Constrained and unconstrained stable discrete minimizations for p-robust local reconstructions in vertex patches in the De Rham complex

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CONSTRAINED AND UNCONSTRAINED STABLE DISCRETE MINIMIZATIONS FOR p-ROBUST LOCAL RECONSTRUCTIONS IN VERTEX PATCHES IN THE DE RHAM COMPLEX*

T. CHAUMONT-FRELET† AND M. VOHRALÍK‡

ABSTRACT. We analyze constrained and unconstrained minimization problems on patches of tetrahedra sharing a common vertex with discontinuous piecewise polynomial data of degree p. We show that the discrete minimizers in the spaces of piecewise polynomials of degree p conforming in the H^1, $\boldsymbol{H}(\mathbf{curl})$, or $\boldsymbol{H}(\mathrm{div})$ spaces are as good as the minimizers in these entire (infinite-dimensional) Sobolev spaces, up to a constant that is independent of p. These results are useful in the analysis and design of finite element methods, namely for devising stable local commuting projectors and establishing local-best–global-best equivalences in a priori analysis and in the context of a posteriori error estimation. Unconstrained minimization in H^1 and constrained minimization in $\boldsymbol{H}(\mathrm{div})$ have been previously treated in the literature. Along with improvement of the results in the H^1 and $\boldsymbol{H}(\mathrm{div})$ cases, our key contribution is the treatment of the $\boldsymbol{H}(\mathbf{curl})$ framework. This enables us to cover the whole De Rham diagram in three space dimensions in a single setting.

KEYWORDS. potential reconstruction, flux reconstruction, a posteriori error estimate, robustness, polynomial degree, best approximation, finite element method.

1. INTRODUCTION

The concept of "equilibrated flux", dating back to at least the seminal paper [29], is the basis for the design of guaranteed a posteriori error estimates for finite element discretization of various PDE problems, see [18, 2, 27, 31, 6, 19, 26, 22, 32, 9] and the references therein. One key feature of this family of estimators is that they can be designed so that they are "polynomial-degree-robust" (or simply, p-robust), meaning that their overestimation factor does not depend on the polynomial degree p of the discretization space. This fact has first been established in [5] when considering a conforming finite element discretization of the two-dimensional Poisson problem. The proof hinges on the following result: if $\mathcal{T}_{\boldsymbol{a}}$ is a vertex patch of triangles sharing a vertex $\boldsymbol{a}$, $\omega_{\boldsymbol{a}}$ is the corresponding domain, $p \geq 0$ is a polynomial degree, and $r_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$ as well as $\boldsymbol{\tau}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$ are given (discontinuous)

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piecewise polynomial data (these notations are rigorously introduced below), there holds

$$\min_{\substack{\boldsymbol{w}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\cdot\boldsymbol{w}_p = r_p}} \|\boldsymbol{w}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{w} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\cdot\boldsymbol{w} = r_p}} \|\boldsymbol{w} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}, \tag{1.1}$$

where the constant C_{st} only depends on the shape-regularity parameter of the patch; crucially, C_{st} is independent of p. The proof of (1.1) hinges on the volume and normal trace p-robust polynomial extensions on a single tetrahedron of [14, Proposition 4.2] and [17, Theorem 7.1], and the result also holds in three space dimensions, see [23, Corollary 3.3]. The (fully computable) minimizer on the left-hand side of (1.1) is directly involved in the construction of the a posteriori error estimator, while the minimum of the right-hand side is not computable but can be straightforwardly related to the discretization error in the patch domain $\omega_{\boldsymbol{a}}$. The constant C_{st} thus naturally enters the efficiency estimate of the estimator, and the p-robustness is a consequence of the fact that it is independent of p.

Constrained minimization problems of the form (1.1) are sufficient for the a posteriori analysis of conforming finite element discretizations. However, when considering nonconforming discretizations [19], another family of minimization problems comes into play. Specifically, the following stability result is of paramount importance: given a (discontinuous) piecewise polynomial $\chi_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ vanishing on $\partial\omega_{\boldsymbol{a}}$, there holds

$$\min_{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_0(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla}(v_p - \chi_p)\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{v \in H^1_0(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla}(v - \chi_p)\|_{\omega_{\boldsymbol{a}}}, \tag{1.2}$$

where $\boldsymbol{\nabla}$ denotes the broken (elementwise) gradient and C_{st} again does not depend on p, see [23, Corollary 3.1] which builds on [15, Theorem 6.1]. Similarly to (1.1), the minimizer of the left-hand side of (1.2) is computed as a part of the estimator construction, while the right-hand side can be linked to the discretization error in the patch domain $\omega_{\boldsymbol{a}}$.

The H^1 and $\boldsymbol{H}(\mathrm{div})$ spaces in (1.1) and (1.2) are naturally involved in the context of the Poisson problem, since the Laplace differential operator is a composition of gradient and divergence. When considering Maxwell's equations and their discretization by Nédélec's elements, minimization problems similar to (1.1) and (1.2) but involving the $\boldsymbol{H}(\mathbf{curl})$ Sobolev space and the curl operator naturally emerge [8, 10, 12]. In particular, an equivalent to (1.1) on a smaller edge patch of tetrahedra has been recently established in [10, Theorem 3.1], building on [14, Proposition 4.2] and [16, Theorem 7.2].

In addition to the analysis of a posteriori error estimators, constrained and unconstrained minimization problems of the form (1.1) and (1.2) are also instrumental in the design of stable local commuting interpolation operators having the projection property under minimal regularity and in the equivalence of "global-best" and "local-best" approximations, see [34, 33, 11, 20].

The goal of the present work is threefold: (i) to establish a $\boldsymbol{H}(\mathbf{curl})$-variant of (1.1) and (1.2) on a vertex patch of tetrahedra; (ii) present a complete theory of constrained and unconstrained local minimization problems in the De Rham complex in three space dimensions, realizing that the $\boldsymbol{H}(\mathbf{curl})$-minimization was the last piece missing; (iii) complement on and improve the results presented in [23] for the treatment of boundary patches.

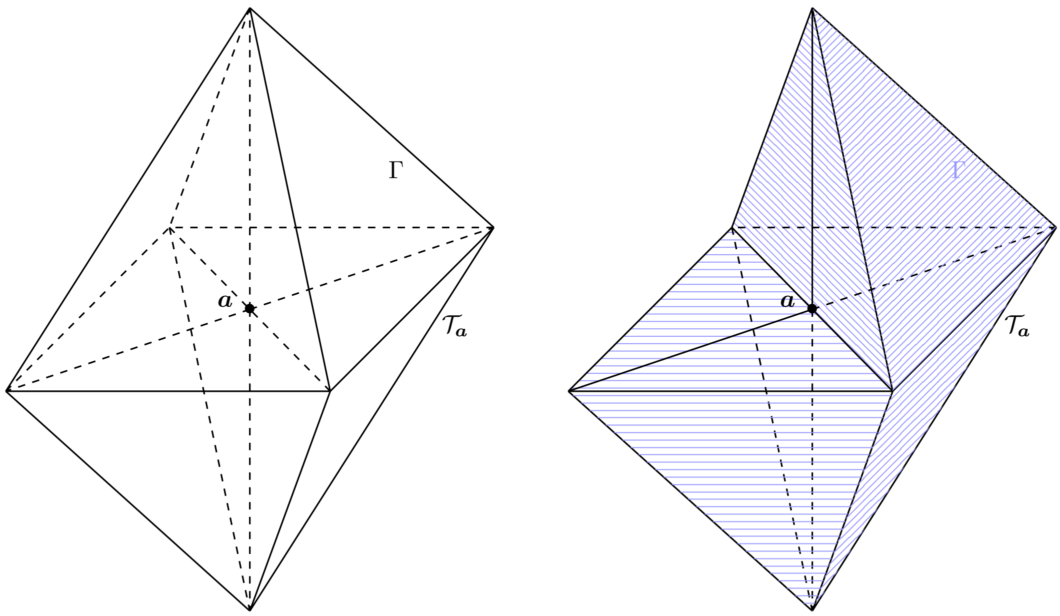


FIGURE 1. Interior patch (left) and boundary patch (right); in both cases, $\Gamma = \partial\omega_{\boldsymbol{a}}$ and $\Gamma_{\boldsymbol{a}} = \emptyset$

The remainder of this work is organized as follows. In Section 2, we precise the setting as well as the notation. Section 3 presents our main results, and we show in Section 4 that these also cover the case of inhomogeneous boundary conditions. Section 5 then collects some technical results and detailed notations used in the bulk of the proofs for interior patches in Section 6. We treat the case of boundary patches in Section 7, and specifically mixed boundary conditions in Section 8. We label as "Proposition" known results, whereas the main new results are named "Theorem" or "Corollary".

2. Setting

2.1. **Vertex patch.** Throughout this work, $\mathcal{T}_{\boldsymbol{a}}$ denotes a patch of tetrahedra, a finite collection of closed nontrivial tetrahedra $K \subset \mathbb{R}^3$ that all have $\boldsymbol{a}$ as vertex, and which is such that for two elements $K_\pm \in \mathcal{T}_{\boldsymbol{a}}$, the intersection $K_- \cap K_+$ is either the vertex $\boldsymbol{a}$, a full edge of both K_- and K_+, or a full face of both K_- and K_+. We also assume that the patch is face connected, meaning that a path between two points in two different tetrahedra in $\mathcal{T}_{\boldsymbol{a}}$ can always pass through interiors of tetrahedra faces. We denote by $\omega_{\boldsymbol{a}}$ the interior of $\cup_{K\in\mathcal{T}_{\boldsymbol{a}}} K$ and we suppose that it has a Lipschitz boundary $\partial\omega_{\boldsymbol{a}}$. Thus, $\omega_{\boldsymbol{a}}$ is in particular simply connected. For a tetrahedron K, $\boldsymbol{n}_K$ is its unit normal vector, outward to K. For the applications we have in mind, this situation appears when $\boldsymbol{a}$ is a vertex of a simplicial mesh $\mathcal{T}_h$ of some computational domain Ω in the context of finite element methods, see, e.g., [13, 4, 21].

Let $\mathcal{F}_{\boldsymbol{a}}$ denote the set of the (closed) faces of the tetrahedra of the patch; with each face $F \in \mathcal{F}_{\boldsymbol{a}}$, we associate a unit normal vector $\boldsymbol{n}_F$ of an arbitrary but fixed orientation. We

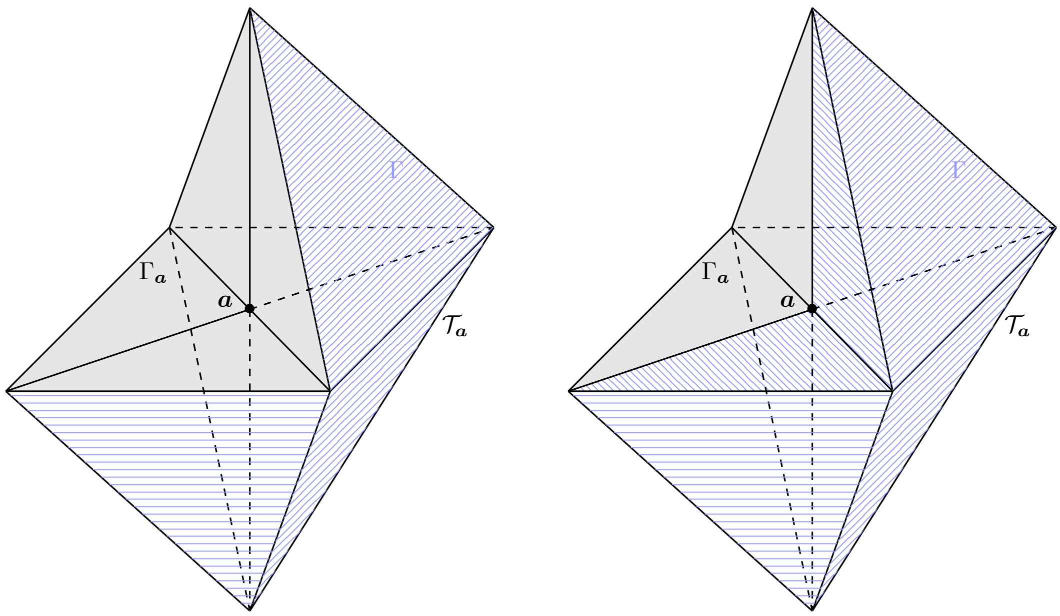


FIGURE 2. Boundary patches with $\Gamma_{\boldsymbol{a}} \neq \emptyset$; $\Gamma_{\boldsymbol{a}}$ corresponding to all faces $F \in \mathcal{F}_{\boldsymbol{a}}$ lying on the boundary of $\omega_{\boldsymbol{a}}$ and sharing the vertex $\boldsymbol{a}$ (left, considered in Sections 3–7), $\Gamma_{\boldsymbol{a}}$ corresponding to some faces $F \in \mathcal{F}_{\boldsymbol{a}}$ lying on the boundary of $\omega_{\boldsymbol{a}}$ and sharing the vertex $\boldsymbol{a}$ (right, considered in Section 8)

will distinguish two situations. When $\omega_{\boldsymbol{a}}$ contains an open ball around $\boldsymbol{a}$, we call $\mathcal{T}_{\boldsymbol{a}}$ an "interior patch" and we set $\Gamma := \partial\omega_{\boldsymbol{a}}$ and $\Gamma_{\boldsymbol{a}} := \emptyset$, see Figure 1, left, for an illustration. When this is not the case, we speak of a "boundary patch". Then, there is a cone $\mathcal{C}$ with the vertex $\boldsymbol{a}$ and a strictly positive solid angle such that $\mathcal{C} \cap \omega_{\boldsymbol{a}} = \emptyset$, forming an "opening". In this case, we distinguish two situations: either 1) $\Gamma := \partial\omega_{\boldsymbol{a}}$ and $\Gamma_{\boldsymbol{a}} := \emptyset$ (see Figure 1, right); or 2) $\Gamma_{\boldsymbol{a}}$ corresponds to all faces $F \in \mathcal{F}_{\boldsymbol{a}}$ lying on the boundary of $\omega_{\boldsymbol{a}}$ and sharing the vertex $\boldsymbol{a}$[1] and $\Gamma := \partial\omega_{\boldsymbol{a}} \setminus \overline{\Gamma_{\boldsymbol{a}}}$ (see Figure 2, left). In all cases, Γ is connected.

2.2. **Shape regularity.** For a tetrahedron K, let h_K and ρ_K respectively denote the diameter of K and the diameter of the largest ball contained in K. The shape-regularity parameter $\kappa_K := h_K/\rho_K$ is then a measure of the "flatness" of K, see, e.g., [13, 21]. If $\mathcal{T}$ is a collection of tetrahedra, we denote by $\kappa_{\mathcal{T}} := \max_{K \in \mathcal{T}} \kappa_K$ the shape-regularity parameter of $\mathcal{T}$.

2.3. **Functional spaces.** If $\omega \subset \mathbb{R}^3$ is a domain (open, bounded, and connected set), $H^1(\omega)$, $\boldsymbol{H}(\mathbf{curl}, \omega)$, and $\boldsymbol{H}(\mathrm{div}, \omega)$ are the usual Sobolev spaces [1, 4, 21, 25], $\boldsymbol{H}^1(\omega) := [H^1(\omega)]^3$, and $\boldsymbol{L}^2(\omega) := [L^2(\omega)]^3$. If $\gamma \subset \partial\omega$ is a relatively open subset of the boundary of ω, $H^1_{0,\gamma}(\omega)$ is the subset of functions of $H^1(\omega)$ with vanishing trace on γ, and $\boldsymbol{H}^1_{0,\gamma}(\omega) :=$

[1] The case where $\Gamma_{\boldsymbol{a}}$ only corresponds to some faces on $\partial\omega_{\boldsymbol{a}}$ and sharing $\boldsymbol{a}$ will be investigated separately in Section 8 below.

$[H^1_{0,\gamma}(\omega)]^3$. In the $\boldsymbol{H}(\mathbf{curl})$ setting, we then denote

$$\boldsymbol{H}_{0,\gamma}(\mathbf{curl},\omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega) \mid \boldsymbol{v}\times\boldsymbol{n}_\omega = \mathbf{0} \text{ on } \gamma\},$$

where the notion of trace is understood by duality, i.e., $\boldsymbol{v}\times\boldsymbol{n}_\omega = \mathbf{0}$ on γ means that

$$(\boldsymbol{\nabla}\times\boldsymbol{v},\boldsymbol{\phi})_\omega - (\boldsymbol{v},\boldsymbol{\nabla}\times\boldsymbol{\phi})_\omega = 0 \qquad \forall \boldsymbol{\phi} \in \boldsymbol{H}^1_{0,\partial\omega\setminus\overline{\gamma}}(\omega).$$

In the $\boldsymbol{H}(\mathrm{div})$ setting, similarly,

$$\boldsymbol{H}_{0,\gamma}(\mathrm{div},\omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega) \mid \boldsymbol{v}\cdot\boldsymbol{n}_\omega = 0 \text{ on } \gamma\},$$

where $\boldsymbol{v}\cdot\boldsymbol{n}_\omega = 0$ on γ means that

$$(\boldsymbol{\nabla}\cdot\boldsymbol{v},\phi)_\omega + (\boldsymbol{v},\boldsymbol{\nabla}\phi)_\omega = 0 \qquad \forall \phi \in H^1_{0,\partial\omega\setminus\overline{\gamma}}(\omega).$$

We refer the reader to [24] for a detailed treatment of boundary conditions in $\boldsymbol{H}(\mathbf{curl},\omega)$ and $\boldsymbol{H}(\mathrm{div},\omega)$. For the sake of simplicity, we also define $L^2_{0,\gamma}(\omega)$ as $L^2(\omega)$ if γ is non-empty, and as the subset of $L^2(\omega)$ with functions of zero mean value on ω if γ is empty. We will also employ the above notations if ω is a (closed) tetrahedron and γ the union of some of its (closed) faces.

