$$

Moreover, as in (A.18), it follows from (A.14), (A.15), and (A.16) that

$$\langle \boldsymbol{w}_h^j \cdot \boldsymbol{n}_{K_j}, \psi_F^k \rangle_{\partial K_j} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \psi_F^k)_{K_j} \qquad 1 \le k \le 4 \tag{A.23}$$

is satisfied on all elements K_j of the patch $\mathcal{T}_{\boldsymbol{a}}$ other than $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Let $\widetilde{\omega}_{\boldsymbol{a}}$ correspond to the patch subdomain $\omega_{\boldsymbol{a}}$ without the element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$. Then, as in (A.20),

$$\begin{aligned}
&-\langle \boldsymbol{w}_h^{j_1}\cdot\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_1}} - \langle \boldsymbol{w}_h^{j_2}\cdot\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_2}} - \langle \boldsymbol{w}_h^{j_3}\cdot\boldsymbol{n}_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}, \psi_F^k\rangle_{F^{j_3}}\\
&\quad= \langle \boldsymbol{w}_h^{j_1}\cdot\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_1}} + \langle \boldsymbol{w}_h^{j_2}\cdot\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_2}} + \langle \boldsymbol{w}_h^{j_2}\cdot\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{F^{j_3}}\\
&\quad= \langle \boldsymbol{w}_h\cdot\boldsymbol{n}_{\widetilde{\omega}_{\boldsymbol{a}}}, \psi_F^k\rangle_{\partial\widetilde{\omega}_{\boldsymbol{a}}} = (\boldsymbol{w}_h, \nabla\psi_F^k)_{\widetilde{\omega}_{\boldsymbol{a}}} + (\nabla\cdot\boldsymbol{w}_h, \psi_F^k)_{\widetilde{\omega}_{\boldsymbol{a}}}\\
&\quad= \sum_{j;\, K_j^\circ\subset\widetilde{\omega}_{\boldsymbol{a}}} \left\{(\boldsymbol{w}_h^j, \nabla\psi_F^k)_{K_j} + (\nabla\cdot\boldsymbol{w}_h^j, \psi_F^k)_{K_j}\right\} = \sum_{j;\, K_j^\circ\subset\widetilde{\omega}_{\boldsymbol{a}}} \langle \boldsymbol{w}_h^j\cdot\boldsymbol{n}_{K_j}, \psi_F^k\rangle_{\partial K_j}\\
&\quad\overset{(A.23)}{=} \sum_{j;\, K_j^\circ\subset\widetilde{\omega}_{\boldsymbol{a}}} (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_j} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla\psi_F^k)_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}} \overset{(A.22)}{=} -(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla\psi_F^k)_{K_{|\mathcal{T}_{\boldsymbol{a}}|}}
\end{aligned}$$

for all $1\le k\le 4$, i.e., (A.21). Thus, there exists a minimizer $\boldsymbol{w}_h^{|\mathcal{T}_{\boldsymbol{a}}|}$ of (A.13) on $K_{|\mathcal{T}_{\boldsymbol{a}}|}$.

Step 2: *Boundary patches with flat boundaries.* We now investigate the case where the vertex $\boldsymbol{a}\in\mathcal{V}_h$ lies on the boundary $\partial\Omega$. We present in this step in detail the case of a boundary patch $\mathcal{T}_{\boldsymbol{a}}$ for which $\Gamma_{\boldsymbol{a}}$, the part of $\partial\omega_{\boldsymbol{a}}$ that contains the faces sharing the vertex $\boldsymbol{a}$, is contained in a plane $\boldsymbol{H}$, which we call a "flat boundary" case. For the sake of simplicity, assume that either $\Gamma_{\boldsymbol{a}}\subset\Gamma_{\rm D}$ or $\Gamma_{\boldsymbol{a}}\subset\Gamma_{\rm N}$ and, without loss generality, that $H=\{\boldsymbol{x}\in\mathbb{R}^3;\ \boldsymbol{x}_3=0\}$. The symmetrization operator around the plane $\boldsymbol{H}$, $\phi:\boldsymbol{x}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\to(\boldsymbol{x}_1,\boldsymbol{x}_2,-\boldsymbol{x}_3)$, as in [21, section 7] and [11, section 7], will be instrumental in the proof. Specifically, we introduce the symmetrized patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}:=\mathcal{T}_{\boldsymbol{a}}\cup\phi(\mathcal{T}_{\boldsymbol{a}})$, with the associated domain $\widetilde{\omega}_{\boldsymbol{a}}$, obtained by mapping the elements of $\mathcal{T}_{\boldsymbol{a}}$ by ϕ. We will employ the Piola mappings from (A.7) to relate the set $W_h(\mathcal{T}_{\boldsymbol{a}},\boldsymbol{\mu}^{\boldsymbol{a}})$ from the announcement of Lemma A.5 to a set $W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$ with an extended datum $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}$. Then the result will follow by Step 1 since $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ is an interior patch.