2.4. **Piecewise polynomial spaces.** Consider a tetrahedron K. For $q \geq 0$, $\mathcal{P}_q(K)$ is the set of polynomials of degree less than or equal to q, and $\boldsymbol{\mathcal{P}}_q(K) := [\mathcal{P}_q(K)]^3$. The spaces of Raviart–Thomas and Nédélec polynomials are then defined by

$$\boldsymbol{\mathcal{RT}}_q(K) := \boldsymbol{\mathcal{P}}_q(K) + \boldsymbol{x}\mathcal{P}_q(K), \qquad \boldsymbol{\mathcal{N}}_q(K) := \boldsymbol{\mathcal{P}}_q(K) + \boldsymbol{x}\times\boldsymbol{\mathcal{P}}_q(K),$$

see [30, 28, 4, 21]. If $\mathcal{T}$ is a collection of tetrahedra, we employ the notations $\mathcal{P}_q(\mathcal{T})$, $\boldsymbol{\mathcal{RT}}_q(\mathcal{T})$, and $\boldsymbol{\mathcal{N}}_q(\mathcal{T})$ for functions whose restrictions to each $K \in \mathcal{T}$ belong respectively to $\mathcal{P}_q(K)$, $\boldsymbol{\mathcal{RT}}_q(K)$, and $\boldsymbol{\mathcal{N}}_q(K)$. Notice that these spaces have no "build-in" continuity conditions (they form the so-called broken spaces); we impose the continuity conditions by an intersection with the Sobolev spaces from Section 2.3.

3. Main results

We start by stating the following result from [23, Corollaries 3.3 and 3.8].[2] Here and below, $C(x)$ means a generic constant only depending on the quantity x. Thus, our results only depend on the shape-regularity $\kappa_{\mathcal{T}_{\boldsymbol{a}}}$ of the patch $\mathcal{T}_{\boldsymbol{a}}$ and not on the underlying polynomial degree p (or mesh size h or any other parameter).

Proposition 3.1 (Constrained minimization in $\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$)**.** *For all $p \geq 0$, $\boldsymbol{\tau}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$, and $r_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap L^2_{0,\Gamma_{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$, we have*

$$\min_{\substack{\boldsymbol{w}_p\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})\\ \boldsymbol{\nabla}\cdot\boldsymbol{w}_p=r_p}} \|\boldsymbol{w}_p-\boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{w}\in\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})\\ \boldsymbol{\nabla}\cdot\boldsymbol{w}=r_p}} \|\boldsymbol{w}-\boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}.$$

Our first new result is an easy consequence of the Proposition 3.1 and treats unconstrained minimization in $\boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$.

[2] Actually, some geometrical situations for boundary patches were excluded in [23] (at most two simplices in the patch $\mathcal{T}_{\boldsymbol{a}}$ or the existence of an interior vertex in Γ; these are now covered by the proof detailed in Section 7 below).

Corollary 3.2 (Unconstrained minimization in $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$)**.** *For all $p \geq 0$ and all $\boldsymbol{\tau}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$, we have*

$$\min_{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} \times \boldsymbol{v}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} \times \boldsymbol{v} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. We proceed as in [11, proof of Theorem 1]. From our assumptions in Section 2.1, $\omega_{\boldsymbol{a}}$ is simply connected, the boundary $\partial\omega_{\boldsymbol{a}}$ is Lipschitz, and Γ is connected. Thus, it follows that the range of the curl operator $\boldsymbol{\nabla}\times$ acting on $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ is exactly the kernel of the divergence operator on $\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$, and a similar property holds for the discrete spaces $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ and $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$, see, e.g., [3, 4, 24]. Then, the result follows from Proposition 3.1, since

$$\begin{aligned}
\min_{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} \times \boldsymbol{v}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} &= \min_{\substack{\boldsymbol{w}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\cdot\boldsymbol{w}_p = 0}} \|\boldsymbol{w}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \\
&\leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{w} \in \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\cdot\boldsymbol{w} = 0}} \|\boldsymbol{w} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \\
&= C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} \times \boldsymbol{v} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}.
\end{aligned}$$

□

The central result of this work is the following theorem which addresses constrained minimization in $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$. Its proof is lengthy, and postponed to Sections 6 and 7.

Theorem 3.3 (Constrained minimization in $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$)**.** *For all $p \geq 0$, $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, and $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot\boldsymbol{j}_p = 0$, we have*

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v}_p = \boldsymbol{j}_p}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{j}_p}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Our last result concerns unconstrained minimization in $H^1(\omega_{\boldsymbol{a}})$. It generalizes the result previously obtained in [23, Corollaries 3.1 and 3.7], which was limited to the case where $\boldsymbol{\chi}_p = \boldsymbol{\nabla}\chi_p$ for $\chi_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ with $\chi_p = 0$ on Γ, and where the geometrical setting of boundary patches had some restrictions.

Corollary 3.4 (Unconstrained minimization in $H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})$)**.** *For all $p \geq 0$ and all $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, we have*

$$\min_{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} v_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{v \in H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} v - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. We proceed as in [11, proof of Theorem 2], similarly as above in Corollary 3.2. Because the patch subdomain $\omega_{\boldsymbol{a}}$ is simply connected, the boundary $\partial\omega_{\boldsymbol{a}}$ is Lipschitz, and the boundary subset Γ is connected, the kernel of the curl operator in $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ is exactly $\boldsymbol{\nabla}(H^1_{0,\Gamma}(\omega_{\boldsymbol{a}}))$, so that

$$\min_{v \in H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} v - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{0}}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Similarly, at the discrete level, the equality

$$\min_{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})} \|\boldsymbol{\nabla} v_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = \min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times \boldsymbol{v}_p = \mathbf{0}}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}$$

holds true, see, e.g., [3, 4, 24]. Then the result follows from Theorem 3.3. □

Remark 3.5 (Converse inequalities)**.** *The converse inequalities to all the statements above trivially hold with constant one.*

Remark 3.6 (Unconstrained $L^2(\omega_{\boldsymbol{a}})$ and constrained $H^1(\omega_{\boldsymbol{a}})$ minimizations)**.** *In principle, we could consider two additional minimization problems with the considered spaces, namely (i) the unconstrained minimization in $L^2(\omega_{\boldsymbol{a}})$; and (ii) the constrained minimization in $H^1(\omega_{\boldsymbol{a}})$. However, these problems are trivial, since in both cases, the continuous and discrete minimizers are the same. Specifically, we have*

$$\min_{q_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \|q_p - r_p\|_{\omega_{\boldsymbol{a}}} = \min_{q \in L^2(\omega_{\boldsymbol{a}})} \|q - r_p\|_{\omega_{\boldsymbol{a}}} = 0$$

for all $r_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$, as well as

$$\min_{\substack{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_{0,\Gamma}(\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} v_p = \boldsymbol{g}_p}} \|v_p - \chi_p\|_{\omega_{\boldsymbol{a}}} = \min_{\substack{v \in H^1_{0,\Gamma}(\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} v = \boldsymbol{g}_p}} \|v - \chi_p\|_{\omega_{\boldsymbol{a}}}$$

for all $\chi_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{g}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ such that $\boldsymbol{\nabla}\times\boldsymbol{g}_p = \mathbf{0}$. We refer to [11, Section 3.3] *for some more considerations in this direction.*

Remark 3.7 (Stable broken polynomial extensions)**.** *The minimization problems considered above can be equivalently formulated as broken polynomial extensions as initially stated in* [5]*, where a discontinuous minimizer with prescribed jumps is sought for instead of a conforming one as above. The two formulations are actually equivalent up to a shift, as shown in* [23, Section 3.1] *or* [10, Lemma 6.8]*.*

4. Inhomogeneous boundary conditions

Proposition 3.1, Corollary 3.2, Theorem 3.3, and Corollary 3.4 are only stated for homogeneous boundary conditions on the Γ part of the boundary of $\omega_{\boldsymbol{a}}$. Supposing inhomogeneous boundary conditions that are suitable piecewise polynomials, these can be lifted to see that the above theory also covers this case. We now present the equivalent reformulations together with their proofs. In place of the boundary data, we rather start directly from the liftings, denoted as $\boldsymbol{\sigma}_p$ and σ_p below. These results are in practice particularly useful in the case of boundary patches, where the inhomogeneous boundary conditions of the patch problems stem from inhomogeneous Dirichlet, Neumann or (homogeneous) Robin boundary conditions of the original partial differential equation (cf. respectively the discussion in [23, Section 4] and in [7]). More precisely, in the applications, inhomogeneous boundary conditions only appear on the part of Γ corresponding to the faces sharing the vertex $\boldsymbol{a}$, which is of course covered by the presentation here.

We start with the $\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$-case of Proposition 3.1. For the datum $\boldsymbol{\sigma}_p$ given below, we say that $\boldsymbol{w}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}}$ on Γ if $\boldsymbol{w} - \boldsymbol{\sigma}_p \in \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$

Corollary 4.1 (Constrained minimization in $\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ with inhomogeneous boundary conditions). *For all $p \geq 0$, $\boldsymbol{\tau}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$, $\boldsymbol{\sigma}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$, and $r_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$ with the additional condition $(\boldsymbol{\sigma}_p{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}}, 1)_{\partial\omega_{\boldsymbol{a}}} = (r_p, 1)_{\omega_{\boldsymbol{a}}}$ if $\Gamma_{\boldsymbol{a}} = \emptyset$, we have*

$$\min_{\substack{\boldsymbol{w}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}{\cdot}\boldsymbol{w}_p = r_p \\ \boldsymbol{w}_p{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{w}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}{\cdot}\boldsymbol{w} = r_p \\ \boldsymbol{w}{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{w} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. We show the equivalence with Proposition 3.1, by a shift by the piecewise polynomial datum $\boldsymbol{\sigma}_p$. Indeed, suppose the setting of Corollary 4.1; the converse direction is similar. Let $\boldsymbol{w} = \boldsymbol{w}^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{w}^0 \in \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$ and $\boldsymbol{w}_p = \boldsymbol{w}_p^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{w}_p^0 \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$. Note that $\boldsymbol{\nabla}{\cdot}\boldsymbol{\sigma}_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$ satisfies $(r_p - \boldsymbol{\nabla}{\cdot}\boldsymbol{\sigma}_p, 1)_{\omega_{\boldsymbol{a}}} = 0$ if $\Gamma_{\boldsymbol{a}} = \emptyset$. Thus, setting $\tilde{r}_p := r_p - \boldsymbol{\nabla}{\cdot}\boldsymbol{\sigma}_p$ and $\tilde{\boldsymbol{\tau}}_p := \boldsymbol{\tau}_p - \boldsymbol{\sigma}_p$, we have $\tilde{r}_p \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap L^2_{0,\Gamma_{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ and $\tilde{\boldsymbol{\tau}}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$. This means that $\tilde{r}_p$ and $\tilde{\boldsymbol{\tau}}_p$ are eligible data for Theorem 3.3, which crucially lead to the same minimization values. □

The unconstrained $\boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$-case of Corollary 3.2 is actually easier, since there is no differential operator constraint. Similarly to the $\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ case, the notation $\boldsymbol{w} \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}}$ on Γ means that $\boldsymbol{w} - \boldsymbol{\sigma}_p \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$.

Corollary 4.2 (Unconstrained minimization in $\boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$ with inhomogeneous boundary conditions). *For all $p \geq 0$, all $\boldsymbol{\sigma}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$, and all $\boldsymbol{\tau}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$, we have*

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{v}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{\nabla}{\times}\boldsymbol{v}_p - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{v} \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{\nabla}{\times}\boldsymbol{v} - \boldsymbol{\tau}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. We show the equivalence with Corollary 3.2, by a shift by the piecewise polynomial datum $\boldsymbol{\sigma}_p$. Suppose the setting of Corollary 4.2; the converse direction is similar. Let $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{v}^0 \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ and $\boldsymbol{v}_p = \boldsymbol{v}_p^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{v}_p^0 \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$. Note that $\boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$ (actually also in $\boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}{\cdot}(\boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p) = 0$). Thus, setting $\tilde{\boldsymbol{\tau}}_p := \boldsymbol{\tau}_p - \boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p$, we have $\tilde{\boldsymbol{\tau}}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$, which is an eligible datum for Corollary 3.2, crucially leading to the same minimization values. □

The constrained $\boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$ case of Theorem 3.3 is similar to the situation of Corollary 4.1:

Corollary 4.3 (Constrained minimization in $\boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$ with inhomogeneous boundary conditions). *For all $p \geq 0$, $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, $\boldsymbol{\sigma}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$, and $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\boldsymbol{j}_p{\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p){\cdot}\boldsymbol{n}_{\omega_{\boldsymbol{a}}}$ on Γ and $\boldsymbol{\nabla}{\cdot}\boldsymbol{j}_p = 0$, we have*

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}{\times}\boldsymbol{v}_p = \boldsymbol{j}_p \\ \boldsymbol{v}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}{\times}\boldsymbol{v} = \boldsymbol{j}_p \\ \boldsymbol{v} \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{\sigma}_p \times \boldsymbol{n}_{\omega_{\boldsymbol{a}}} \text{ on } \Gamma}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. We show the equivalence with Theorem 3.3, again by a shift by the piecewise polynomial datum $\boldsymbol{\sigma}_p$. Suppose the setting of Corollary 4.3; the converse direction is similar. Let $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{v}^0 \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$ and $\boldsymbol{v}_p = \boldsymbol{v}_p^0 + \boldsymbol{\sigma}_p$ with $\boldsymbol{v}_p^0 \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})$. Note that $\boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}{\cdot}(\boldsymbol{\nabla}{\times}\boldsymbol{\sigma}_p) = 0$

and $(\boldsymbol{\nabla}\times\boldsymbol{\sigma}_p)\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = \boldsymbol{j}_p\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}}$ on Γ. Thus, setting $\tilde{\boldsymbol{j}}_p := \boldsymbol{j}_p - \boldsymbol{\nabla}\times\boldsymbol{\sigma}_p$ and $\tilde{\boldsymbol{\chi}}_p := \boldsymbol{\chi}_p - \boldsymbol{\sigma}_p$, we have $\tilde{\boldsymbol{j}}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{j}}_p = 0$ and $\tilde{\boldsymbol{\chi}}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$. This means that $\tilde{\boldsymbol{j}}_p$ and $\tilde{\boldsymbol{\chi}}_p$ are eligible data for Theorem 3.3, which crucially lead to the same minimization values. □

We now finally present how Corollary 3.4 covers inhomogeneous boundary conditions:

Corollary 4.4 (Unconstrained minimization in $H^1(\omega_{\boldsymbol{a}})$ with inhomogeneous boundary conditions)**.** *For all $p \geq 0$, $\sigma_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega_{\boldsymbol{a}})$, and all $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, we have*

$$\min_{\substack{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega_{\boldsymbol{a}}) \\ v_p = \sigma_p \text{ on } \Gamma}} \|\boldsymbol{\nabla} v_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{v \in H^1(\omega_{\boldsymbol{a}}) \\ v = \sigma_p \text{ on } \Gamma}} \|\boldsymbol{\nabla} v - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Proof. The proof passes through equivalence with Corollary 3.4. It is similar as above but again easier, since there is no differential operator constraint. Every $v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1(\omega_{\boldsymbol{a}})$ with $v_p = \sigma_p$ on Γ can be written as $v_p = v_p^0 + \sigma_p$, where $v_p^0 \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})$, and similarly for $v = v^0 + \sigma_p$ with $v^0 \in H^1_{0,\Gamma}(\omega_{\boldsymbol{a}})$. Then, $\tilde{\boldsymbol{\chi}}_p := \boldsymbol{\chi}_p - \boldsymbol{\nabla}\sigma_p$ lies in $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$ and forms an eligible datum for Corollary 3.4, which leads to the same minimization values. □

The remainder of this manuscript is dedicated to establishing Theorem 3.3.

5. Detailed notation and preliminary results

5.1. **Tangential traces.** Consider a tetrahedron K and $\mathcal{F} \subset \mathcal{F}_K$, a (sub)set of its faces. The definition of tangential traces of $\boldsymbol{H}(\mathbf{curl}, K)$ functions on the faces $F \in \mathcal{F}$ is a subtle matter. As we only work with piecewise polynomial traces, one way is to proceed with the liftings as in Corollaries 4.2 and 4.3. We rather proceed here following [8, 10], introducing the more general concept of a weak tangential trace, solely working with the boundary data (called $\boldsymbol{r}_p$ here) and not directly their Nédélec liftings ($\boldsymbol{\sigma}_p$ in Corollaries 4.2 and 4.3).

If $\boldsymbol{v} \in \boldsymbol{H}^1(K)$ and $F \in \mathcal{F}_K$, we denote by

$$\pi_F^{\boldsymbol{\tau}}(\boldsymbol{v}) := (\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}_F)\boldsymbol{n}_F)|_F \in \boldsymbol{L}^2(F) \tag{5.1}$$

its "usual" tangential trace on the face F (the orientation of $\boldsymbol{n}_F$ is not important here). We then define surface Nédélec spaces on faces as traces of volume Nédélec polynomials, setting

$$\boldsymbol{\mathcal{N}}_p(F) := \{\pi_F^{\boldsymbol{\tau}}(\boldsymbol{v}) \mid \boldsymbol{v} \in \boldsymbol{\mathcal{N}}_p(K)\},$$

and, if $\boldsymbol{w} \in \boldsymbol{\mathcal{N}}_p(F)$,

$$\mathrm{curl}_F\, \boldsymbol{w} := (\boldsymbol{\nabla}\times\boldsymbol{v})|_F\cdot\boldsymbol{n}_F,$$

where $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}_p(K)$ is such that $\boldsymbol{w} = \pi_F^{\boldsymbol{\tau}}(\boldsymbol{v})$ (the orientation of $\boldsymbol{n}_F$ counts here). One easily checks that for any face F (which is geometrically a triangle), these definitions are independent of the choice of the tetrahedron K such that $F \in \mathcal{F}_K$.

For a collection of faces $\mathcal{F} \subset \mathcal{F}_K$, we introduce

$$\boldsymbol{\mathcal{N}}_p(\mathcal{F}) := \prod_{F \in \mathcal{F}} \boldsymbol{\mathcal{N}}_p(F)$$

and

$$\boldsymbol{\mathcal{N}}_p(\Gamma_{\mathcal{F}}) := \{\boldsymbol{w} \in \boldsymbol{\mathcal{N}}_p(\mathcal{F}) \mid \exists \boldsymbol{v} \in \boldsymbol{\mathcal{N}}_p(K);\ \boldsymbol{w}|_F = \pi_F^{\boldsymbol{\tau}}(\boldsymbol{v})\ \forall F \in \mathcal{F}\}\,. \tag{5.2}$$

Notice that there is an induced tangential trace compatibility condition on each edge shared by faces of $\mathcal{F}$ in the definition of $\boldsymbol{\mathcal{N}}_p(\Gamma_{\mathcal{F}})$.

We then define a weak notion of tangential trace using integration by parts. Specifically, if $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, K)$ and $\boldsymbol{r}_p \in \boldsymbol{\mathcal{N}}_p(\Gamma_{\mathcal{F}})$ for some $p \geq 0$, the statement "$\boldsymbol{v}|_{\mathcal{F}}^{\boldsymbol{\tau}} = \boldsymbol{r}_p$" means that

$$(\boldsymbol{\nabla}\times\boldsymbol{v}, \boldsymbol{\phi})_K - (\boldsymbol{v}, \boldsymbol{\nabla}\times\boldsymbol{\phi})_K = \sum_{F\in\mathcal{F}} (\boldsymbol{r}_p, \boldsymbol{\phi} \times \boldsymbol{n}_K)_F \quad \forall \boldsymbol{\phi} \in \boldsymbol{H}^1_{\boldsymbol{\tau},\mathcal{F}^{\mathrm{c}}}(K), \tag{5.3}$$

where

$$\boldsymbol{H}^1_{\boldsymbol{\tau},\mathcal{F}^{\mathrm{c}}}(K) := \left\{\boldsymbol{w} \in \boldsymbol{H}^1(K) \mid \pi_F^{\boldsymbol{\tau}}(\boldsymbol{w}) = \boldsymbol{0} \quad \forall F \in \mathcal{F}^{\mathrm{c}} := \mathcal{F}_K \setminus \mathcal{F}\right\}.$$

Notice that when $\boldsymbol{v} \in \boldsymbol{H}^1(K)$, $\boldsymbol{v}|_{\mathcal{F}}^{\boldsymbol{\tau}} = \boldsymbol{r}_p$ if and only if $\pi_F^{\boldsymbol{\tau}}(\boldsymbol{v}) = \boldsymbol{r}_p|_F$ for all $F \in \mathcal{F}$.

Remark 5.1 (Compatibility of the weak definitions of tangential traces)**.** *Let $\Gamma_{\mathcal{F}}$ be the portion of the boundary of K corresponding to the faces in $\mathcal{F}$ and $\Gamma_{\mathcal{F}}^{\mathrm{c}}$ the corresponding complement (both open). We note that when $\boldsymbol{r}_p = \boldsymbol{0}$, the subspace of $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, K)$ such that $\boldsymbol{v}|_{\mathcal{F}}^{\boldsymbol{\tau}} = \boldsymbol{0}$ is identical with $\boldsymbol{H}_{0,\Gamma_{\mathcal{F}}}(\mathbf{curl}, K)$ from Section 2.3, where test functions $\boldsymbol{w} \in \boldsymbol{H}^1(K)$ such that $\boldsymbol{w} = \boldsymbol{0}$ on $\Gamma_{\mathcal{F}}^{\mathrm{c}}$ are used. Using test functions $\boldsymbol{w} \in \boldsymbol{H}^1(K)$ such that only $\pi_F^{\boldsymbol{\tau}}(\boldsymbol{w}) = \boldsymbol{0}$ on $\Gamma_{\mathcal{F}}^{\mathrm{c}}$ will be exploited below for the Piola transforms.*

5.2. **Piola mappings.** Consider two tetrahedra $K_{\mathrm{in}}, K_{\mathrm{out}} \in \mathcal{T}_{\boldsymbol{a}}$ and an invertible affine transformation $\psi : K_{\mathrm{in}} \to K_{\mathrm{out}}$. Such a transformation can be uniquely identified by specifying which vertex of K_{in} is mapped to which vertex of K_{out}. We denote by $\mathbb{J}$ the (constant) Jacobian matrix of ψ and we let $\varepsilon := \mathrm{sign}(\det \mathbb{J})$.

We associate with ψ two "Piola" mappings for vector-valued functions $\boldsymbol{v} : K_{\mathrm{in}} \to \mathbb{R}^3$ defined by

$$\psi^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{J}^{-T}\left(\boldsymbol{v} \circ \psi^{-1}\right), \qquad \psi^{\mathrm{d}}(\boldsymbol{v}) := (\det \mathbb{J})^{-1}\mathbb{J}\left(\boldsymbol{v} \circ \psi^{-1}\right). \tag{5.4}$$

These mappings commute with the curl operator in the sense that

$$\boldsymbol{\nabla}\times(\psi^{\mathrm{c}}(\boldsymbol{v})) = \psi^{\mathrm{d}}(\boldsymbol{\nabla}\times\boldsymbol{v}) \tag{5.5}$$

whenever $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, K_{\mathrm{in}})$. In addition, if $\boldsymbol{v}_{\mathrm{in}} \in \boldsymbol{H}(\mathbf{curl}, K_{\mathrm{in}})$ and $\boldsymbol{w}_{\mathrm{out}} \in \boldsymbol{H}(\mathbf{curl}, K_{\mathrm{out}})$ we have

$$(\psi^{\mathrm{c}}(\boldsymbol{v}_{\mathrm{in}}), \boldsymbol{\nabla}\times\boldsymbol{w}_{\mathrm{out}})_{K_{\mathrm{out}}} = \varepsilon(\boldsymbol{v}_{\mathrm{in}}, \boldsymbol{\nabla}\times\left((\psi^{\mathrm{c}})^{-1}(\boldsymbol{w}_{\mathrm{out}})\right))_{K_{\mathrm{in}}}. \tag{5.6}$$

Finally, we use the fact that the Piola mappings are stable in the sense that

$$\|\psi^{\mathrm{c}}(\boldsymbol{v})\|_{K_{\mathrm{out}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})\|\boldsymbol{v}\|_{K_{\mathrm{in}}} \quad \forall \boldsymbol{v} \in \boldsymbol{L}^2(K_{\mathrm{in}}). \tag{5.7}$$

We refer the reader to [21, Section 9] for an in-depth presentation of Piola mappings and proofs of the properties stated above.

5.3. **Stability in one tetrahedron.** We close this section with a simple extension of a result from [8, Theorem 2], corresponding to Theorem 3.3 (or more precisely Corollary 4.3) where the vertex patch $\mathcal{T}_{\boldsymbol{a}}$ is replaced by a single tetrahedron K.

Definition 5.2 (Compatible data in a tetrahedron)**.** *Let K be a tetrahedron. Consider a (sub)set $\mathcal{F} \subset \mathcal{F}_K$ of the faces of K. We say that $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(K)$ and $\boldsymbol{r}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{F})$ are compatible data if*

$$\nabla\cdot \boldsymbol{j}_p = 0, \tag{5.8a}$$

$$\boldsymbol{r}_p \in \boldsymbol{\mathcal{N}}_p(\Gamma_{\mathcal{F}}), \tag{5.8b}$$

$$\boldsymbol{j}_p\cdot\boldsymbol{n}_F = \operatorname{curl}_F(\boldsymbol{r}_p|_F) \quad \forall F \in \mathcal{F}. \tag{5.8c}$$

Lemma 5.3 (Stable minimization in a tetrahedron)**.** *Consider a tetrahedron K and a (sub)set $\mathcal{F}$ of its faces. For all $p \geq 0$, for all compatible data $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{r}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{F})$ as per Definition 5.2, and for all $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(K)$, we have*

$$\min_{\substack{\boldsymbol{v}_p\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_p=\boldsymbol{j}_p\\ \boldsymbol{v}_p|^{\boldsymbol{\tau}}_{\mathcal{F}}=\boldsymbol{r}_p}} \|\boldsymbol{v}_p-\boldsymbol{\chi}_p\|_K \leq C(\kappa_K) \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl},K)\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}_p\\ \boldsymbol{v}|^{\boldsymbol{\tau}}_{\mathcal{F}}=\boldsymbol{r}_p}} \|\boldsymbol{v}-\boldsymbol{\chi}_p\|_K.$$

Proof. The proof proceeds by a shift by $\boldsymbol{\chi}_p$, similarly to that of Corollary 4.2. Let us introduce $\widetilde{\boldsymbol{j}}_p := \boldsymbol{j}_p - \nabla\times\boldsymbol{\chi}_p$ and $\widetilde{\boldsymbol{r}}_p|_F := \boldsymbol{r}_p|_F - \pi^{\boldsymbol{\tau}}_F(\boldsymbol{\chi}_p)$ for all $F \in \mathcal{F}$. The new data $\widetilde{\boldsymbol{j}}_p$ and $\widetilde{\boldsymbol{r}}_p$ are compatible as per Definition 5.2, since $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(K)$. We now have from [8, Theorem 2] (which corresponds to Lemma 5.3 when $\boldsymbol{\chi}_p = \boldsymbol{0}$) that

$$\min_{\substack{\widetilde{\boldsymbol{v}}_p\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\widetilde{\boldsymbol{v}}_p=\widetilde{\boldsymbol{j}}_p\\ \widetilde{\boldsymbol{v}}_p|^{\boldsymbol{\tau}}_{\mathcal{F}}=\widetilde{\boldsymbol{r}}_p}} \|\widetilde{\boldsymbol{v}}_p\|_K \leq C(\kappa_K) \min_{\substack{\widetilde{\boldsymbol{v}}\in\boldsymbol{H}(\mathbf{curl},K)\\ \nabla\times\widetilde{\boldsymbol{v}}=\widetilde{\boldsymbol{j}}_p\\ \widetilde{\boldsymbol{v}}|^{\boldsymbol{\tau}}_{\mathcal{F}}=\widetilde{\boldsymbol{r}}_p}} \|\widetilde{\boldsymbol{v}}\|_K.$$

Denote respectively by $\boldsymbol{v}_p^\star, \widetilde{\boldsymbol{v}}_p^\star \in \boldsymbol{\mathcal{N}}_p(K)$ and $\boldsymbol{v}^\star, \widetilde{\boldsymbol{v}}^\star \in \boldsymbol{H}(\mathbf{curl}, K)$ the (unique) minimizers of the above left- and right-hand sides. Then the respective inequalities write as $\|\boldsymbol{v}_p^\star - \boldsymbol{\chi}_p\|_K \leq C(\kappa_K)\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_K$ and $\|\widetilde{\boldsymbol{v}}_p^\star\|_K \leq C(\kappa_K)\|\widetilde{\boldsymbol{v}}^\star\|_K$ and a shift by $\boldsymbol{\chi}_p$ shows that actually $\widetilde{\boldsymbol{v}}_p^\star = \boldsymbol{v}_p^\star - \boldsymbol{\chi}_p$ and $\widetilde{\boldsymbol{v}}^\star = \boldsymbol{v}^\star - \boldsymbol{\chi}_p$. □

6. Proof of Theorem 3.3 for interior patches

We first consider interior patches, i.e., the case where $\omega_{\boldsymbol{a}}$ contains an open ball around $\boldsymbol{a}$ (so that $\boldsymbol{a} \notin \partial\omega_{\boldsymbol{a}}$), where $\Gamma = \partial\omega_{\boldsymbol{a}}$ and $\Gamma_{\boldsymbol{a}} = \emptyset$.

We follow the approach introduced in [5] and extended in [23] and [10], so that our proof relies on an explicit construction of a discrete element $\boldsymbol{\xi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$ satisfying $\nabla\times\boldsymbol{\xi}_p = \boldsymbol{j}_p$ and

$$\|\boldsymbol{\xi}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}_p}} \|\boldsymbol{v}-\boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

To construct this element, we pass through the patch one tetrahedron at a time, following a suitable enumeration $K_1, K_2, \ldots, K_{|\mathcal{T}_{\boldsymbol{a}}|}$. At each step $1 \leq j \leq |\mathcal{T}_{\boldsymbol{a}}|$, $\boldsymbol{\xi}_p|_{K_j}$ is defined as the minimizer of an element-wise constrained minimization problem like in Lemma 5.3, with carefully chosen boundary data.

For this argument to function, we need a suitable enumeration of the tetrahedra of the patch, to pass in the right order. This is elaborated in Section 6.1. Then, we need to ensure that data we prescribe for the minimization problem in each element K_j are compatible as per Definition 5.2. It turns out that two arduous cases appear. First, the argument becomes subtle when K_j is the last element closing a loop around an edge e of the patch. Similarly, the last element $K_{|\mathcal{T}_{\boldsymbol{a}}|}$ of the patch must be carefully addressed. Section 6.2 and 6.3 provide intermediate results to treat these two cases.

6.1. **Enumeration of the elements in the patch.** For $K \in \mathcal{T}_{\boldsymbol{a}}$, we denote by $\mathcal{F}_K^{\rm int}$ the set of faces of K sharing the vertex $\boldsymbol{a}$. If e is an edge of the patch having $\boldsymbol{a}$ as a vertex, we denote by $\mathcal{T}_e \subset \mathcal{T}_{\boldsymbol{a}}$ the edge patch of elements sharing the edge e and by ω_e the associated open subdomain.

We call an enumeration of the patch $\mathcal{T}_{\boldsymbol{a}}$ an ordering of its elements $K_1, \dots, K_{|\mathcal{T}_{\boldsymbol{a}}|}$. For such an enumeration, for $1 \leq j \leq |\mathcal{T}_{\boldsymbol{a}}|$, we denote by $\mathcal{F}_j^\sharp \subset \mathcal{F}_{K_j}^{\rm int}$ the set of faces of K_j shared with an already enumerated element K_i with $i < j$, and we set $\mathcal{F}_j^\flat := \mathcal{F}_{K_j}^{\rm int} \setminus \mathcal{F}_j^\sharp$. The result in [23, Lemma B.1] provides us with a suitable enumeration featuring the key properties listed below.

Proposition 6.1 (Patch enumeration)**.** *There exists an enumeration $\{K_1, \dots, K_{|\mathcal{T}_{\boldsymbol{a}}|}\}$ of the vertex patch $\mathcal{T}_{\boldsymbol{a}}$ such that:*

(i) *For $1 < j \leq |\mathcal{T}_{\boldsymbol{a}}|$, if there are at least two faces in $\mathcal{F}_j^\sharp$ intersecting in an edge, then all the elements sharing this edge come sooner in the enumeration, i.e., if $|\mathcal{F}_j^\sharp| \geq 2$ with $F^1, F^2 \in \mathcal{F}_j^\sharp$, then letting $e := F^1 \cap F^2$, $K_i \in \mathcal{T}_e \setminus \{K_j\}$ implies that $i < j$.*

(ii) *For all $1 < j < |\mathcal{T}_{\boldsymbol{a}}|$, there are one or two neighbors of K_j which have been already enumerated and correspondingly two or one neighbors of K_j which have not been enumerated yet, i.e., $|\mathcal{F}_j^\sharp| \in \{1,2\}$ (so that $|\mathcal{F}_j^\flat| = 3 - |\mathcal{F}_j^\sharp| \in \{1,2\}$ as well) for all but the first and the last element. In particular, $\mathcal{F}_j^\sharp$ is empty if and only if $j = 1$ and $\mathcal{F}_j^\sharp$ contains all the interior faces of K_j (so that $\mathcal{F}_j^\flat$ is empty) if and only if $j = |\mathcal{T}_{\boldsymbol{a}}|$.*

6.2. **Two-color refinement of edge patches.** This section recalls the following useful result to deal with the last element of an edge patch of [23, Lemma B.2]:

Proposition 6.2 (Two-color refinement around edges)**.** *Fix a tetrahedron $K_j \in \mathcal{T}_{\boldsymbol{a}}$ and an edge e of K_j having $\boldsymbol{a}$ as one endpoint. Then there exists a conforming refinement $\mathcal{T}_e'$ of $\mathcal{T}_e$ composed of tetrahedra such that*

(i) *$\mathcal{T}_e'$ contains K_j.*

(ii) *All the tetrahedra in $\mathcal{T}_e'$ have e as an edge, and their two other vertices lie on $\partial\omega_{\boldsymbol{a}}$.*

(iii) *There holds $\kappa_{\mathcal{T}_e'} \leq 2\kappa_{\mathcal{T}_e}$.*

(iv) *Collecting all the vertices of* $\mathcal{T}'_e$ *that are not endpoints of* e *in the set* $\mathcal{V}'_e$*, there is a two-color map* $\mathtt{col} : \mathcal{V}'_e \to \{1, 2\}$ *so that for all* $\kappa \in \mathcal{T}'_e$*, the two vertices of* κ *that are not endpoints of* e*, say* $\{\boldsymbol{a}^n_\kappa\}_{1\le n\le 2}$*, satisfy* $\mathtt{col}(\boldsymbol{a}^n_\kappa) = n$*.*

Above, $\mathcal{T}'_e$ can be taken as $\mathcal{T}_e$ when the number of tetrahedra in $\mathcal{T}_e$ is pair. When the number of tetrahedra in $\mathcal{T}_e$ is impair, it is enough to cut one of the tetrahedra in $\mathcal{T}_e$, different from K_j, into two tetrahedra still sharing the edge e.

6.3. **Three-color refinement of vertex patches.** Here, we present the following technical result to address the last element of the vertex patch from [23, Lemma B.3]:

Proposition 6.3 (Three-color patch refinement)**.** *Fix a tetrahedron* $K_j \in \mathcal{T}_{\boldsymbol{a}}$*. There exists a conforming refinement* $\mathcal{T}'_{\boldsymbol{a}}$ *of* $\mathcal{T}_{\boldsymbol{a}}$ *composed of tetrahedra such that*

(i) $\mathcal{T}'_{\boldsymbol{a}}$ *contains* K_j*.*

(ii) *All the tetrahedra in* $\mathcal{T}'_{\boldsymbol{a}}$ *have* $\boldsymbol{a}$ *as a vertex, and their three other vertices lie on* $\partial\omega_{\boldsymbol{a}}$*.*

(iii) *There holds* $\kappa_{\mathcal{T}'_{\boldsymbol{a}}} \le C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})$*.*

(iv) *Collecting all the vertices of* $\mathcal{T}'_{\boldsymbol{a}}$ *distinct from* $\boldsymbol{a}$ *in the set* $\mathcal{V}'_{\boldsymbol{a}}$*, there is a three-color map* $\mathtt{col} : \mathcal{V}'_{\boldsymbol{a}} \to \{1, 2, 3\}$ *so that for all* $\kappa \in \mathcal{T}'_{\boldsymbol{a}}$*, the three vertices of* κ *distinct from* $\boldsymbol{a}$*, say* $\{\boldsymbol{a}^n_\kappa\}_{1\le n\le 3}$*, satisfy* $\mathtt{col}(\boldsymbol{a}^n_\kappa) = n$*.*

6.4. **Proof of Theorem 3.3.** We are now ready to prove Theorem 3.3 for interior patches.