Step 2a: *The case* $\boldsymbol{a}\in\Gamma_{\rm D}$. We start by defining the extended datum $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}\in\boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}})$ from $\boldsymbol{\mu}^{\boldsymbol{a}}$: We simply set $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}:=\boldsymbol{\mu}^{\boldsymbol{a}}$ in $\omega_{\boldsymbol{a}}$ and $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}:=\phi^{\rm d}(\boldsymbol{\mu}^{\boldsymbol{a}})$ on $\widetilde{\omega}_{\boldsymbol{a}}\setminus\omega_{\boldsymbol{a}}$. Let $\widetilde{q}_h\in\mathcal{P}_1(\widetilde{\mathcal{T}}_{\boldsymbol{a}})\cap H^1_*(\widetilde{\omega}_{\boldsymbol{a}})$. Recalling that ϕ is a symmetrization operator, its (constant) Jacobian matrix has a negative determinant. As a result, using (A.10) and (A.9),

$$\begin{aligned}
(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}},\nabla\widetilde{q}_h)_{\widetilde{\omega}_{\boldsymbol{a}}} &= (\boldsymbol{\mu}^{\boldsymbol{a}},\nabla\widetilde{q}_h)_{\omega_{\boldsymbol{a}}} + (\phi^{\rm d}(\boldsymbol{\mu}^{\boldsymbol{a}}),\nabla\widetilde{q}_h)_{\widetilde{\omega}_{\boldsymbol{a}}\setminus\omega_{\boldsymbol{a}}} = (\boldsymbol{\mu}^{\boldsymbol{a}},\nabla\widetilde{q}_h)_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\mu}^{\boldsymbol{a}},(\phi^{\rm c})^{-1}(\nabla\widetilde{q}_h))_{\omega_{\boldsymbol{a}}}\\
&= (\boldsymbol{\mu}^{\boldsymbol{a}},\nabla\widetilde{q}_h)_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\mu}^{\boldsymbol{a}},\nabla((\phi^{\rm g})^{-1}(\widetilde{q}_h)))_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\mu}^{\boldsymbol{a}},\nabla q_h)_{\omega_{\boldsymbol{a}}}
\end{aligned}$$

with $q_h:=\widetilde{q}_h-(\phi^{\rm g})^{-1}(\widetilde{q}_h)$. Because $(\phi^{\rm g})^{-1}$ preserves the trace on $\boldsymbol{H}$, we see that $q_h=0$ on $\boldsymbol{H}$ (see (A.8)), and since it also maps piecewise polynomials to piecewise polynomials (see (A.11)), $q_h\in\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})\cap H^1_*(\omega_{\boldsymbol{a}})$ (recall from section 2.7 that $H^1_*(\omega_{\boldsymbol{a}})=\{v\in H^1(\omega_{\boldsymbol{a}});\ v=0 \text{ on } \gamma_{\rm D}=\Gamma_{\boldsymbol{a}}\}$ here). Hence, $(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}},\nabla\widetilde{q}_h)_{\widetilde{\omega}_{\boldsymbol{a}}}=0$ by our assumption $(\boldsymbol{\mu}^{\boldsymbol{a}},\nabla q_h)_{\omega_{\boldsymbol{a}}}=0$. Thus, $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}$ satisfies the assumption of Lemma A.5 on the interior patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, and therefore Step 1 ensures that the set $W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$ is nonempty.