Proof of Theorem 3.3 for interior patches. Denote by

$$\boldsymbol{v}^\star := \arg \min_{\substack{\boldsymbol{v}\in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl},\omega_{\boldsymbol{a}})\\ \boldsymbol{\nabla}\times\boldsymbol{v}=\boldsymbol{j}_p}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}$$

the continuous minimizer.

We rely on the enumeration K_j, $1 \le j \le |\mathcal{T}_{\boldsymbol{a}}|$, from Proposition 6.1. Following [5, 10, 23], we construct an admissible $\boldsymbol{\xi}_p$ from the discrete minimization set $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$ by sequential element-wise minimizations following this enumeration. Specifically, for each element K_j, $1 \le j \le |\mathcal{T}_{\boldsymbol{a}}|$, we define $F^{\rm ext}_j := \partial K_j \cap \partial\omega_{\boldsymbol{a}}$ and the set of faces $\mathcal{F}_j := \{F^{\rm ext}_j\} \cup \mathcal{F}^\sharp_j$ consisting of the face $F^{\rm ext}_j$ on the patch boundary and of the faces of K_j with neighbors that come sooner in the enumeration, with a smaller number. We also denote the local volume data by $\boldsymbol{j}^j_p := \boldsymbol{j}_p|_{K_j} \in \boldsymbol{\mathcal{RT}}_p(K_j)$ and $\boldsymbol{\chi}^j_p := \boldsymbol{\chi}_p|_{K_j} \in \boldsymbol{\mathcal{N}}_p(K_j)$. We will then iteratively define a boundary datum $\boldsymbol{r}^j_p \in \boldsymbol{\mathcal{N}}_p(\Gamma_{\mathcal{F}_j})$, leading to the definition

$$\boldsymbol{\xi}_p|_{K_j} := \boldsymbol{\xi}^j_p := \arg \min_{\substack{\boldsymbol{v}_p\in\boldsymbol{\mathcal{N}}_p(K_j)\\ \boldsymbol{\nabla}\times\boldsymbol{v}_p=\boldsymbol{j}^j_p\\ \boldsymbol{v}_p|^{\tau}_{\mathcal{F}_j}=\boldsymbol{r}^j_p}} \|\boldsymbol{v}_p - \boldsymbol{\chi}^j_p\|_{K_j}. \tag{6.1}$$

Our goal is then to show, by induction, that at each step j, the data $\boldsymbol{j}^j_p$ and $\boldsymbol{r}^j_p$ are admissible in the sense of Definition 5.2, so that problem (6.1) is well-posed, and that

$$\|\boldsymbol{\xi}^j_p - \boldsymbol{\chi}^j_p\|_{K_j} \le C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}. \tag{6.2}$$

Then, relying on the boundary data $\boldsymbol{r}_p^j$, we will establish that $\boldsymbol{\xi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$, showing that $\boldsymbol{\xi}_p$ belongs to the discrete minimization set in Theorem 3.3, which will conclude the proof since then

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_p = \boldsymbol{j}_p}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{\xi}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = \left\{ \sum_{j=1}^{|\mathcal{T}_{\boldsymbol{a}}|} \|\boldsymbol{\xi}_p^j - \boldsymbol{\chi}_p^j\|_{K_j}^2 \right\}^{\frac{1}{2}}$$

$$\leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) |\mathcal{T}_{\boldsymbol{a}}|^{\frac{1}{2}} \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) |\mathcal{T}_{\boldsymbol{a}}|^{\frac{1}{2}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v} = \boldsymbol{j}_p}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}},$$

and the number $|\mathcal{T}_{\boldsymbol{a}}|$ of tetrahedra in the patch $\mathcal{T}_{\boldsymbol{a}}$ is bounded by constant only depending on the shape-regularity parameter $\kappa_{\mathcal{T}_{\boldsymbol{a}}}$.

Step 1. We start by defining the boundary data $\boldsymbol{r}_p^j$ used in (6.1). We let (i) $\boldsymbol{r}_p^j|_{F_j^{\mathrm{ext}}} := \mathbf{0}$ on the external face F_j^{ext}; and (ii) on each face $F_{i,j} \in \mathcal{F}_j^\sharp$ shared by K_j and K_i, $i < j$, we set $\boldsymbol{r}_p^j|_{F_{i,j}} := \pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i)$, which we can do since $\boldsymbol{\xi}_p^i$ is already defined on the simplices K_i with a smaller index i.

Step 2. We now verify that the data constructed in Step 1 are admissible as per Definition 5.2, so that problem (6.1) is well-posed. Notice that since $\boldsymbol{\nabla} \cdot \boldsymbol{j}_p^j = \boldsymbol{\nabla} \cdot (\boldsymbol{j}|_{K_j}) = 0$, (5.8a) is satisfied by construction. Considering (5.8c), we have $\mathcal{F}_j = \{F_j^{\mathrm{ext}}\} \cup \mathcal{F}_j^\sharp$. For the exterior face F_j^{ext}, the associated data $\boldsymbol{r}_p^j|_{F_j^{\mathrm{ext}}} = \mathbf{0}$ always vanishes, and $\boldsymbol{j}_p \cdot \boldsymbol{n}_{F_j^{\mathrm{ext}}} = \mathrm{curl}_{F_j^{\mathrm{ext}}} \boldsymbol{r}_p^j|_{F_j^{\mathrm{ext}}} = 0$ holds true since $\boldsymbol{j}_p \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \cap \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$ by assumption. According to the enumeration from Proposition 6.1, $\mathcal{F}_j^\sharp$ is empty on the first element K_1, so there is nothing more to verify for (5.8c) when $j = 1$. On the other hand, when $j > 1$, the remaining faces in $\mathcal{F}_j^\sharp$ are of the form $F_{i,j} = \partial K_i \cap \partial K_j$, where K_i has been previously visited, $i < j$. We then have

$$\mathrm{curl}_{F_{i,j}}(\boldsymbol{r}_p^j|_{F_{i,j}}) = \mathrm{curl}_{F_{i,j}}(\pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i)) = (\boldsymbol{\nabla} \times \boldsymbol{\xi}_p^i)|_{F_{i,j}} \cdot \boldsymbol{n}_{F_{i,j}} = \boldsymbol{j}_p^i \cdot \boldsymbol{n}_{F_{i,j}} = \boldsymbol{j}_p^j \cdot \boldsymbol{n}_{F_{i,j}}$$

since $\boldsymbol{j}_p \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}}) \cap \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})$ and since $\boldsymbol{\nabla} \times \boldsymbol{\xi}_p^i = \boldsymbol{j}_p^i$ on K_i by induction. As a result, we are left to check (5.8b). To do so, following (5.2), we need to find $\boldsymbol{R}_p^j \in \boldsymbol{\mathcal{N}}_p(K_j)$ such that $\boldsymbol{r}_p^j|_F = \pi_F^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j)$ for all faces $F \in \mathcal{F}_j$. We distinguish 4 subcases for this purpose.

Step 2a. In the first element K_1, we have $\mathcal{F}_1 = \{F_1^{\mathrm{ext}}\}$ and $\boldsymbol{r}_p^1|_{F_1^{\mathrm{ext}}} = \mathbf{0}$. It is clear that $\boldsymbol{r}_p^1|_{F_1^{\mathrm{ext}}} = \pi_{F_1^{\mathrm{ext}}}^{\boldsymbol{\tau}}(\mathbf{0})$ which shows (5.8b) for $\boldsymbol{R}_p^1 := \mathbf{0}$.

Step 2b. We then consider the case where the element K_j, $1 < j < |\mathcal{T}_{\boldsymbol{a}}|$, is such that $|\mathcal{F}_j^\sharp| = 1$, i.e., there is a single element K_i with $i < j$ such that $\mathcal{F}_j^\sharp = \{F_{i,j}\}$, $F_{i,j} = \partial K_i \cap \partial K_j$. There exists a unique affine mapping $\psi_{i,j} : K_i \to K_j$ that leaves the face $F_{i,j}$ invariant, and we set $\boldsymbol{R}_p^j := \psi_{i,j}^{\mathrm{c}}(\boldsymbol{\xi}_p^i) \in \boldsymbol{\mathcal{N}}_p(K_j)$. Since the Piola mapping preserves tangential traces, maps F_i^{ext} onto F_j^{ext}, and leaves $F_{i,j}$ invariant, we clearly have $\pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i) = \boldsymbol{r}_p^j|_{F_{i,j}}$ and $\pi_{F_j^{\mathrm{ext}}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \mathbf{0}$ since $\pi_{F_i^{\mathrm{ext}}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i) = \mathbf{0}$. This shows that $\boldsymbol{r}_p^j|_F = \pi_F^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j)$ for all $F \in \mathcal{F}_j$, so that (5.8b) is satisfied in view of definition (5.2).

Step 2c. The next case is an element K_j with $1 < j < |\mathcal{T}_{\boldsymbol{a}}|$ such that $|\mathcal{F}_j^\sharp| = 2$. We will use an argument similar to the one above in Step 2b, relying this time on Piola mappings from all tetrahedra sharing the edge e common to the two faces in $\mathcal{F}_j^\sharp$. First, since $|\mathcal{F}_j^\sharp| = 2$, Proposition 6.1 implies that K_j is the lastly enumerated element of the edge patch $\mathcal{T}_e$. Invoking Proposition 6.2, there is a refined edge patch $\mathcal{T}_e'$ such that $\mathcal{T}_e' = \mathcal{T}_e$ if the number of tetrahedra in $\mathcal{T}_e$ is pair, whereas one of the tetrahedra in $\mathcal{T}_e$, different from K_j, has been cut into two if $|\mathcal{T}_e|$ is impair. In any case, $\mathcal{T}_e' = \{\kappa_1, \ldots, \kappa_n\}$, $\kappa_n = K_j$, and the vertices of $\mathcal{T}_e'$ that are not endpoints of the edge e are colored by two colors (alternating along the numbering $1, \ldots, n$).

Let $\psi_{\ell,n} : \kappa_\ell \to \kappa_n$ be the unique invertible affine mapping of the tetrahedron κ_ℓ to the tetrahedron κ_n preserving the two endpoints of the edge e and the colors of the two other vertices; this in particular means that the faces $F_\ell^{\rm ext}$ are mapped to $F_j^{\rm ext}$, the two faces in $\mathcal{F}_j^\sharp$ are left invariant, and the other faces sharing the edge e have their remaining vertex mapped while preserving its color. Denote by $\varepsilon_{\ell,n}$ the sign of determinant of the Jacobian of $\psi_{\ell,n}$. Let finally, for $1 \le \ell \le n-1$, $\iota(\ell)$ be the index of the element $K_{\iota(\ell)} \in \mathcal{T}_e$ such that $\kappa_\ell \subset K_{\iota(\ell)}$ (if the number of tetrahedra in $\mathcal{T}_e$ is pair, we can actually always write $\kappa_\ell = K_{\iota(\ell)}$; if not, a strict inclusion only holds on the two subsimplices of the simplex that has been cut). This allows for the following "folding" Piola mappings definition:

$$\boldsymbol{R}_p^j := -\sum_{\ell=1}^{n-1} \varepsilon_{\ell,n} \psi_{\ell,n}^{\rm c}(\boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}) \in \boldsymbol{\mathcal{N}}_p(K_j). \tag{6.3}$$

As K_j is the last element of the edge patch $\mathcal{T}_e$, for all $1 \le \ell \le n-1$, $\boldsymbol{\xi}_p^{\iota(\ell)}$ have been previously defined, and this in such a way that (i) their tangential traces vanish on $\partial\omega_{\boldsymbol{a}}$; and (ii) their tangential traces match on faces shared by two previously enumerated elements. Now, since the faces in $F_\ell^{\rm ext} \subset \partial\omega_{\boldsymbol{a}}$ are mapped to $F_j^{\rm ext}$, $\pi_{F_j^{\rm ext}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \boldsymbol{0}$ follows from $\pi_{F_\ell^{\rm ext}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}) = \boldsymbol{0}$. Similarly, all the faces sharing the edge e other than the two faces in $\mathcal{F}_j^\sharp$ are mapped twice, with two opposite signs in view of $\varepsilon_{\ell,n}$ (indeed, $\varepsilon_{\ell_-,n} + \varepsilon_{\ell_+,n} = 0$ if the two elements κ_{ℓ_-} and κ_{ℓ_+} from $\mathcal{T}_e'$ share a common face), leaving only the contributions from the neighbors from the two faces in $\mathcal{F}_j^\sharp$. Thus $\pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i) = \boldsymbol{r}_p^j|_{F_{i,j}}$ for the (two) faces $F_{i,j} \in \mathcal{F}_j^\sharp$, and (5.8b) is satisfied.

Step 2d. We finish with the last element K_j, $j = |\mathcal{T}_{\boldsymbol{a}}|$. In this case we have $|\mathcal{F}_j^\sharp| = 3$. In extension of Step 2c, we rely on Piola mappings from all the tetrahedra of the patch $\mathcal{T}_{\boldsymbol{a}}$ other than K_j. Following Proposition 6.3, we invoke for this purpose a three-color patch refinement $\mathcal{T}_{\boldsymbol{a}}'$ such that $\mathcal{T}_{\boldsymbol{a}}' = \{\kappa_1, \ldots, \kappa_n\}$, $\kappa_n = K_j$.

Let $\psi_{\ell,n} : \kappa_\ell \to \kappa_n$ be the unique invertible affine mapping of the tetrahedron κ_ℓ to the tetrahedron κ_n preserving the vertex $\boldsymbol{a}$ and the colors of the three other vertices; this in particular means that the faces $F_\ell^{\rm ext}$ are mapped to $F_j^{\rm ext}$ and the other faces have their vertices mapped while preserving their color. Denote by $\varepsilon_{\ell,n}$ the sign of determinant of the Jacobian of $\psi_{\ell,n}$. Let finally, for $1 \le \ell \le n-1$, $\iota(\ell)$ be the index of the element $K_{\iota(\ell)} \in \mathcal{T}_{\boldsymbol{a}}$

such that $\kappa_\ell \subset K_{\iota(\ell)}$. This allows for the following "folding" Piola mappings definition:

$$\boldsymbol{R}_p^j := -\sum_{\ell=1}^{n-1} \varepsilon_{\ell,n} \psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}) \in \boldsymbol{\mathcal{N}}_p(K_j). \tag{6.4}$$

As above in Step 2c, we observe that (i) all $\boldsymbol{\xi}_p^{\iota(\ell)}$ have been previously defined and have a zero/matching tangential trace; (ii) each boundary face of $\mathcal{T}_{\boldsymbol{a}}'$ (except of F_j^{ext}) is mapped to F_j^{ext}; (iii) each interior face of $\mathcal{T}_{\boldsymbol{a}}'$ other than the three faces from $\mathcal{F}_j^\sharp$ is mapped twice, each time with an opposite sign; and (iv) the three faces from $\mathcal{F}_j^\sharp$ are only mapped once. This yields $\pi_{F_j^{\mathrm{ext}}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \boldsymbol{0}$ together with $\pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{R}_p^j) = \pi_{F_{i,j}}^{\boldsymbol{\tau}}(\boldsymbol{\xi}_p^i) = \boldsymbol{r}_p^j|_{F_{i,j}}$ for the (three) faces $F_{i,j} \in \mathcal{F}_j^\sharp$, so that (5.8b) follows.

Step 3. We now show (6.2), that is, at each step $1 \leq j \leq |\mathcal{T}_{\boldsymbol{a}}|$, the element $\boldsymbol{\xi}_p^j$ given by (6.1) is stable as compared to the continuous minimizer $\boldsymbol{v}^\star$. Let

$$\boldsymbol{V}(K_j) := \left\{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, K_j) \mid \boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{j}_p^j,\ \boldsymbol{v}|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{r}_p^j\right\}. \tag{6.5}$$

From Step 2, we know that this set is nonempty. To show (6.2), we will construct for every $1 \leq j \leq |\mathcal{T}_{\boldsymbol{a}}|$ an element $\boldsymbol{w}_j^\star \in \boldsymbol{V}(K_j)$ such that

$$\|\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j\|_{K_j} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}. \tag{6.6}$$

Estimate (6.2) then follows from Lemma 5.3 since

$$\begin{aligned}\|\boldsymbol{\xi}_p^j - \boldsymbol{\chi}_p^j\|_{K_j} &= \min_{\boldsymbol{w}_p \in \boldsymbol{V}(K_j)\cap\boldsymbol{\mathcal{N}}_p(K_j)} \|\boldsymbol{w}_p - \boldsymbol{\chi}_p^j\|_{K_j} \\ &\leq C(\kappa_{K_j}) \min_{\boldsymbol{w}\in\boldsymbol{V}(K_j)} \|\boldsymbol{w} - \boldsymbol{\chi}_p^j\|_{K_j} \leq C(\kappa_{K_j}) \|\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j\|_{K_j}.\end{aligned}$$

Step 3a. In the first element K_1, we actually readily observe that $\boldsymbol{w}_j^\star := \boldsymbol{v}^\star|_{K_1}$ belongs to the minimization set $\boldsymbol{V}(K_1)$, so that (6.6) is immediately satisfied with the constant $C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) = 1$.

Step 3b. We next consider those elements K_j, $1 < j < |\mathcal{T}_{\boldsymbol{a}}|$, for which $|\mathcal{F}_j^\sharp| = 1$, and we denote by $1 \leq i < j$ the index such that $\mathcal{F}_j^\sharp = \{F_{i,j}\}$ with $F_{i,j} = \partial K_i \cap \partial K_j$. As in Step 2b, we consider the affine map $\psi_{i,j} : K_i \to K_j$ that leaves the face $F_{i,j}$ invariant, and we set

$$\boldsymbol{w}_j^\star := \boldsymbol{v}^\star|_{K_j} - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i} - \boldsymbol{\xi}_p^i). \tag{6.7}$$

We now show that $\boldsymbol{w}_j^\star$ belongs to $\boldsymbol{V}(K_j)$ given by (6.5). First, by the Piola mapping, $\boldsymbol{w}_j^\star \in \boldsymbol{H}(\mathbf{curl}, K_j)$. Moreover, recalling (5.5), it is clear that

$$\boldsymbol{\nabla}\times\boldsymbol{w}_j^\star = \boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{K_j}) - \psi_{ij}^{\mathrm{d}}(\boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{K_i} - \boldsymbol{\xi}_p^i)) = \boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{K_j}) = \boldsymbol{j}_p^j.$$

Finally, roughly speaking, the fact that $\boldsymbol{w}_j^\star|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{r}_p^j$ follows from (6.7) since all $\boldsymbol{v}^\star|_{K_j}$, $\boldsymbol{v}^\star|_{K_i}$, and $\boldsymbol{\xi}_p^i$ have a zero tangential trace on $\partial\omega_{\boldsymbol{a}}$ and the tangential trace of $\boldsymbol{v}^\star$ is continuous across $F_{i,j}$, so that its contribution vanishes in $\boldsymbol{w}_j^\star$ and only the desired contribution from $\boldsymbol{\xi}_p^i$ is left. However, in contrast to Step 2b carried out for piecewise polynomials, we cannot rigorously prove this in this strong sense because we cannot easily localize the notion of tangential trace to one face for $\boldsymbol{H}(\mathbf{curl})$ functions. As a result, we have to resort to the weak notion of tangential trace introduced in (5.3). For this purpose, we first note that, following Step 2b,

$$\boldsymbol{w}_j^\star = \boldsymbol{v}^\star|_{K_j} - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i}) + \boldsymbol{R}_p^j,$$

with $\boldsymbol{R}_p^j|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{r}_p^j$. Thus, we need to show that $(\boldsymbol{v}^\star|_{K_j} - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i}))|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{0}$. Recall that $\mathcal{F}_j = \{F_j^{\mathrm{ext}}\} \cup \mathcal{F}_j^\sharp = \{F_j^{\mathrm{ext}}, F_{i,j}\}$. Following (5.3), let $\boldsymbol{\phi} \in \boldsymbol{H}^1_{\boldsymbol{\tau},\mathcal{F}_j^{\mathrm{c}}}(K_j)$. Letting $\psi_{j,i} = (\psi_{i,j})^{-1}$, the function

$$\widetilde{\boldsymbol{\phi}}|_{K_j} := \boldsymbol{\phi}, \qquad \widetilde{\boldsymbol{\phi}}|_{K_i} := \psi_{j,i}^{\mathrm{c}}(\boldsymbol{\phi}) \tag{6.8}$$

belongs to $\boldsymbol{H}_{0,\partial(K_i\cup K_j)\setminus\partial\omega_{\boldsymbol{a}}}(\mathbf{curl}, K_i \cup K_j)$. Then, noticing that the sign of the determinant of the Jacobian of $\psi_{i,j}$ is negative, (5.6) allows us to write

$$\begin{aligned}
&(\boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{K_j} - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i})), \boldsymbol{\phi})_{K_j} - (\boldsymbol{v}^\star|_{K_j} - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i}), \boldsymbol{\nabla}\times\boldsymbol{\phi})_{K_j} \\
&= (\boldsymbol{\nabla}\times\boldsymbol{v}^\star, \boldsymbol{\phi})_{K_j} - (\boldsymbol{v}^\star, \boldsymbol{\nabla}\times\boldsymbol{\phi})_{K_j} + (\boldsymbol{\nabla}\times\boldsymbol{v}^\star, \psi_{j,i}^{\mathrm{c}}(\boldsymbol{\phi}))_{K_i} - (\boldsymbol{v}^\star, \boldsymbol{\nabla}\times\psi_{j,i}^{\mathrm{c}}(\boldsymbol{\phi}))_{K_i} \\
&= (\boldsymbol{\nabla}\times\boldsymbol{v}^\star, \widetilde{\boldsymbol{\phi}})_{K_i\cup K_j} - (\boldsymbol{v}^\star, \boldsymbol{\nabla}\times\widetilde{\boldsymbol{\phi}})_{K_i\cup K_j} = 0,
\end{aligned}$$

since $\boldsymbol{v}^\star|_{K_i\cup K_j} \in \boldsymbol{H}_{0,\partial(K_i\cup K_j)\cap\partial\omega_{\boldsymbol{a}}}(\mathbf{curl}, K_i \cup K_j)$.

Finally, we have

$$\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j = (\boldsymbol{v}^\star|_{K_j} - \boldsymbol{\chi}_p^j) - \psi_{i,j}^{\mathrm{c}}(\boldsymbol{v}^\star|_{K_i} - \boldsymbol{\chi}_p^i) + \psi_{i,j}^{\mathrm{c}}(\boldsymbol{\xi}_p^i - \boldsymbol{\chi}_p^i),$$

and recalling (5.7), it follows that

$$\|\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j\|_{K_j} \le \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p^j\|_{K_j} + C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})\big(\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p^i\|_{K_i} + \|\boldsymbol{\xi}_p^i - \boldsymbol{\chi}_p^i\|_{K_i}\big) \le C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}},$$

since (6.2) holds in K_i by induction. Thus (6.6) holds.

Step 3c. The next situation is the case of an element K_j, $1 < j < |\mathcal{T}_{\boldsymbol{a}}|$, with $|\mathcal{F}_j^\sharp| = 2$. We keep the notation of Step 2c for the two-color refinement $\mathcal{T}_e'$ of the patch around the edge e and the associated affine mappings $\psi_{\ell,n}$. We set, in extension of (6.7) from the previous Step 3b and following the recipe (6.3),

$$\boldsymbol{w}_j^\star := \boldsymbol{v}^\star|_{K_j} + \sum_{\ell=1}^{n-1} \varepsilon_{\ell,n}\psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{v}^\star|_{\kappa_\ell} - \boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}). \tag{6.9}$$

We now again show that $\boldsymbol{w}_j^\star \in \boldsymbol{V}(K_j)$ given by (6.5). First, by the Piola mappings, $\boldsymbol{w}_j^\star \in \boldsymbol{H}(\mathbf{curl}, K_j)$. Moreover, since $\boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{K_j}) = \boldsymbol{j}_p^j$ and $\boldsymbol{\nabla}\times(\boldsymbol{v}^\star|_{\kappa_\ell}) = \boldsymbol{\nabla}\times(\boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}) = \boldsymbol{j}_p^{\iota(\ell)}|_{\kappa_\ell}$, it is clear that $\boldsymbol{w}_j^\star$ satisfies the curl constraint of $\boldsymbol{V}(K_j)$, $\boldsymbol{\nabla}\times\boldsymbol{w}_j^\star = \boldsymbol{j}_p^j$. We then turn to the trace constraint $\boldsymbol{w}_j^\star|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{r}_p^j$. We rewrite (6.9), using (6.3), as

$$\boldsymbol{w}_j^\star = \sum_{\ell=1}^{n} \varepsilon_{\ell,n}\psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{v}^\star|_{\kappa_\ell}) - \sum_{\ell=1}^{n-1} \varepsilon_{\ell,n}\psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}) = \sum_{\ell=1}^{n} \varepsilon_{\ell,n}\psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{v}^\star|_{\kappa_\ell}) + \boldsymbol{R}_p^j,$$

with $\psi_{n,n}^{\mathrm{c}}$ identity and $\varepsilon_{n,n} = 1$. Since $\boldsymbol{R}_p^j|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{r}_p^j$ from Step 2c, we merely need to show that $\big(\sum_{\ell=1}^{n} \varepsilon_{\ell,n}\psi_{\ell,n}^{\mathrm{c}}(\boldsymbol{v}^\star|_{\kappa_\ell})\big)|_{\mathcal{F}_j}^{\boldsymbol{\tau}} = \boldsymbol{0}$. Intuitively, this is rather clear; following the reasoning of Step 2c, (i) all the faces F_ℓ^{ext} are mapped to F_j^{ext}, yielding a zero tangential trace; (ii) all the faces sharing the edge e, including the two faces in $\mathcal{F}_j^\sharp$, are mapped twice with two opposite signs, yielding a zero tangential trace. To show this rigorously, we again rely on the characterization (5.3). Recalling that $\mathcal{F}_j = \{F_j^{\rm ext}\} \cup \mathcal{F}_j^\sharp$, consider thus $\boldsymbol{\phi} \in \boldsymbol{H}^1_{\boldsymbol{\tau},\mathcal{F}_j^{\rm c}}(K_j)$. In extension of (6.8), let us define

$$\widetilde{\boldsymbol{\phi}}|_{\kappa_\ell} := \psi^{\rm c}_{n,\ell}(\boldsymbol{\phi}) \qquad 1 \leq \ell \leq n. \tag{6.10}$$

By the two-coloring of Proposition 6.2, as in Step 2c, this "unfolding" of $\boldsymbol{\phi}$ gives $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{H}_{0,\partial\omega_e\setminus\partial\omega_{\boldsymbol{a}}}(\mathbf{curl},\omega_e)$. Then, using (5.6), we have

$$\begin{aligned}
&\left(\boldsymbol{\nabla}\times\left(\sum_{\ell=1}^{n}\varepsilon_{\ell,n}\psi^{\rm c}_{\ell,n}(\boldsymbol{v}^\star|_{\kappa_\ell})\right),\boldsymbol{\phi}\right)_{K_j} - \left(\sum_{\ell=1}^{n}\varepsilon_{\ell,n}\psi^{\rm c}_{\ell,n}(\boldsymbol{v}^\star|_{\kappa_\ell}),\boldsymbol{\nabla}\times\boldsymbol{\phi}\right)_{K_j}\\
&= \sum_{\ell=1}^{n}\left\{(\boldsymbol{\nabla}\times\boldsymbol{v}^\star,\psi^{\rm c}_{n,\ell}(\boldsymbol{\phi}))_{\kappa_\ell} - (\boldsymbol{v}^\star,\boldsymbol{\nabla}\times(\psi^{\rm c}_{n,\ell}(\boldsymbol{\phi})))_{\kappa_\ell}\right\}\\
&= (\boldsymbol{\nabla}\times\boldsymbol{v}^\star,\widetilde{\boldsymbol{\phi}})_{\omega_e} - (\boldsymbol{v}^\star,\boldsymbol{\nabla}\times\widetilde{\boldsymbol{\phi}})_{\omega_e} = 0,
\end{aligned}$$

since $\boldsymbol{v}^\star|_{\omega_e} \in \boldsymbol{H}_{0,\partial\omega_e\cap\partial\omega_{\boldsymbol{a}}}(\mathbf{curl},\omega_e)$, so that indeed $\boldsymbol{w}_j^\star \in \boldsymbol{V}(K_j)$.

We conclude this step by showing that (6.6) holds true. First, we write that

$$\begin{aligned}
\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j = \boldsymbol{v}^\star|_{K_j} - \boldsymbol{\chi}_p^j &+ \sum_{\ell-1}^{n-1}\varepsilon_{\ell,n}\psi^{\rm c}_{\ell,n}(\boldsymbol{v}^\star|_{\kappa_\ell} - \boldsymbol{\chi}_p^{\iota(\ell)}|_{\kappa_\ell})\\
&+ \sum_{\ell-1}^{n-1}\varepsilon_{\ell,n}\psi^{\rm c}_{\ell,n}(\boldsymbol{\chi}_p^{\iota(\ell)}|_{\kappa_\ell} - \boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell}),
\end{aligned}$$

and it follows that, recalling (5.7),

$$\begin{aligned}
\|\boldsymbol{w}_j^\star - \boldsymbol{\chi}_p^j\|_{K_j} &\leq \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p^j\|_{K_j} + \sum_{\ell-1}^{n-1}\|\psi^{\rm c}_{\ell,n}(\boldsymbol{v}^\star|_{\kappa_\ell} - \boldsymbol{\chi}_p^{\iota(\ell)}|_{\kappa_\ell})\|_{K_j} + \sum_{\ell-1}^{n-1}\|\psi^{\rm c}_{\ell,n}(\boldsymbol{\chi}_p^{\iota(\ell)}|_{\kappa_\ell} - \boldsymbol{\xi}_p^{\iota(\ell)}|_{\kappa_\ell})\|_{K_j}\\
&\leq \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p^j\|_{K_j} + C(\kappa_{\mathcal{T}_e'})\left(\sum_{\ell-1}^{n-1}\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p^{\iota(\ell)}\|_{\kappa_\ell} + \sum_{\ell-1}^{n-1}\|\boldsymbol{\chi}_p^{\iota(\ell)} - \boldsymbol{\xi}_p^{\iota(\ell)}\|_{\kappa_\ell}\right)\\
&\leq C(\kappa_{\mathcal{T}_e'})\left(\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} + \sum_{\ell-1}^{n-1}\|\boldsymbol{\chi}_p^{\iota(\ell)} - \boldsymbol{\xi}_p^{\iota(\ell)}\|_{K_{\iota(\ell)}}\right).
\end{aligned}$$

Then (6.6) follows since $\kappa_{\mathcal{T}_e'} \leq 2\kappa_{\mathcal{T}_e} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})$ and, by induction, (6.2) holds for for $i = \iota(\ell) < j$, $1 \leq \ell \leq n-1$.

Step 3d. The proof for the last element is analogous to that of Step 3c. We in particular still rely on (6.9) and (6.10) where, this time, the three-color patch refinement $\mathcal{T}_{\boldsymbol{a}}' = \{\kappa_1,\ldots,\kappa_n\}$, $\kappa_n = K_j$, of Proposition 6.3 is employed. Here, $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{H}(\mathbf{curl},\omega_{\boldsymbol{a}})$, whereas, since $\Gamma = \partial\omega_{\boldsymbol{a}}$ for the considered interior patch case, $\boldsymbol{v}^\star \in \boldsymbol{H}_{0,\partial\omega_{\boldsymbol{a}}}(\mathbf{curl},\omega_{\boldsymbol{a}})$ by definition.

- **Step 4.** We finally define $\boldsymbol{\xi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$ by setting $\boldsymbol{\xi}_p|_{K_j} := \boldsymbol{\xi}_p^j$ for $1 \leq j \leq |\mathcal{T}_{\boldsymbol{a}}|$ and verify that $\boldsymbol{\xi}_p \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$. By construction, the tangential trace of each $\boldsymbol{\xi}_p^j$ vanishes on F_j^{ext}, and if $F_{i,j}$ is the face shared by two tetrahedra K_i and K_j, the tangential traces of $\boldsymbol{\xi}_p^i$ and $\boldsymbol{\xi}_p^j$ match on $F_{i,j}$. It follows that $\boldsymbol{\xi}_p \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$. Since, by construction, $\boldsymbol{\nabla}\times(\boldsymbol{\xi}_p|_{K_j}) = \boldsymbol{\nabla}\times\boldsymbol{\xi}_p^j = \boldsymbol{j}_p^j = \boldsymbol{j}_p|_{K_j}$ for all $K_j \in \mathcal{T}_{\boldsymbol{a}}$, this means that that $\boldsymbol{\nabla}\times\boldsymbol{\xi}_p = \boldsymbol{j}_p$ globally in $\omega_{\boldsymbol{a}}$, which concludes the proof. □

7. Proof of Theorem 3.3 for boundary patches

In this section, we study the boundary patches by extending the approach of [23, Section 7]. We will more precisely be led to consider a boundary patch $\mathcal{T}_{\boldsymbol{a}}$ around the vertex $\boldsymbol{a}$, the corresponding open domain $\omega_{\boldsymbol{a}}$, and $\Gamma = \partial\omega_{\boldsymbol{a}} \setminus \overline{\Gamma_{\boldsymbol{a}}}$. Recall from Section 2.1 that we consider two situations: either $\Gamma_{\boldsymbol{a}} = \emptyset$ (so that $\Gamma = \partial\omega_{\boldsymbol{a}}$, see Figure 1, right), or $\Gamma_{\boldsymbol{a}}$ corresponds to all faces $F \in \mathcal{F}_{\boldsymbol{a}}$ lying on the boundary of $\omega_{\boldsymbol{a}}$ and sharing the vertex $\boldsymbol{a}$ (see Figure 2, left). The case of Figure 2, right, will be investigated separately in Section 8 below. We will additionally work with different patches obtained by geometrical mappings, some of those will be boundary and some interior in the terminology of Section 2.1.

Let $\mathcal{T}_{\boldsymbol{a}}$ be a vertex patch in the sense of Section 2.1, interior or boundary. For a given $p \geq 0$ and $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$ and $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\text{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot\boldsymbol{j}_p = 0$, let

$$\boldsymbol{v}_p^\star := \arg \min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v}_p = \boldsymbol{j}_p}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}, \quad \boldsymbol{v}^\star := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{j}_p}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \tag{7.1}$$

be respectively the discrete and continuous minimizers from Theorem 3.3. We will consider here the best uniform constant $C_{\text{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma}$ in the inequality

$$\|\boldsymbol{v}_p^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\text{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma} \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}},$$

i.e.

$$C_{\text{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma} := \sup_{\substack{\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\text{div}, \omega_{\boldsymbol{a}}); \boldsymbol{\nabla}\cdot\boldsymbol{j}_p = 0 \\ \boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})}} \frac{\|\boldsymbol{v}_p^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}}{\|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}}. \tag{7.2}$$

For interior patches, we have shown in Section 6 that $C_{\text{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma}$ is uniformly bounded by a constant only dependent on the patch shape-regularity parameter $\kappa_{\mathcal{T}_{\boldsymbol{a}}}$. For boundary patches, it is clear that $C_{\text{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma}$ is bounded for each p, and our goal here is to show that it is actually uniformly bounded in p, again by $C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})$ only.

7.1. Equivalent patches.

We will first need the concept of equivalent patches, which, roughly speaking, corresponds to patches having the same mesh topology. Let $\mathcal{T}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{T}}_{\boldsymbol{b}}$ be two vertex patches around two possibly different vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ in the sense of Section 2.1. $\mathcal{T}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{T}}_{\boldsymbol{b}}$ can be interior or boundary, though this concept will only be used for boundary patches.

Definition 7.1 (Equivalent patches). *Two vertex patches $\mathcal{T}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{T}}_{\boldsymbol{b}}$ around the vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ and covering the domains $\omega_{\boldsymbol{a}}$ and $\widetilde{\omega}_{\boldsymbol{b}}$ are said to be equivalent if there exists a bijection $\psi : \omega_{\boldsymbol{a}} \to \widetilde{\omega}_{\boldsymbol{b}}$ such that $\psi|_K$ is affine and $\psi(K) \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}$ for all $K \in \mathcal{T}_{\boldsymbol{a}}$. Note that ψ necessary preserves the topology, i.e., $\boldsymbol{b} = \psi(\boldsymbol{a})$, if a (boundary) face $F \in \mathcal{F}_{\boldsymbol{a}}$ shares $\boldsymbol{a}$,* *then* $\psi(F) \in \widetilde{\mathcal{F}}_{\boldsymbol{b}}$ *is a (boundary) face that shares* $\boldsymbol{b}$*, and if* $K, L \in \mathcal{T}_{\boldsymbol{a}}$ *are neighbors over a face* F*, then* $\psi(K), \psi(L) \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}$ *are neighbors over the face* $\psi(F)$*.*

The stability constants of equivalent patches are tightly linked together. Actually, they simply differ up to a factor depending only on the shape regularity parameter of the two patches.