We now consider an arbitrary element $\widetilde{\boldsymbol{w}}_h\in W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$ and set $\boldsymbol{w}_h:=\widetilde{\boldsymbol{w}}_h|_{\omega_{\boldsymbol{a}}}$. Since $\widetilde{\boldsymbol{w}}_h\in\boldsymbol{\mathcal{RT}}_1(\widetilde{\mathcal{T}}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}})$, it is clear that $\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, namely as no normal trace boundary conditions are required on $\Gamma_{\boldsymbol{a}}\subset H$. Indeed, in this case, $\Gamma_{\boldsymbol{a}}=\gamma_{\rm D}$ in the notation of section 2.7, so that $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}}=0 \text{ on } \partial\omega_{\boldsymbol{a}}\setminus\Gamma_{\boldsymbol{a}}\}$. Moreover, $\nabla\cdot\boldsymbol{w}_h=\nabla\cdot\widetilde{\boldsymbol{w}}_h=0$ on $\omega_{\boldsymbol{a}}$. Finally, $(\boldsymbol{w}_h,\boldsymbol{r}_h)_K=(\boldsymbol{\mu}^{\boldsymbol{a}},\boldsymbol{r}_h)_K$ for all $\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^3$ and all $K\in\mathcal{T}_{\boldsymbol{a}}$ since $\mathcal{T}_{\boldsymbol{a}}\subset\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and simply $(\boldsymbol{w}_h,\boldsymbol{r}_h)_K=(\widetilde{\boldsymbol{w}}_h,\boldsymbol{r}_h)_K$, $(\boldsymbol{\mu}^{\boldsymbol{a}},\boldsymbol{r}_h)_K=(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}},\boldsymbol{r}_h)_K$, and $\widetilde{\boldsymbol{w}}_h\in W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$, so that $(\widetilde{\boldsymbol{w}}_h,\boldsymbol{r}_h)_K=(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}},\boldsymbol{r}_h)_K$. This concludes the proof that the set $W_h(\mathcal{T}_{\boldsymbol{a}},\boldsymbol{\mu}^{\boldsymbol{a}})$ is nonempty in this case.

Step 2b: *The case* $\boldsymbol{a} \in \Gamma_{\mathrm{N}}$. In this case, we extend the datum $\boldsymbol{\mu}^{\boldsymbol{a}}$ by setting $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}} := \boldsymbol{\mu}^{\boldsymbol{a}}$ on $\omega_{\boldsymbol{a}}$ and $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}} := \mathbf{0}$ on $\widetilde{\omega}_{\boldsymbol{a}} \setminus \omega_{\boldsymbol{a}}$. If $\widetilde{q}_h \in \mathcal{P}_1(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap H^1_*(\widetilde{\omega}_{\boldsymbol{a}})$, we have

$$(\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}, \nabla \widetilde{q}_h)_{\widetilde{\omega}_{\boldsymbol{a}}} = (\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla \widetilde{q}_h)_{\omega_{\boldsymbol{a}}} = 0$$

since $\widetilde{q}_h|_{\omega_{\boldsymbol{a}}} \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega_{\boldsymbol{a}})$, whose gradients have the same span as those of $\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, the zero mean value subspace of $\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega_{\boldsymbol{a}})$ following section 2.7 in this case. It thus follows from Step 1 that $W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}}, \widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$ is nonempty.

Consider an element $\widetilde{\boldsymbol{w}}_h \in W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}}, \widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$, and set $\boldsymbol{w}_h := \widetilde{\boldsymbol{w}}_h|_{\omega_{\boldsymbol{a}}} - (\phi^{\mathrm{d}})^{-1}(\widetilde{\boldsymbol{w}}_h|_{\widetilde{\omega}_{\boldsymbol{a}} \setminus \omega_{\boldsymbol{a}}})$. We need to show that $\boldsymbol{w}_h \in W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$. Recall that here the functions in $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ need to satisfy the no-flow boundary condition on the whole patch boundary $\partial\omega_{\boldsymbol{a}}$ and in particular on $\boldsymbol{H}$: $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}});\, \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}\}$ from section 2.7 in this case. Since the Piola mapping $(\phi^{\mathrm{d}})^{-1}$ maps piecewise Raviart–Thomas polynomials to piecewise Raviart–Thomas polynomials (cf. (A.11)) and preserves the divergence (cf. (A.9)) and the normal trace (cf. (A.8)), it is clear that $\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\nabla\cdot\boldsymbol{w}_h = 0$. It remains to show that $(\boldsymbol{w}_h, \boldsymbol{r}_h)_K = (\boldsymbol{\mu}^{\boldsymbol{a}}, \boldsymbol{r}_h)_K$ for all $\boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3$ and all $K \in \mathcal{T}_{\boldsymbol{a}}$. Let $K \in \mathcal{T}_{\boldsymbol{a}}$ and $\boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3$, and let $\widetilde{K}$ be the tetrahedron corresponding to K by the symmetry map ϕ. Then

$$\begin{aligned}
(\boldsymbol{w}_h, \boldsymbol{r}_h)_K &= (\widetilde{\boldsymbol{w}}_h, \boldsymbol{r}_h)_K - ((\phi^{\mathrm{d}})^{-1}(\widetilde{\boldsymbol{w}}_h), \boldsymbol{r}_h)_K = (\widetilde{\boldsymbol{w}}_h, \boldsymbol{r}_h)_K + (\widetilde{\boldsymbol{w}}_h, \phi^{\mathrm{c}}(\boldsymbol{r}_h))_{\widetilde{K}} \\
&= (\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}, \boldsymbol{r}_h)_K + (\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}, \phi^{\mathrm{c}}(\boldsymbol{r}_h))_{\widetilde{K}} = (\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}, \boldsymbol{r}_h)_K = (\boldsymbol{\mu}^{\boldsymbol{a}}, \boldsymbol{r}_h)_K,
\end{aligned}$$

where we have used (A.10), the fact that the Piola mapping ϕ^{c} maps piecewise constant vectors onto piecewise constant vectors (this can be seen from the definition (A.7) of ϕ^{c} since its Jacobian matrix is constant here), that $\widetilde{\boldsymbol{w}}_h \in W_h(\widetilde{\mathcal{T}}_{\boldsymbol{a}}, \widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}})$, and finally that $\widetilde{\boldsymbol{\mu}}^{\boldsymbol{a}}$ is the extension of $\boldsymbol{\mu}^{\boldsymbol{a}}$ by zero to the symmetrized patch. This concludes the proof that $W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$ is nonempty in this case.

Step 3: *General boundary patches.* For general boundary patches, the proof follows the lines of Step 2 while employing the extension and restriction operators introduced in [11, section 7] instead of the (simpler) symmetrization operator ϕ of Step 2. We do not give details here. ∎

A.6. Proof of Lemma A.4. We can now finally establish a proof of Lemma A.4.

Proof of Lemma A.4. *Step* 1: Existence and uniqueness. The minimization set in (A.5) is the set $W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$ of Lemma A.5 with $\boldsymbol{\mu}^{\boldsymbol{a}} := \boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$. Since the minimization functional in (A.5) is convex, it is sufficient to show that $W_h(\mathcal{T}_{\boldsymbol{a}}, \boldsymbol{\mu}^{\boldsymbol{a}})$ is nonempty to ensure the existence and uniqueness of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$ from (A.5). From Lemma A.5, we need to show that $(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} = 0$ for all $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$. This is actually a direct consequence of assumption (A.1c). Indeed, from the divergence constraint in (A.3) and since $q' \geq 1$ and $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$, we have

$$\begin{aligned}
(\boldsymbol{\mu}^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} - (\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} + (\nabla\cdot\bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} \\
&= (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} + (\Pi_{q'}(g^{\boldsymbol{a}}), q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} \overset{(\mathrm{A.1c})}{=} 0.
\end{aligned}$$

Step 2: *Stability bound.* We now proceed with the proof of the stability (A.6).