Lemma 7.2 (Equivalent patches)**.** *If* $\mathcal{T}_{\boldsymbol{a}}$ *and* $\widetilde{\mathcal{T}}_{\boldsymbol{b}}$ *are equivalent patches in the sense of Definition 7.1, then, for all* $p \geq 0$*,*

$$C_{\mathrm{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}, \kappa_{\widetilde{\mathcal{T}}_{\boldsymbol{b}}}) C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\boldsymbol{b}},\widetilde{\Gamma}}, \tag{7.3}$$

where $\widetilde{\Gamma} := \psi(\Gamma)$ *and* ψ *is the bijection in Definition 7.1.*

Proof. Fix a polynomial degree $p \geq 0$. Consider data $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot \boldsymbol{j}_p = 0$ and $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$. We define $\widetilde{\boldsymbol{j}}_p := \psi^{\mathrm{d}}(\boldsymbol{j}_p)$ and $\widetilde{\boldsymbol{\chi}}_p := \psi^{\mathrm{c}}(\boldsymbol{\chi}_p)$, where ψ^{d} and ψ^{c} are the Piola mappings from (5.4). Because the mapping ψ is piecewise affine, we have $\widetilde{\boldsymbol{j}}_p \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{b}}) \cap \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{b}})$ and $\widetilde{\boldsymbol{\chi}}_p \in \boldsymbol{\mathcal{N}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{b}})$. As a result, if we denote by $\widetilde{\boldsymbol{v}}^{\star}$ and $\widetilde{\boldsymbol{v}}_p^{\star}$ the continuous and discrete minimizers on $\widetilde{\mathcal{T}}_{\boldsymbol{b}}$ with data $\widetilde{\boldsymbol{j}}_p$ and $\widetilde{\boldsymbol{\chi}}_p$, we have

$$\|\widetilde{\boldsymbol{v}}_p^{\star} - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{b}}} \leq C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\boldsymbol{b}},\widetilde{\Gamma}} \|\widetilde{\boldsymbol{v}}^{\star} - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{b}}}.$$

Then, letting $|\!|\!|\cdot|\!|\!|$ be the usual operator norm from $\boldsymbol{L}^2(\omega_{\boldsymbol{a}})$ to $\boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{b}})$ (or vice-versa), we have on the one hand, since $\widetilde{\boldsymbol{v}}^{\star}$ is the minimizer and $\psi^{\mathrm{c}}(\boldsymbol{v}^{\star}) \in \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathbf{curl}, \widetilde{\omega}_{\boldsymbol{b}})$ with $\boldsymbol{\nabla}\times(\psi^{\mathrm{c}}(\boldsymbol{v}^{\star})) = \widetilde{\boldsymbol{j}}_p$ that

$$\|\widetilde{\boldsymbol{v}}^{\star} - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{b}}} \leq \|\psi^{\mathrm{c}}(\boldsymbol{v}^{\star}) - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{b}}} = \|\psi^{\mathrm{c}}(\boldsymbol{v}^{\star} - \boldsymbol{\chi}_p)\|_{\widetilde{\omega}_{\boldsymbol{b}}} \leq |\!|\!|\psi^{\mathrm{c}}|\!|\!| \|\boldsymbol{v}^{\star} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}},$$

and on the other hand that

$$\|\boldsymbol{v}_p^{\star} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq \|(\psi^{\mathrm{c}})^{-1}(\widetilde{\boldsymbol{v}}_p^{\star} - \widetilde{\boldsymbol{\chi}}_p)\|_{\omega_{\boldsymbol{a}}} \leq |\!|\!|(\psi^{\mathrm{c}})^{-1}|\!|\!| \|\widetilde{\boldsymbol{v}}_p^{\star} - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{b}}}.$$

It follows that

$$\|\boldsymbol{v}_p^{\star} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\boldsymbol{b}},\widetilde{\Gamma}} |\!|\!|\psi^{\mathrm{c}}|\!|\!| |\!|\!|(\psi^{\mathrm{c}})^{-1}|\!|\!| \|\boldsymbol{v}^{\star} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

Since the data was arbitrary, (7.3) follows from the estimate

$$|\!|\!|\psi^{\mathrm{c}}|\!|\!| |\!|\!|(\psi^{\mathrm{c}})^{-1}|\!|\!| \leq \bar{\kappa}_{\mathcal{T}_{\boldsymbol{a}}}^4 \bar{\kappa}_{\widetilde{\mathcal{T}}_{\boldsymbol{b}}}^4$$

with

$$\bar{\kappa}_{\mathcal{T}_{\boldsymbol{a}}} := \frac{\max_{K \in \mathcal{T}_{\boldsymbol{a}}} h_K}{\min_{K \in \mathcal{T}_{\boldsymbol{a}}} \rho_K}, \quad \bar{\kappa}_{\widetilde{\mathcal{T}}_{\boldsymbol{b}}} := \frac{\max_{K \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}} h_K}{\min_{K \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}} \rho_K}$$

that may be easily obtained from standard estimates on the Jacobian matrices $\mathbb{J}$ defining the affine mappings (see, e.g., [13, Theorem 3.1.2]). The conclusion then follows since $\bar{\kappa}_{\mathcal{T}_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}})$ and $\bar{\kappa}_{\widetilde{\mathcal{T}}_{\boldsymbol{b}}} \leq C(\kappa_{\widetilde{\mathcal{T}}_{\boldsymbol{b}}})$. □

7.2. **Extensions of patches.** We will next need the concept of "patch extension". Specifically, if a patch $\mathcal{T}_{\boldsymbol{a}}$ can be extended into another patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}} \supset \mathcal{T}_{\boldsymbol{a}}$ in a suitable way, then the stability constant $C_{\mathrm{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma}$ of $\mathcal{T}_{\boldsymbol{a}}$ given by (7.2) will be controlled by that of $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$. Here, $\mathcal{T}_{\boldsymbol{a}}$ will typically be a boundary patch and $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ either boundary or interior. The precise definition is as follows.

Definition 7.3 (Patch extension). *Consider two patches $\mathcal{T}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ around the same vertex $\boldsymbol{a}$, with associated domains $\omega_{\boldsymbol{a}}$ and $\widetilde{\omega}_{\boldsymbol{a}}$. We say that $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ is an extension of $\mathcal{T}_{\boldsymbol{a}}$ if the following holds:*

(1) $\mathcal{T}_{\boldsymbol{a}} \subset \widetilde{\mathcal{T}}_{\boldsymbol{a}}$.
(2) *If $\Gamma_{\boldsymbol{a}} \neq \emptyset$ corresponds to boundary faces of $\mathcal{T}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$ with $\Gamma = \partial\omega_{\boldsymbol{a}} \setminus \overline{\Gamma_{\boldsymbol{a}}} \neq \partial\omega_{\boldsymbol{a}}$, cf. Figure 2, left, we let $\widetilde{\Gamma}_{\boldsymbol{a}}$ correspond to boundary faces of $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$ and $\widetilde{\Gamma} := \partial\widetilde{\omega}_{\boldsymbol{a}} \setminus \widetilde{\Gamma}_{\boldsymbol{a}}$. We introduce the trivial restriction operator $\mathscr{R}_0 : \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}}) \ni \widetilde{\boldsymbol{v}} \to \widetilde{\boldsymbol{v}}|_{\omega_{\boldsymbol{a}}} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$. Besides, there exist extension operators $\mathscr{E}^{\mathrm{c}}, \mathscr{E}^{\mathrm{d}} : \boldsymbol{L}^2(\omega_{\boldsymbol{a}}) \to \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}})$ such that*
   (a) $\mathscr{E}^{\mathrm{c}}(\boldsymbol{v})|_{\omega_{\boldsymbol{a}}} = \mathscr{E}^{\mathrm{d}}(\boldsymbol{v})|_{\omega_{\boldsymbol{a}}} = \boldsymbol{v}$ *for all* $\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$;
   (b) $\mathscr{E}^{\mathrm{c}} : \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \to \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathbf{curl}, \widetilde{\omega}_{\boldsymbol{a}})$ *and* $\mathscr{E}^{\mathrm{d}} : \boldsymbol{H}_{0,\Gamma}(\mathrm{div}, \omega_{\boldsymbol{a}}) \to \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}})$;
   (c) $\mathscr{E}^{\mathrm{c}} : \boldsymbol{\mathcal{N}}_q(\mathcal{T}_{\boldsymbol{a}}) \to \boldsymbol{\mathcal{N}}_q(\widetilde{\mathcal{T}}_{\boldsymbol{a}})$ *and* $\mathscr{E}^{\mathrm{d}} : \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_{\boldsymbol{a}}) \to \boldsymbol{\mathcal{RT}}_q(\widetilde{\mathcal{T}}_{\boldsymbol{a}})$;
   (d) $\boldsymbol{\nabla} \times (\mathscr{E}^{\mathrm{c}}(\boldsymbol{v})) = \mathscr{E}^{\mathrm{d}}(\boldsymbol{\nabla} \times \boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \omega_{\boldsymbol{a}})$.
(3) *If $\Gamma_{\boldsymbol{a}} = \emptyset$ with $\Gamma = \partial\omega_{\boldsymbol{a}} \setminus \overline{\Gamma_{\boldsymbol{a}}} = \partial\omega_{\boldsymbol{a}}$, cf. Figure 1, right, we let $\widetilde{\Gamma}_{\boldsymbol{a}} := \emptyset$ and $\widetilde{\Gamma} := \partial\widetilde{\omega}_{\boldsymbol{a}}$. We introduce the trivial extension operator $\mathscr{E}_0 : \boldsymbol{L}^2(\omega_{\boldsymbol{a}}) \ni \boldsymbol{v} \to \widetilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}})$ where $\widetilde{\boldsymbol{v}}$ is the extension of $\boldsymbol{v}$ by $\mathbf{0}$. Besides, there exist restriction operators $\mathscr{R}^{\mathrm{c}}, \mathscr{R}^{\mathrm{d}} : \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}}) \to \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$ such that*
   (a) *If $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}})$ is the extension by $\mathbf{0}$ of $\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$, then $\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}) = \mathscr{R}^{\mathrm{d}}(\widetilde{\boldsymbol{v}}) = \boldsymbol{v}$;*
   (b) $\mathscr{R}^{\mathrm{c}} : \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathbf{curl}, \widetilde{\omega}_{\boldsymbol{a}}) \to \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$ *and* $\mathscr{R}^{\mathrm{d}} : \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}}) \to \boldsymbol{H}_{0,\Gamma}(\mathrm{div}, \omega_{\boldsymbol{a}})$;
   (c) $\mathscr{R}^{\mathrm{c}} : \boldsymbol{\mathcal{N}}_q(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \to \boldsymbol{\mathcal{N}}_q(\mathcal{T}_{\boldsymbol{a}})$ *and* $\mathscr{R}^{\mathrm{d}} : \boldsymbol{\mathcal{RT}}_q(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \to \boldsymbol{\mathcal{RT}}_q(\mathcal{T}_{\boldsymbol{a}})$;
   (d) $\boldsymbol{\nabla} \times (\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}})) = \mathscr{R}^{\mathrm{d}}(\boldsymbol{\nabla} \times \widetilde{\boldsymbol{v}})$ *for all* $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}(\mathbf{curl}, \widetilde{\omega}_{\boldsymbol{a}})$.

As we state below, extensions can be composed, so that it is possible to extend an initial patch several times.

Lemma 7.4 (Composition of extensions). *If, in the sense of Definition 7.3, $\widetilde{\mathcal{T}}_{\boldsymbol{a}}^1$ is an extension of $\mathcal{T}_{\boldsymbol{a}}$ with operators $\mathscr{E}_1^{\mathrm{c}}$ and $\mathscr{R}_1^{\mathrm{c}}$ and $\widetilde{\mathcal{T}}_{\boldsymbol{a}}^2$ is an extension of $\widetilde{\mathcal{T}}_{\boldsymbol{a}}^1$ with operators $\mathscr{E}_{1,2}^{\mathrm{c}}$ and $\mathscr{R}_{1,2}^{\mathrm{c}}$, then $\widetilde{\mathcal{T}}_{\boldsymbol{a}}^2$ is an extension of $\mathcal{T}_{\boldsymbol{a}}$ with operators $\mathscr{E}_2^{\mathrm{c}} := \mathscr{E}_{1,2}^{\mathrm{c}} \circ \mathscr{E}_1^{\mathrm{c}}$ and $\mathscr{R}_2^{\mathrm{c}} := \mathscr{R}_1^{\mathrm{c}} \circ \mathscr{R}_{1,2}^{\mathrm{c}}$. The same holds for the operators $\mathscr{E}^{\mathrm{d}}$ and $\mathscr{R}^{\mathrm{d}}$.*

Crucially, if we can prove the stability of discrete minimization in an extension of a given patch, then it also holds on the original patch. Indeed, we have the following inequality with the constant that only depends on the norms of the extension and restriction operators of Definition 7.3.

Lemma 7.5 (Patch extensions)**.** *Consider a vertex patch* $\mathcal{T}_{\boldsymbol{a}}$ *and an extension* $\widetilde{\mathcal{T}}_{\boldsymbol{a}} \supset \mathcal{T}_{\boldsymbol{a}}$ *in the sense of Definition 7.3. Then, for all* $p \geq 0$*, we have*

$$C_{\mathrm{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma} \leq \max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|) C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\Gamma}}.$$

Proof. Let $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot \boldsymbol{j}_p = 0$ and $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, and recall the discrete and continuous minimizers $\boldsymbol{v}_p^\star$ and $\boldsymbol{v}^\star$ from (7.1).

Case 1. Let us first consider the case where $\Gamma_{\boldsymbol{a}} = \emptyset$ and $\Gamma = \partial\omega_{\boldsymbol{a}}$, see Figure 1, right. Due to the boundary condition, we have from Definition 7.3 $\widetilde{\boldsymbol{j}}_p := \mathscr{E}_0(\boldsymbol{j}_p) \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot \widetilde{\boldsymbol{j}}_p = 0$ and $\widetilde{\boldsymbol{\chi}}_p := \mathscr{E}_0(\boldsymbol{\chi}_p) \in \boldsymbol{\mathcal{N}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})$. Denote by $\widetilde{\boldsymbol{v}}^\star$ and $\widetilde{\boldsymbol{v}}_p^\star$ the associated continuous and discrete minimizers associated with these data in the extended patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and the homogeneous boundary condition prescribed on $\widetilde{\Gamma} = \partial\widetilde{\omega}_{\boldsymbol{a}}$. Since $\mathscr{E}_0(\boldsymbol{v}^\star) \in \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\times(\mathscr{E}_0(\boldsymbol{v}^\star)) = \widetilde{\boldsymbol{j}}_p$, we have

$$C^{-1}_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\boldsymbol{a}},\widetilde{\Gamma}} \|\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \leq \|\widetilde{\boldsymbol{v}}^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \leq \|\mathscr{E}_0(\boldsymbol{v}^\star) - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} = \|\mathscr{E}_0(\boldsymbol{v}^\star - \boldsymbol{\chi}_p)\|_{\widetilde{\omega}_{\boldsymbol{a}}} = \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}.$$

On the other hand, using properties 3d and 3a of Definition 7.3, we have $\boldsymbol{\nabla}\times(\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_p^\star)) = \mathscr{R}^{\mathrm{d}}(\widetilde{\boldsymbol{j}}_p) = \boldsymbol{j}_p$ and $\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{\chi}}_p) = \boldsymbol{\chi}_p$. Moreover, from properties 3b and 3c, we also have $\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_p^\star) \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$. Thus

$$\|\boldsymbol{v}_p^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq \|\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_p^\star) - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = \|\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p)\|_{\omega_{\boldsymbol{a}}} \leq \|\mathscr{R}^{\mathrm{c}}\| \|\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}},$$

which concludes the proof.

Case 2. We now consider the case where $\Gamma_{\boldsymbol{a}} \neq \emptyset$ and $\Gamma = \partial\omega_{\boldsymbol{a}} \setminus \overline{\Gamma_{\boldsymbol{a}}} \neq \partial\omega_{\boldsymbol{a}}$, see Figure 2, left. We extend the data as $\widetilde{\boldsymbol{j}}_p := \mathscr{E}^{\mathrm{d}}(\boldsymbol{j}_p) \in \boldsymbol{\mathcal{RT}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot \widetilde{\boldsymbol{j}}_p = 0$ and $\widetilde{\boldsymbol{\chi}}_p := \mathscr{E}^{\mathrm{c}}(\boldsymbol{\chi}_p) \in \boldsymbol{\mathcal{N}}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})$. Since $\mathscr{E}^{\mathrm{c}}(\boldsymbol{v}^\star) \in \boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\times(\mathscr{E}^{\mathrm{c}}(\boldsymbol{v}^\star)) = \widetilde{\boldsymbol{j}}_p$, we have

$$\begin{aligned} C^{-1}_{\mathrm{st},p,\widetilde{\mathcal{T}},\widetilde{\Gamma}} \|\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} &\leq \|\widetilde{\boldsymbol{v}}^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \leq \|\mathscr{E}^{\mathrm{c}}(\boldsymbol{v}^\star) - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \\ &= \|\mathscr{E}^{\mathrm{c}}(\boldsymbol{v}^\star - \boldsymbol{\chi}_p)\|_{\widetilde{\omega}_{\boldsymbol{a}}} \leq \|\mathscr{E}^{\mathrm{c}}\| \|\boldsymbol{v}^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}. \end{aligned}$$

On the other hand, we have $\mathscr{R}_0(\widetilde{\boldsymbol{v}}_p^\star) \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\times(\mathscr{R}_0(\widetilde{\boldsymbol{v}}_p^\star)) = \mathscr{R}_0(\widetilde{\boldsymbol{j}}_p) = \boldsymbol{j}_p$ and $\mathscr{R}_0(\widetilde{\boldsymbol{\chi}}_p) = \boldsymbol{\chi}_p$, so that

$$\|\boldsymbol{v}_p^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq \|\widetilde{\boldsymbol{v}}_p^\star - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} = \|\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\omega_{\boldsymbol{a}}} \leq \|\widetilde{\boldsymbol{v}}_p^\star - \widetilde{\boldsymbol{\chi}}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}},$$

which concludes the proof. □

7.3. **Parachute and flattenable patches.** The next central concept is the one of a "parachute patch" where all the vertices except the central vertex lie in the same plane. As a result, the faces not sharing the central vertex are easily identified with a two-dimensional planar triangular mesh, which makes the reasoning easier. An illustration is given in Figures 3 and 5, left.