Step 2a: *Generic stability bound.* Set again $\boldsymbol{\mu}^{\boldsymbol{a}} := \boldsymbol{\tau}_h^{\boldsymbol{a}} - \bar{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$, and denote by $\boldsymbol{\mu}_h^{\boldsymbol{a}}$ the $\boldsymbol{L}^2(\omega_{\boldsymbol{a}})$-orthogonal projection of $\boldsymbol{\mu}^{\boldsymbol{a}}$ onto $[\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3$. Considering the Euler (–Lagrange) equations associated with (A.5), it is clear that we can equivalently replace $\boldsymbol{\mu}^{\boldsymbol{a}}$ by $\boldsymbol{\mu}_h^{\boldsymbol{a}}$ in the definition (A.5) of $\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}$. Furthermore, because (A.5) is a quadratic minimization problem with linear constraints, the operator $T : [\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3 \ni \boldsymbol{\mu}_h^{\boldsymbol{a}} \to \bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} \in$

$\boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ (well-defined from Step 1) is linear. Since both $[\mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}})]^3$ and $\boldsymbol{\mathcal{RT}}_1(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ are finite-dimensional spaces, the operator T is continuous, and there exists a constant $C(\mathcal{T}_{\boldsymbol{a}})$ such that

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq C(\mathcal{T}_{\boldsymbol{a}})\|\boldsymbol{\mu}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq C(\mathcal{T}_{\boldsymbol{a}})\|\boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}, \tag{A.24}$$

where we used the fact that $\boldsymbol{\mu}_h^{\boldsymbol{a}}$ is defined from $\boldsymbol{\mu}^{\boldsymbol{a}}$ by projection in the last inequality. The constant $C(\mathcal{T}_{\boldsymbol{a}})$ is independent of the polynomial degree q (recall that (A.5) works with $\boldsymbol{\mathcal{RT}}_1$ elements only) but depends on the patch $\mathcal{T}_{\boldsymbol{a}}$ in an unspecified way. To make the dependence explicit, we resort in the next step to a reference patch.

Step 2b: *Explicit stability bound.* For a fixed shape-regularity parameter $\kappa_{\mathcal{T}_h}$ from section 2.4, there exists a maximal number of elements $N(\kappa_{\mathcal{T}_h})$ allowed in any patch $\mathcal{T}_{\boldsymbol{a}}$. In turn, for any $N(\kappa_{\mathcal{T}_h})$, there exists a finite set of reference patches $\{\widehat{\mathcal{T}}\}$ such that for all vertex patches $\mathcal{T}_{\boldsymbol{a}}$, there exists a reference patch $\widehat{\mathcal{T}}$ and a bi-Lipschitz mapping $\phi: \omega_{\boldsymbol{a}} \to \widehat{\omega}$ ($\widehat{\omega}$ being the open domain associated with $\widehat{\mathcal{T}}$) such that $\phi|_K$ is an affine mapping between the tetrahedron $K \in \mathcal{T}_{\boldsymbol{a}}$ and a tetrahedron $\widehat{K} \in \widehat{\mathcal{T}}$. The associated Piola mapping ϕ^{d} from (A.7) will be useful.

Crucially, we observe that for all $\widehat{K} \in \widehat{\mathcal{T}}$, $\boldsymbol{v} \in \boldsymbol{L}^2(K)$, and $\widehat{\boldsymbol{r}}_h \in [\mathcal{P}_0(\widehat{K})]^3$, there exists $\boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3$ such that $(\phi^{\mathrm{d}}(\boldsymbol{v}), \widehat{\boldsymbol{r}}_h)_{\widehat{K}} = (\boldsymbol{v}, \boldsymbol{r}_h)_K$ since, elementwise, the Piola transform amounts to a multiplication by a constant matrix and a change of coordinates. It follows that ϕ^{d} maps the minimization set of (A.5) on $\mathcal{T}_{\boldsymbol{a}}$ into the minimization set of the equivalent problem set on $\widehat{\mathcal{T}}$ with constraints $\phi^{\mathrm{d}}(\boldsymbol{\mu}^{\boldsymbol{a}})$.

Now, on the reference patch $\widehat{\mathcal{T}}$, if $\widehat{\boldsymbol{\epsilon}}_h$ is the minimizer of (A.5) with the datum $\phi^{\mathrm{d}}(\boldsymbol{\mu}^{\boldsymbol{a}})$, we conclude from Step 2a that

$$\|\widehat{\boldsymbol{\epsilon}}_h\|_{\widehat{\omega}} \leq C(\kappa_{\mathcal{T}_h})\|\phi^{\mathrm{d}}(\boldsymbol{\mu}^{\boldsymbol{a}})\|_{\widehat{\omega}} \leq C(\kappa_{\mathcal{T}_h})\|\phi^{\mathrm{d}}\|\|\boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}.$$

On the other hand, since $(\phi^{\mathrm{d}})^{-1}(\widehat{\boldsymbol{\epsilon}}_h)$ belongs the minimization set on $\mathcal{T}_{\boldsymbol{a}}$, we have

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|(\phi^{\mathrm{d}})^{-1}(\widehat{\boldsymbol{\epsilon}}_h) - \boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|(\phi^{\mathrm{d}})^{-1}\|\|\widehat{\boldsymbol{\epsilon}}_h\|_{\widehat{\omega}} + \|\boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}},$$

so that

$$\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq (1 + C(\kappa_{\mathcal{T}_h})\|(\phi^{\mathrm{d}})^{-1}\|\|\phi^{\mathrm{d}}\|)\|\boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}.$$

At this point, we conclude the proof since $\|(\phi^{\mathrm{d}})^{-1}\|\|\phi^{\mathrm{d}}\|$ only depends on $\kappa_{\mathcal{T}_h}$ due to (A.12) and $\|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\bar{\boldsymbol{\epsilon}}_h^{\boldsymbol{a}} - \boldsymbol{\mu}^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$. ◻

Appendix B. Decomposition of a divergence-free piecewise polynomial with an elementwise orthogonality into local divergence-free contributions.

Let $q \geq 0$ be a fixed integer, and recall the notation of section 2; namely, $\boldsymbol{I}_{K,q}^{\boldsymbol{\mathcal{RT}}}$ is the canonical q-degree Raviart–Thomas interpolate on the given mesh element $K \in \mathcal{T}_h$ from (2.4), and $\lesssim$ means smaller or equal to up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$. The following result is of independent interest.

THEOREM B.1 (decomposition of a divergence-free Raviart–Thomas piecewise polynomial with an elementwise orthogonality constraint into local divergence-free contributions). *Let*

$$\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla \cdot \boldsymbol{\delta}_h = 0 \tag{B.1}$$

be a divergence-free q-degree Raviart–Thomas piecewise polynomial that is elementwise orthogonal to vector-valued constants:

$$(\boldsymbol{\delta}_h, \boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_h. \tag{B.2}$$

Then there exists a unique solution to the q'-degree Raviart–Thomas elementwise minimizations, $q' = q$ or $q' = q+1$,

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K^2 \tag{B.3}$$

for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$. This yields patchwise divergence-free contributions

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{q'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h, \tag{B.4}$$

decomposing $\boldsymbol{\delta}_h$ as

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{B.5}$$

Moreover, for all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, there hold the local stability estimates

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}} - \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K \lesssim \|\boldsymbol{\delta}_h\|_K, \tag{B.6a}$$

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_K \lesssim_{q'} \|\boldsymbol{\delta}_h\|_K, \tag{B.6b}$$

where $\lesssim_{q'}$ means $\lesssim$ for $q' = q+1$ and up to a constant only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$ and the degree q when $q' = q$.