Definition 7.6 (Parachute patch)**.** *A parachute patch* $\mathcal{T}_{\mathbf{0}}$ *is a boundary patch around the vertex* $\mathbf{0} \in \mathbb{R}^3$ *with associated domain* $\omega_{\mathbf{0}}$ *such that all the non-central vertices of the patch* $\boldsymbol{b} \neq \mathbf{0}$ *lie in the plane* $H := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x}_3 = 1\}$*. In this case, we denote by* $\lceil \mathcal{T}_{\mathbf{0}} \rceil$ *the planar*

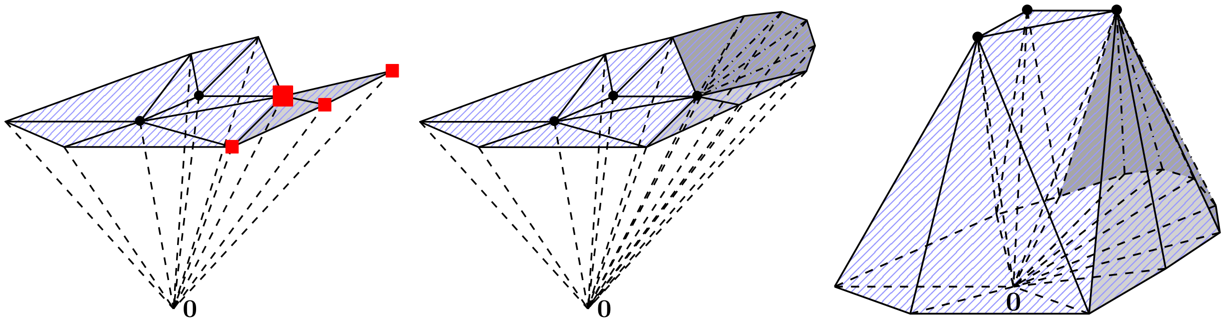


FIGURE 3. Parachute patch (left), its flattenable extension (middle), and its flattening (right). The portion of the boundary of $\omega_{\mathbf{0}}$ corresponding to those faces F that do not share the vertex $\mathbf{0}$, is highlighted by (light-blue) dashing. Triangles $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ that have no vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$ are highlighted by a light grey fill and the corresponding vertices by (red) squares (left). Triangles added by the extension around the "problematic" vertex marked by the large (red) square from Lemma 7.10, Case 2, are highlighted by a dark grey fill (middle, right).

triangular mesh induced by $\mathcal{T}_{\mathbf{0}}$ *on* H *and by* $\lceil \omega_{\mathbf{0}} \rceil \subset \mathbb{R}^2$ *the corresponding two-dimensional planar domain.*

Crucially, every boundary patch is equivalent to a "reference" parachute patch, as we next demonstrate.

Lemma 7.7 (Reference parachute patches). *Consider a shape-regularity parameter* $\kappa > 0$. *There exists a finite set of reference parachute patches* $\widehat{\mathscr{T}_\kappa} = \{\widehat{\mathcal{T}_{\mathbf{0}}}\}$ *such that if* $\mathcal{T}_{\boldsymbol{a}}$ *is a boundary patch with shape-regularity parameter* $\kappa_{\mathcal{T}_{\boldsymbol{a}}} \leq \kappa$, *then there exists exactly one* $\widehat{\mathcal{T}_{\mathbf{0}}} \in \widehat{\mathscr{T}_\kappa}$ *that is equivalent to* $\mathcal{T}_{\boldsymbol{a}}$ *in the sense of Definition 7.1.*

Proof. For each $\kappa > 0$, we denote by $N(\kappa)$ the maximum number of tetrahedra in a boundary patch $\mathcal{T}_{\boldsymbol{a}}$ with shape-regularity parameter $\kappa_{\mathcal{T}_{\boldsymbol{a}}} \leq \kappa$. For each $N \in \mathbb{N}$, let $\lceil \widehat{\mathscr{T}_N} \rceil$ denote the set of possible topologies of conforming planar triangular meshes with N elements. Then $\widehat{\mathscr{T}_\kappa}$ is defined by extending the elements of $\lceil \widehat{\mathscr{T}_{N(\kappa)}} \rceil$ into a parachute patch.

Fix $\kappa > 0$. If $\mathcal{T}_{\boldsymbol{a}}$ is a boundary patch with $\kappa_{\mathcal{T}_{\boldsymbol{a}}} \leq \kappa$, then $\mathcal{T}_{\boldsymbol{a}}$ has at most $N(\kappa)$ tetrahedra K. Because $\mathcal{T}_{\boldsymbol{a}}$ is a boundary patch, there is a cone $\mathcal{C}$ with the vertex $\boldsymbol{a}$ and a strictly positive solid angle such that $\mathcal{C} \cap \omega_{\boldsymbol{a}} = \emptyset$ (forming an "opening"). Thus, $\mathcal{T}_{\boldsymbol{a}}$ can be transformed into a first parachute patch $\mathcal{T}_{\mathbf{0}}$ with the corresponding planar mesh $\lceil \mathcal{T}_{\mathbf{0}} \rceil$. This can be done by transforming $\boldsymbol{a}$ into $\mathbf{0}$ by translation, dilating the solid angle of the opening until all the vertices lie on the same side of a plane, rotating, and then shortening the sizes of edges connecting each non-central vertex to the central one to align them onto the plane H. Then, there exists a parachute patch $\widehat{\mathcal{T}_{\mathbf{0}}} \in \widehat{\mathscr{T}_\kappa}$ such that $\lceil \widehat{\mathcal{T}_{\mathbf{0}}} \rceil$ has the same topology as $\lceil \mathcal{T}_{\mathbf{0}} \rceil$. Finally, $\mathcal{T}_{\mathbf{0}}$ is equivalent to $\widehat{\mathcal{T}_{\mathbf{0}}}$, and therefore $\mathcal{T}_{\boldsymbol{a}}$ is equivalent to $\widehat{\mathcal{T}_{\mathbf{0}}} \in \widehat{\mathscr{T}_\kappa}$. □

We next identify a class of parachute patches that can be "flattened".[3] An illustration of a "flattenable patch" is given in Figure 3, middle.

Definition 7.8 (Flattenable patch). *A flattenable patch is a parachute patch $\mathcal{T}_{\mathbf{0}}$ such that every triangle $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ has at least one vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$.*

Crucially, we can easily show the stability of discrete minimization of flattenable patches, as they are equivalent to interior patches.

Lemma 7.9 (Stable discrete minimization for flattenable patches). *If $\mathcal{T}_{\mathbf{0}}$ is a flattenable patch in the sense of Definition 7.8 and either $\Gamma_{\mathbf{0}} = \emptyset$ and $\Gamma = \partial\omega_{\mathbf{0}}$ (as in Figure 1, right) or $\Gamma_{\mathbf{0}}$ corresponds to all faces $F \in \mathcal{F}_{\mathbf{0}}$ lying on the boundary of $\omega_{\mathbf{0}}$ and sharing the vertex $\mathbf{0}$ and $\Gamma = \partial\omega_{\mathbf{0}} \setminus \overline{\Gamma_{\mathbf{0}}}$ (as in Figure 2, left), then for all $p \geq 0$, we have*

$$C_{\mathrm{st},p,\mathcal{T}_{\mathbf{0}},\Gamma} \leq C(\kappa_{\mathcal{T}_{\mathbf{0}}}),$$

where the constant on the right-hand side only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_{\mathbf{0}}}$.

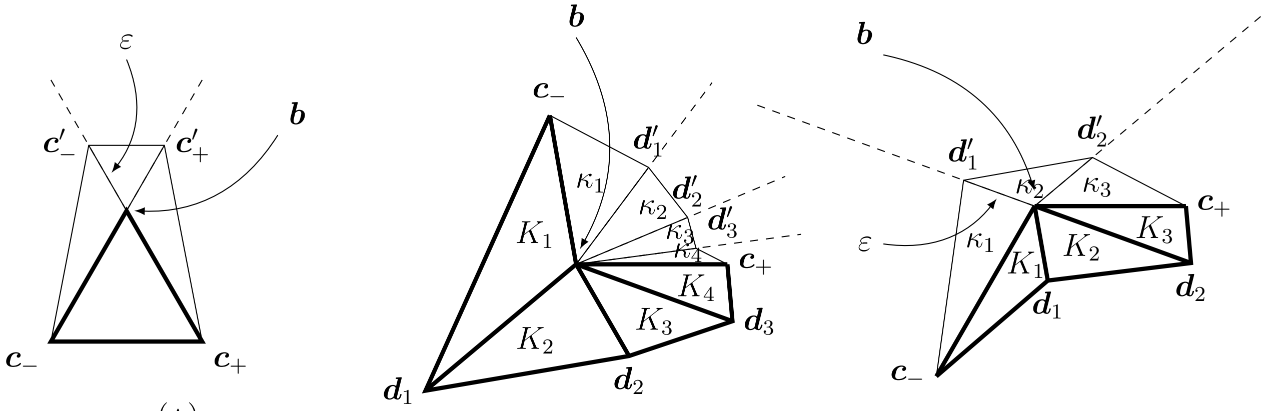
Proof. We follow the lines of [23, Section 7]. First, by the characterization of Definition 7.8, and relying on Definition 7.1, $\mathcal{T}_{\mathbf{0}}$ is equivalent to a "flattened" patch $\mathcal{T}_{\mathbf{0}}'$ where all faces $F \in \mathcal{F}_{\mathbf{0}}$ lying on the boundary of $\omega_{\mathbf{0}}$ and sharing the vertex $\mathbf{0}$ lie on the hyperplane $\{\boldsymbol{x}_3 = 0\}$. An illustration is given in Figure 3 middle for $\mathcal{T}_{\mathbf{0}}$ and right for $\mathcal{T}_{\mathbf{0}}'$. Following Definition 7.3, we can then extend $\mathcal{T}_{\mathbf{0}}'$ into a patch $\widetilde{\mathcal{T}}_{\mathbf{0}}'$ by symmetry around the plane $\{\boldsymbol{x}_3 = 0\}$. This is done the following way: we denote by $\phi(\boldsymbol{x}) := (\boldsymbol{x}_1, \boldsymbol{x}_2, -\boldsymbol{x}_3)$ the symmetrization operator around $\{\boldsymbol{x}_3 = 0\}$, and use the associated Piola mappings ϕ^{c} and ϕ^{d} from (5.4). We then set $\mathscr{E}^{\bullet}(\boldsymbol{v}) = \boldsymbol{v} + \phi^{\bullet}(\boldsymbol{v})$ for $\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\mathbf{0}}')$ with $\bullet = \mathrm{c}, \mathrm{d}$ if $\Gamma_{\boldsymbol{a}} \neq \emptyset$ and $\mathscr{R}^{\bullet}(\widetilde{\boldsymbol{v}}) = \widetilde{\boldsymbol{v}}|_{\omega_{\mathbf{0}}'} - \phi^{\bullet}(\widetilde{\boldsymbol{v}}|_{\widetilde{\omega}_{\mathbf{0}}' \setminus \omega_{\mathbf{0}}'})$ for $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\widetilde{\omega}_{\mathbf{0}}')$ otherwise. The resulting patch $\widetilde{\mathcal{T}}_{\mathbf{0}}'$ is interior, so that $C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\mathbf{0}}',\widetilde{\Gamma}'} \leq C(\kappa_{\widetilde{\mathcal{T}}_{\mathbf{0}}'})$ for all $p \geq 0$ in view of the validity of Theorem 3.3 for interior patches established in Section 6. Moreover, from Lemmas 7.2 and 7.5,

$$C_{\mathrm{st},p,\mathcal{T}_{\mathbf{0}},\Gamma} \leq C(\kappa_{\mathcal{T}_{\mathbf{0}}}, \kappa_{\mathcal{T}_{\mathbf{0}}'}) C_{\mathrm{st},p,\mathcal{T}_{\mathbf{0}}',\Gamma'} \leq C(\kappa_{\mathcal{T}_{\mathbf{0}}}, \kappa_{\mathcal{T}_{\mathbf{0}}'}) \max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|) C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\mathbf{0}}',\widetilde{\Gamma}'}.$$

Thus, the result follows since $C(\kappa_{\mathcal{T}_{\mathbf{0}}}, \kappa_{\mathcal{T}_{\mathbf{0}}'}) \max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|)$ as well as $\kappa_{\widetilde{\mathcal{T}}_{\mathbf{0}}'}$ only depend on $\kappa_{\mathcal{T}_{\mathbf{0}}}$. Indeed, since ϕ is an isometry here, a careful inspection of (5.4) reveals that $\|\phi^{\bullet}\| = 1$. □

7.4. **Transformation of a general parachute patch to a flattenable patch.** At this point, we know that every boundary patch is equivalent to a parachute patch, and we have established the discrete stable minimization result for flattenable patches. The missing link is to establish a connection between a general parachute patch and a flattenable patch. This is done through the concept of patch extension around "problematic" vertices: the vertices of triangles $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ that have no vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$, highlighted by the (red) squares in Figures 3 and 5, left.

[3]This corresponds to the situation that "all the faces in $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}$ have at least one vertex lying in the interior of $\partial\omega_{\boldsymbol{a}}^{\mathrm{ext}}$" of [23].



(A) (B) (C)

FIGURE 4. Extensions around problematic vertices

Lemma 7.10 (Extension around problematic vertices). *Consider a parachute patch $\mathcal{T}_{\mathbf{0}}$ such that there is a triangle $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ which has no vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$ (such as the vertices highlighted by the (red) squares in Figure 3). Consider such a vertex $\boldsymbol{b} \neq \mathbf{0}$: it is is connected to $\mathbf{0}$ trough an edge lying in $\partial\omega_{\mathbf{0}}$. Then, in the sense of Definition 7.3, we can extend the patch into $\widetilde{\mathcal{T}}_{\mathbf{0}} := \mathcal{T}_{\mathbf{0}} \cup \{\kappa_1, \dots, \kappa_\ell\}$ in such a way that $[\mathbf{0}, \boldsymbol{b}]$ is in the interior of $\widetilde{\omega}_{\mathbf{0}}$ and $\kappa_1, \dots, \kappa_\ell$ share the vertex $\boldsymbol{b}$. In addition, the corresponding operators $\mathscr{E}^{\mathrm{c}}, \mathscr{E}^{\mathrm{d}} : \boldsymbol{L}^2(\omega_{\mathbf{0}}) \to \boldsymbol{L}^2(\widetilde{\omega}_{\mathbf{0}})$ and $\mathscr{R}^{\mathrm{c}}, \mathscr{R}^{\mathrm{d}} : \boldsymbol{L}^2(\widetilde{\omega}_{\mathbf{0}}) \to \boldsymbol{L}^2(\omega_{\mathbf{0}})$ can be constructed such that* $\max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|)$ *only depends on $\kappa_{\mathcal{T}_{\mathbf{0}}}$.*

Proof. Consider a vertex $\boldsymbol{b} \in \mathcal{T}_{\mathbf{0}}$ as described above.

Case 1. We first consider the case where $\boldsymbol{b}$ only belongs to one tetrahedron, such as the vertex highlighted by the small (red) square and lying at the tip in Figures 3 and 5, left. We are going to extend $\mathcal{T}_{\mathbf{0}}$ here with three new tetrahedra. The process for creating these tetrahedra is depicted in Figure 4a. Denote by $[\mathbf{0}, \boldsymbol{b}, \boldsymbol{c}_-, \boldsymbol{c}_+]$ the vertices of the tetrahedron K sharing $\boldsymbol{b}$. We can simply consider the triangle $\{\boldsymbol{b}, \boldsymbol{c}_-, \boldsymbol{c}_+\}$ in the plane H. We define $\boldsymbol{c}'_-$ and $\boldsymbol{c}'_+$ by symmetry around $\boldsymbol{b}$and adjust the size of the edges $[\boldsymbol{b}, \boldsymbol{c}'_\pm]$ to a small enough ε so that the introduced edges do not intersect the remainder of the patch (this is always possible since the patch is Lipschitz by assumption). We then add the triangles $\kappa_- := \{\boldsymbol{b}, \boldsymbol{c}_-, \boldsymbol{c}'_-\}$, $\kappa_0 := \{\boldsymbol{b}, \boldsymbol{c}'_-, \boldsymbol{c}'_+\}$, and $\kappa_+ := \{\boldsymbol{b}, \boldsymbol{c}'_+, \boldsymbol{c}_+\}$ to the surface mesh $\lceil \mathcal{T}_{\mathbf{0}} \rceil$.