Remark B.2 (the two settings $q' = q$ or $q' = q+1$ in Theorem B.1). With the choice $q' = q$, the contributions $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ in Theorem B.1 stay in the same degree Raviart–Thomas space as the datum $\boldsymbol{\delta}_h$, but, unfortunately, the stability (B.6b) is not necessarily q-robust. For q-robustness, the choice $q' = q+1$, increasing the degree of $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ by one, is to be used. Note that in this case, the Raviart–Thomas interpolator $\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}$ can be disregarded since then $\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K) = (\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K$.

Proof. Let $\boldsymbol{\delta}_h$ satisfy (B.1) and (B.2). We address (B.3)–(B.6) in four steps.

Step 1: *Proof of the well-posedness of* (B.3). Fix $K \in \mathcal{T}_h$ and $\boldsymbol{a} \in \mathcal{V}_K$. The existence and uniqueness of $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ from (B.3) are classical following, e.g., [6], when the Neumann compatibility condition $\langle \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} = 0$ is satisfied. This can be shown via (2.4a), the Green theorem, the assumption $\nabla\cdot\boldsymbol{\delta}_h = 0$ in (B.1), and the elementwise orthogonality assumption (B.2) (note that $(\nabla\psi^{\boldsymbol{a}})|_K \in [\mathcal{P}_0(K)]^3$) as

$$\begin{aligned}\langle \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} &= \langle \psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\cdot\boldsymbol{n}_K, 1\rangle_{\partial K} = \langle \boldsymbol{\delta}_h\cdot\boldsymbol{n}_K, \psi^{\boldsymbol{a}}\rangle_{\partial K} \\ &= (\nabla\cdot\boldsymbol{\delta}_h, \psi^{\boldsymbol{a}})_K + (\boldsymbol{\delta}_h, \nabla\psi^{\boldsymbol{a}})_K = 0.\end{aligned}$$

Step 2: *Proof of the stability estimates* (B.6). Still for a fixed $K \in \mathcal{T}_h$ and $\boldsymbol{a} \in \mathcal{V}_K$, consider the problem

$$\hat{\boldsymbol{\delta}}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \\ \nabla\cdot\boldsymbol{v}_h = (-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h)|_K \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K}} \|\boldsymbol{v}_h\|_K^2. \tag{B.7}$$

This problem is again well-posed since from (B.2), $(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, 1)_K = (\boldsymbol{\delta}_h, \nabla\psi^{\boldsymbol{a}})_K = 0$; moreover, $(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h)|_K \in \mathcal{P}_q(K) \subset \mathcal{P}_{q'}(K)$ since from $\nabla\cdot\boldsymbol{\delta}_h = 0$, it follows that $\boldsymbol{\delta}_h|_K \in [\mathcal{P}_q(K)]^3$ (see, e.g., [6, Corollary 2.3.1]). It follows that $\hat{\boldsymbol{\delta}}_h^{\boldsymbol{a}}|_K = \boldsymbol{\delta}_h^{\boldsymbol{a}}|_K -$

$\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)$; indeed, the commuting property (2.5) yields, on the simplex K, $\nabla\cdot(\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)) = \mathcal{P}_{q'}(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)) = \mathcal{P}_{q'}(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h) = \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h$. Problem (B.7) fits the framework of [21, Lemma A.3] with $r_F = 0$, $r_K = (-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h)|_K$, and $p = q'$, so that

$$\|\boldsymbol{\delta}^{\boldsymbol{a}}_h - \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K = \|\hat{\boldsymbol{\delta}}^{\boldsymbol{a}}_h\|_K = \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{q'}(K)\\ \nabla\cdot\boldsymbol{v}_h=-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=0 \text{ on } \partial K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \nabla\cdot\boldsymbol{v}=-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h\\ \boldsymbol{v}\cdot\boldsymbol{n}_K=0 \text{ on } \partial K}} \|\boldsymbol{v}\|_K = \|\nabla\zeta_K\|_K.$$

Here, by the primal-dual equivalence, $\zeta_K \in H^1_*(K)$ is such that

$$(\nabla\zeta_K, \nabla v)_K = -(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, v)_K \qquad \forall v \in H^1_*(K)$$

with $H^1_*(K) := \{v \in H^1(K);\ (v,1)_K = 0\}$, where the Poincaré inequality gives $\|v\|_K \lesssim h_K\|\nabla v\|_K$. Then the Cauchy–Schwarz inequality and shape regularity yield

$$\|\nabla\zeta_K\|_K = \max_{\substack{v\in H^1_*(K)\\ \|\nabla v\|_K=1}} (\nabla\zeta_K, \nabla v)_K = \max_{\substack{v\in H^1_*(K)\\ \|\nabla v\|_K=1}} -(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\delta}_h, v)_K \lesssim \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,K}\|\boldsymbol{\delta}_h\|_K h_K \lesssim \|\boldsymbol{\delta}_h\|_K.$$

Combining the two above estimates gives the desired stability result (B.6a). The other stability result (B.6b) follows from (B.6a) by the triangle inequality together with the non-q-robust stability bound $\|\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K \lesssim_{q'} \|\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K \leq \|\boldsymbol{\delta}_h\|_K$ when $q' = q$, whereas $\|\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\|_K = \|\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h\|_K \leq \|\boldsymbol{\delta}_h\|_K$ when $q' = q+1$.

Step 3: *Proof of the patchwise properties* (B.4). The first property in (B.4) follows from the prescription of the normal components in (B.3), whereas the second one is the divergence prescription in (B.3).

Step 4: *Proof of the decomposition* (B.5). Finally, in order to prove (B.5), set $\tilde{\boldsymbol{\delta}}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{\delta}^{\boldsymbol{a}}_h$. Now fix an element $K \in \mathcal{T}_h$, and remark that from the normal trace constraint in (B.3) and the linearity of the interpolator $\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}$, on ∂K,

$$\begin{aligned}\tilde{\boldsymbol{\delta}}_h|_K\cdot\boldsymbol{n}_K &= \sum_{\boldsymbol{a}\in\mathcal{V}_K}\boldsymbol{\delta}^{\boldsymbol{a}}_h|_K\cdot\boldsymbol{n}_K = \sum_{\boldsymbol{a}\in\mathcal{V}_K}\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K)\cdot\boldsymbol{n}_K = \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}\left[\sum_{\boldsymbol{a}\in\mathcal{V}_K}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K\right]\cdot\boldsymbol{n}_K\\ &= \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}(\boldsymbol{\delta}_h|_K)\cdot\boldsymbol{n}_K = \boldsymbol{\delta}_h|_K\cdot\boldsymbol{n}_K\end{aligned}$$

also using the partition of unity (2.1). Similarly, by the divergence constraint in (B.3) and $\nabla\cdot\boldsymbol{\delta}_h = 0$ from (B.1), on K,

$$\nabla\cdot\tilde{\boldsymbol{\delta}}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_K}\nabla\cdot\boldsymbol{\delta}^{\boldsymbol{a}}_h = 0 = \nabla\cdot\boldsymbol{\delta}_h.$$

Consequently, $(\tilde{\boldsymbol{\delta}}_h - \boldsymbol{\delta}_h)|_K \in \boldsymbol{\mathcal{RT}}_{q'}(K)$ has zero normal trace and divergence. Moreover, the Euler conditions of problem (B.3) state

$$(\boldsymbol{\delta}^{\boldsymbol{a}}_h - \boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}((\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)|_K), \boldsymbol{v}_h)_K = 0 \quad \forall\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0 \text{ and } \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0$$

on ∂K. Summing this over all vertices $\boldsymbol{a} \in \mathcal{V}_K$ and using again the linearity of $\boldsymbol{I}^{\boldsymbol{\mathcal{RT}}}_{K,q'}$,

$$(\tilde{\boldsymbol{\delta}}_h - \boldsymbol{\delta}_h, \boldsymbol{v}_h)_K = 0 \qquad \forall\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{q'}(K) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0 \text{ and } \boldsymbol{v}_h\cdot\boldsymbol{n}_K = 0 \text{ on } \partial K,$$

so that indeed $\tilde{\boldsymbol{\delta}}_h = \boldsymbol{\delta}_h$ on any mesh element $K \in \mathcal{T}_h$. ∎

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