If $\Gamma_{\mathbf{0}} = \emptyset$, we need to define the restriction operator $\mathscr{R}^{\mathrm{c}}$. We set

$$\mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}}) := \boldsymbol{v}|_{\omega_{\mathbf{0}}} - \psi_-^{\mathrm{c}}(\boldsymbol{v}|_{\kappa_-}) + \psi_0^{\mathrm{c}}(\boldsymbol{v}|_{\kappa_0}) - \psi_+^{\mathrm{c}}(\boldsymbol{v}|_{\kappa_+}),$$

where ψ_- is the affine mapping that preserves $\boldsymbol{b}$ and $\boldsymbol{c}_-$ and maps $\boldsymbol{c}'_-$ to $\boldsymbol{c}_+$, ψ_0 preserves $\boldsymbol{b}$ and maps $\boldsymbol{c}'_\pm$ to $\boldsymbol{c}_\mp$, and ψ_+ preserve $\boldsymbol{b}$ and $\boldsymbol{c}_+$ and maps $\boldsymbol{c}'_+$ to $\boldsymbol{c}_-$ and where $\psi_\bullet^{\mathrm{c}}$ are the corresponding Piola mappings from (5.4). We employ an analogous definition for $\mathscr{R}^{\mathrm{d}}$. If $\Gamma_{\mathbf{0}} \neq \emptyset$, we need to define the extension operator $\mathscr{E}^{\mathrm{c}}$. We do this by

$$\mathscr{E}^{\mathrm{c}}(\boldsymbol{v})|_{\kappa_-} := (\psi_-^{\mathrm{c}})^{-1}(\boldsymbol{v}|_K), \quad \mathscr{E}^{\mathrm{c}}(\boldsymbol{v})|_{\kappa_0} := (\psi_0^{\mathrm{c}})^{-1}(\boldsymbol{v}|_K), \quad \mathscr{E}^{\mathrm{c}}(\boldsymbol{v})|_{\kappa_+} := (\psi_+^{\mathrm{c}})^{-1}(\boldsymbol{v}|_K),$$

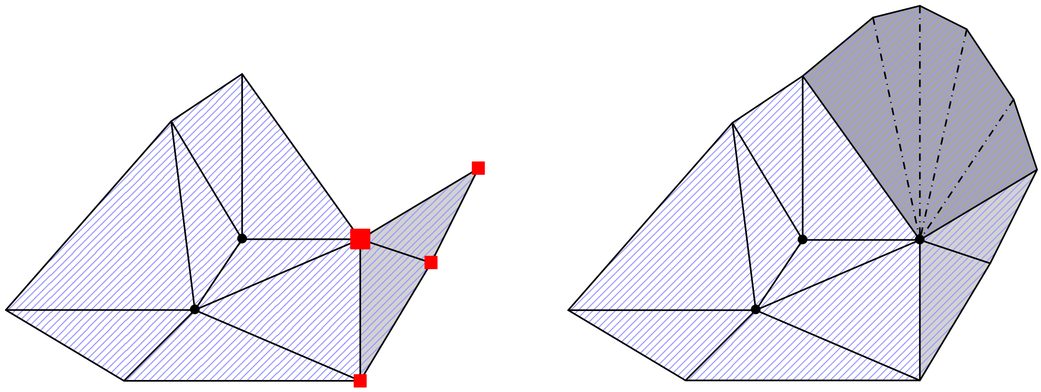


FIGURE 5. Planar triangulations $\lceil \mathcal{T}_{\mathbf{0}} \rceil$ corresponding to Figure 3. Triangles $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ that have no vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$ (highlighted by a light grey fill and the corresponding vertices by (red) squares, left). Triangles added by the extension around the "problematic" vertex marked by the large (red) square from Lemma 7.10, Case 2 (highlighted by a dark grey fill, right).

and by identity elsewhere, as requested in 2a of Definition 7.3, and similarly for $\mathscr{E}^{\mathrm{d}}$. These definitions crucially ensure that conditions of Definition 7.3 are satisfied, namely in what concerns the normal and tangential traces continuity and the homogeneous boundary conditions. Moreover, as they only combine Piola mappings on three tetrahedra, $\max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|)$ only depends on $\kappa_{\mathcal{T}_{\mathbf{0}}}$.

Case 2. We now consider the case where $\boldsymbol{b}$ is shared by more than one tetrahedron, such as the vertex highlighted by the large (red) square in Figures 3 and 5, left. Figures 4b and 4c describe the process. In this case, we denote by $[\boldsymbol{c}_-, \boldsymbol{d}_1, \dots, \boldsymbol{d}_n, \boldsymbol{c}_+]$ the vertices connected to $\boldsymbol{b}$ in the plane H. We create vertices $\boldsymbol{d}_1', \dots, \boldsymbol{d}_n'$ by symmetry and angular dilatation around $\boldsymbol{b}$, and we adjust the length ε of the edges connecting the $\boldsymbol{d}_j'$ to $\boldsymbol{b}$ so that the new triangles do not intersect the remainder of the mesh. We then add the triangles $\kappa_1 := \{\boldsymbol{b}, \boldsymbol{c}_-, \boldsymbol{d}_1'\}$, $\kappa_{j+1} := \{\boldsymbol{b}, \boldsymbol{d}_j', \boldsymbol{d}_{j+1}'\}$, $1 \le j \le n-1$, and $\kappa_{n+1} := \{\boldsymbol{b}, \boldsymbol{d}_n', \boldsymbol{c}_+\}$ to the surface mesh. For the situation of Figure 3, left, this is illustrated in Figure 5. The extension (if $\Gamma_{\mathbf{0}} \neq \emptyset$) and restriction (if $\Gamma_{\mathbf{0}} = \emptyset$) operators can be naturally constructed by symmetry:

$$\mathscr{E}^{\mathrm{c}}(\boldsymbol{v})|_{\kappa_j} := \psi_j^{\mathrm{c}}(\boldsymbol{v}|_{K_j}) \qquad \mathscr{R}^{\mathrm{c}}(\widetilde{\boldsymbol{v}})|_{K_j} := \widetilde{\boldsymbol{v}}|_{K_j} - (\psi_j^{\mathrm{c}})^{-1}(\widetilde{\boldsymbol{v}}|_{\kappa_j})$$

and by identity elsewhere, where $\psi_j : K_j \to \kappa_j$ preserves $\boldsymbol{b}$ (as well as $\boldsymbol{c}_-$ for K_1 and $\boldsymbol{c}_+$ for K_{n+1}) and maps $\boldsymbol{d}_j$ to $\boldsymbol{d}_j'$ and $\boldsymbol{d}_{j+1}$ to $\boldsymbol{d}_{j+1}'$, and similarly for $\mathscr{E}^{\mathrm{d}}$ and $\mathscr{R}^{\mathrm{d}}$. Here again, the fact that $\max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|)$ only depends on $\kappa_{\mathcal{T}_{\mathbf{0}}}$ again comes from these Piola mappings. □

Here is our connection between a general parachute patch and a flattenable patch:

Lemma 7.11 (Transformation of a general parachute patch to a flattenable patch)**.** *Every parachute patch $\mathcal{T}_{\mathbf{0}}$ has a flattenable extension $\widetilde{\mathcal{T}}_{\mathbf{0}}$ in the sense of Definition 7.3 such that the corresponding operators $\mathscr{E}^{\mathrm{c}}$ and $\mathscr{R}^{\mathrm{c}}$ satisfy* $\max(\|\mathscr{E}^{\mathrm{c}}\|, \|\mathscr{R}^{\mathrm{c}}\|) \leq C(\kappa_{\mathcal{T}_{\mathbf{0}}})$.

Proof. Consider a parachute patch $\mathcal{T}_{\mathbf{0}}$. Let $\{\boldsymbol{b}_1, \dots, \dots \boldsymbol{b}_n\}$ be the vertices of triangles $T \in \lceil \mathcal{T}_{\mathbf{0}} \rceil$ that have no vertex in the interior of $\lceil \omega_{\mathbf{0}} \rceil$. If $n = 0$, then $\mathcal{T}_{\mathbf{0}}$ is flattenable by Definition 7.8 and no extension is necessary. If $n > 0$, consider $\mathcal{T}_{\mathbf{0}}^1$, the extension of Lemma 7.10 for the vertex $\boldsymbol{b}_1$. In $\mathcal{T}_{\mathbf{0}}^1$, $\boldsymbol{b}_1$ is now an internal vertex. In addition, the triangles added to $\lceil \mathcal{T}_{\mathbf{0}} \rceil$ all have a vertex in the interior of $\lceil \omega_{\mathbf{0}}^1 \rceil$ (the vertex $\boldsymbol{b}_1$). As a result, $\mathcal{T}_{\mathbf{0}}^1$ has at most $n-1$ problematic vertices. (In Figure 3, left, there were 3 problematic vertices, but there is no more left after the extension of Lemma 7.10 applied to the vertex marked by the large (red) square, see Figure 3, middle. Thus, more than one problematic vertex can be removed at a time.) Repeating this operation at most n times, we obtain a flattenable patch. Moreover, by Lemma 7.4, this is an extension of $\mathcal{T}_{\mathbf{0}}$, and since the maximal number of vertices in $\mathcal{T}_{\mathbf{0}}$ (and thus n) only depends on $\kappa_{\mathcal{T}_{\mathbf{0}}}$, we obtain the statement. □

7.5. **Proof of Theorem 3.3 for boundary patches.** We are now finally ready to prove Theorem 3.3 for boundary patches.

Proof of Theorem 3.3 for boundary patches. First, Lemma 7.7 states that $\mathcal{T}_{\boldsymbol{a}}$ is equivalent to a reference parachute patch $\widehat{\mathcal{T}}_{\mathbf{0}} \in \widehat{\mathscr{T}}_{\kappa_{\mathcal{T}_{\boldsymbol{a}}}}$, and Lemma 7.2 ensures that

$$C_{\mathrm{st},p,\mathcal{T}_{\boldsymbol{a}},\Gamma} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) C_{\mathrm{st},p,\widehat{\mathcal{T}}_{\mathbf{0}},\widehat{\Gamma}},$$

since $\kappa_{\widehat{\mathcal{T}}_{\mathbf{0}}}$ only depends on $\kappa_{\mathcal{T}_{\boldsymbol{a}}}$. We then follow Lemma 7.11 to extend $\widehat{\mathcal{T}}_{\mathbf{0}}$ into a flattenable patch $\widetilde{\mathcal{T}}_{\mathbf{0}}$ with extension and restriction operator norms only depending on $\kappa_{\widehat{\mathcal{T}}_{\mathbf{0}}}$, and hence, on $\kappa_{\mathcal{T}_{\boldsymbol{a}}}$, so that

$$C_{\mathrm{st},p,\widehat{\mathcal{T}}_{\mathbf{0}},\widehat{\Gamma}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\mathbf{0}},\widetilde{\Gamma}}.$$

Then, the result follows from Lemma 7.9 which states that

$$C_{\mathrm{st},p,\widetilde{\mathcal{T}}_{\mathbf{0}},\widetilde{\Gamma}} \leq C(\kappa_{\widetilde{\mathcal{T}}_{\mathbf{0}}}) \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}).$$

□

8. Mixed boundary conditions

We now investigate the case of mixed boundary conditions. We thus consider a boundary patch $\mathcal{T}_{\boldsymbol{a}}$ with central vertex $\boldsymbol{a}$ and associated domain $\omega_{\boldsymbol{a}}$, where $\Gamma_{\boldsymbol{a}}$ corresponds to some (neither no, nor not all) faces on $\partial\omega_{\boldsymbol{a}}$ and sharing $\boldsymbol{a}$, as in Figure 2, right. Then, Γ contains all faces on $\partial\omega_{\boldsymbol{a}}$ that do not share the vertex $\boldsymbol{a}$ but also additional faces sharing $\boldsymbol{a}$. As before, we have $\Gamma_{\boldsymbol{a}} = \partial\omega_{\boldsymbol{a}} \setminus \Gamma$. The set $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$ gathering the faces sharing the vertex $\boldsymbol{a}$ and contained in Γ will be useful.

Lemma 8.1 (Planar $\Gamma_{\boldsymbol{a}}$)**.** *Assume that $\mathcal{T}_{\boldsymbol{a}}$ is a vertex patch with mixed boundary conditions as described above with $\Gamma_{\boldsymbol{a}}$ contained in a plane H, such that $\omega_{\boldsymbol{a}}$ is entirely lying on one* *side of H and no face in $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ is contained in H. Then, for all $p \geq 0$, $\boldsymbol{\chi}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})$, and $\boldsymbol{j}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\nabla}\cdot\boldsymbol{j}_p = 0$, we have*

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v}_p = \boldsymbol{j}_p}} \|\boldsymbol{v}_p - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}} \leq C(\kappa_{\mathcal{T}_{\boldsymbol{a}}}) \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla}\times\boldsymbol{v} = \boldsymbol{j}_p}} \|\boldsymbol{v} - \boldsymbol{\chi}_p\|_{\omega_{\boldsymbol{a}}}. \tag{8.1}$$

Proof. We employ a technique of patch extension similar to the proof of Lemma 7.5. Without loss of generality, we assume that $H = \{\boldsymbol{x} \in \mathbb{R}^3;\ \boldsymbol{x}_3 = 0\}$ and denote by $\phi : \mathbb{R}^3 \ni \boldsymbol{x} \to (\boldsymbol{x}_1, \boldsymbol{x}_2, -\boldsymbol{x}_3) \in \mathbb{R}^3$ the symmetrization operator around H. Since ϕ is an isometry, the Piola mappings ϕ^{c} and ϕ^{d} have both a norm equal to 1, as in Lemma 7.9.

We then introduce the patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}} := \mathcal{T}_{\boldsymbol{a}} \cup \phi(\mathcal{T}_{\boldsymbol{a}})$, symmetrized around H. This indeed correctly defines a patch: as $\omega_{\boldsymbol{a}}$ only lies on one side of H, $\phi(\mathcal{T}_{\boldsymbol{a}})$ does not overlap with $\mathcal{T}_{\boldsymbol{a}}$. Notice that it is still a boundary patch since $\Gamma_{\boldsymbol{a}}^{\mathrm{D}} \neq \emptyset$. For $\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$, the extension operators $\mathscr{E}^{\bullet}(\boldsymbol{v})|_{\omega_{\boldsymbol{a}}} = \boldsymbol{v}$, $\mathscr{E}^{\bullet}(\boldsymbol{v})|_{\widetilde{\omega}_{\boldsymbol{a}}} = \phi^{\mathrm{c}}(\boldsymbol{v})$ for $\bullet = \mathrm{c}$ or d together with the trivial restriction operator $\mathscr{R}_0 : \boldsymbol{L}^2(\widetilde{\omega}_{\boldsymbol{a}}) \ni \widetilde{\boldsymbol{v}} \to \widetilde{\boldsymbol{v}}|_{\omega_{\boldsymbol{a}}} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}})$ will be useful.

The argument we employed in Lemma 7.5, case 2, still applies verbatim here, with the norms $\|\mathscr{E}^{\mathrm{c}}\| = \sqrt{2}$ and $\|\mathscr{R}^{\mathrm{c}}\| = 1$. The key difference is, however, that $\mathscr{E}^{\mathrm{c}}$ maps $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl}, \omega_{\boldsymbol{a}})$ into $\boldsymbol{H}_{0,\widetilde{\Gamma}}(\mathbf{curl}, \widetilde{\omega}_{\boldsymbol{a}})$ where $\widetilde{\Gamma}$ is here the whole boundary of $\partial\widetilde{\omega}_{\boldsymbol{a}}$. Indeed, we do have $\phi(\Gamma) = \partial\widetilde{\omega}_{\boldsymbol{a}}$. Thus, the resulting problem on $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ does not have mixed boundary conditions, and is consequently covered by the proof of Theorem 3.3 from Section 7. This concludes the proof. ☐

Lemma 8.2 (Wedge $\Gamma_{\boldsymbol{a}}$)**.** *Assume that $\mathcal{T}_{\boldsymbol{a}}$ is a vertex patch with mixed boundary conditions as described above and such that (i) $\Gamma_{\boldsymbol{a}}$ is contained in two planes H and H' intersecting at an angle $\pi/2$; (ii) $\omega_{\boldsymbol{a}}$ is contained between the two planes, in the sense that it is contained in a quarter space; (iii) no face in $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$ is contained in H or H'. Then, (8.1) holds true.*

Proof. We first symmetrize the patch around H' following the argument in the proof of Lemma 8.1. Then, it is easily seen that the resulting problem in the patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ satisfies the assumptions of Lemma 8.1 as the resulting $\widetilde{\Gamma}_{\boldsymbol{a}}$ is then contained in H. Applying Lemma 8.1 to the problem set over $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ concludes the proof. ☐

Finally, we arrive at the following conditional[4] result, which is established in the same way as Lemma 7.2. For the sake of shortness, we skip the proof here. Examples of patches with mixed boundary conditions covered by this result are shown in Figure 6.

Theorem 8.3 (Mixed boundary patches)**.** *Consider a vertex patch $\mathcal{T}_{\boldsymbol{a}}$ with domain $\omega_{\boldsymbol{a}}$ and $\Gamma \subset \partial\omega_{\boldsymbol{a}}$. Recall the above notation for $\Gamma_{\boldsymbol{a}}$. If there exists a bilipschitz mapping $\phi : \omega_{\boldsymbol{a}} \to \widetilde{\omega}_{\boldsymbol{a}}$ such that $\phi|_K$ is affine for all $K \in \mathcal{T}_{\boldsymbol{a}}$ and such that $\widetilde{\Gamma}_{\boldsymbol{a}} := \phi(\Gamma_{\boldsymbol{a}})$ and $\widetilde{\omega}_{\boldsymbol{a}}$ enter the setting of either Lemma 8.1 or Lemma 8.2, then (8.1) holds true.*

[4] We believe that the conditions of Theorem 8.3 cover all cases where $\Gamma_{\boldsymbol{a}}$ is connected, but we were not able to come up with a rigorous proof.

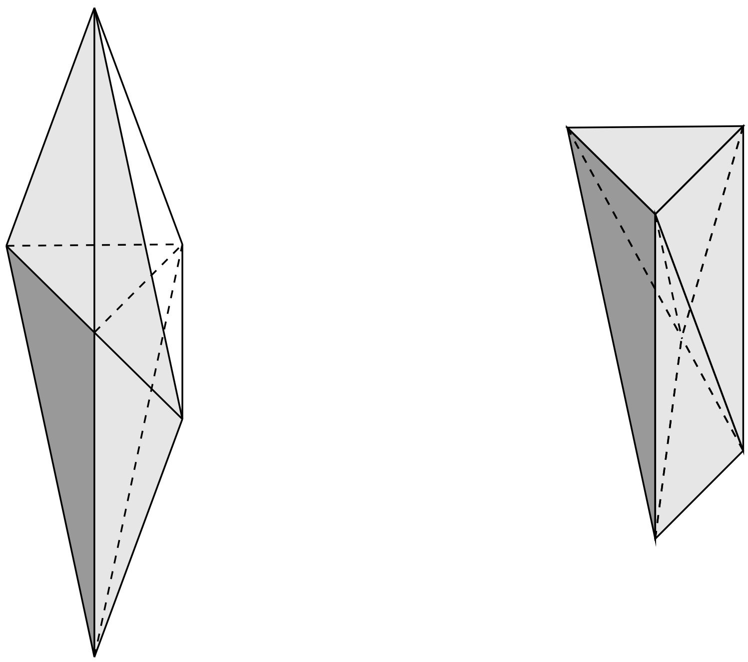


FIGURE 6. Example of patches with mixed boundary conditions covered by our analysis: The original patch is on the left and its derformed version on the right. If the dark grey boundary correspond to $\Gamma_{\boldsymbol{a}}$ (and thus, the light grey is $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$) then the deformed patch fits the setting of Lemma 8.1. Conversly, if the light grey represents $\Gamma_{\boldsymbol{a}}$ (and the dark grey, $\mathcal{F}_{\boldsymbol{a}}^{\mathrm{D}}$), then the deformed patch is covered by Lemma 8.2.

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