Two Types of Guaranteed (and Robust) *A Posteriori* Estimates for Finite Volume Methods

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ABSTRACT. *We present in this contribution the basic ideas of two types of* a posteriori *error estimates for finite volume discretizations of inhomogeneous and anisotropic pure diffusion problems. In both cases, our element estimators represent local lower bounds for the energy error and can thus be used as efficient indicators for adaptive mesh refinement. Moreover, our estimates are fully computable and thus allow us to guarantee the overall discretization error. Finally, in the second approach, the effectivity index, i.e., the ratio of the estimated and actual error, is independent of discontinuities in a scalar diffusion tensor. This full robustness is particularly important for the cases with singular solutions. Any cell- or vertex-centered finite volume scheme is included in our analysis, the estimators have a clear physical interpretation, are easily and locally computable, and numerical experiments confirm their excellent accuracy.*

KEYWORDS: *inhomogeneous and anisotropic diffusion, finite volume method, harmonic averaging,* a posteriori *error estimates, guaranteed upper bound, robustness*

1. Introduction

We consider in this paper the problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, && \text{[1a]}\\
p &= g \quad \text{on } \Gamma_{\mathrm{D}}, && \text{[1b]}\\
-\mathbf{S}\nabla p\cdot\mathbf{n} &= u \quad \text{on } \Gamma_{\mathrm{N}}, && \text{[1c]}
\end{aligned}
$$

where $\mathbf{S}$ is (an inhomogeneous and anisotropic) diffusion–dispersion tensor, f is a source term, and g and u prescribe the Dirichlet and Neumann boundary conditions, respectively. We suppose that $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a polygonal (polyhedral) domain, that $\Gamma_{\mathrm{D}}\cap\Gamma_{\mathrm{N}}=\emptyset$, $\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}=\Gamma:=\partial\Omega$, and that $|\Gamma_{\mathrm{D}}|\neq 0$, where $|\Gamma_{\mathrm{D}}|$ is the measure of the set Γ_{D}. Finally, $\mathbf{n}$ stands for the unit normal vector of $\partial\Omega$, outward to Ω.

One of the first *a posteriori* error estimates for finite volume methods, in the L^1-norm for nonlinear time-dependent convection–diffusion problems, was given by Ohlberger [OHL 01]. Energy norm estimates for linear elliptic equations including local efficiency results were then derived by, e.g., Achdou *et al.* [ACH 03] or Nicaise [NIC 05]. We present here the basic results of [VOH 06] and [VOH 08] that allow us to tightly control the overall error, including the cases with singular solutions due to inhomogeneities and anisotropies, and to refine the meshes adaptively and thus to increase the efficiency of the calculations. We in particular compare the two approaches, where the first one is *a priori* suited for cell-centered finite volume methods and the second one for vertex-centered finite volume methods. We also extend the results of [VOH 08] to full diffusion tensors and inhomogeneous Dirichlet and Neumann boundary conditions. For references, complete description of the results, all proofs, detailed numerical experiments, and extensions to the convection–diffusion–reaction case and general polygonal/polyhedral meshes including the nonmatching ones for cell-centered schemes, the reader is referred to [VOH 06] and [VOH 08].

2. Notation, continuous and discrete problems, and some useful inequalities

2.1. *Notation and assumptions*

Let $\mathcal{T}_h$ denote a conforming simplicial mesh of Ω, $\mathcal{V}_h$ its vertices, $\mathcal{E}_h$ its sides (edges if $d = 2$, faces if $d = 3$), and $\mathcal{E}_h^{\mathrm{int}}$ ($\mathcal{E}_h^{\mathrm{ext}}$) all interior (exterior) sides. We will next use $\mathcal{E}_h^{\mathrm{N}}$ for the sides contained (only entirely) in $\overline{\Gamma_{\mathrm{N}}}$, $\mathcal{E}_K$ for all sides of $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{int}}$ for $\sigma \in \mathcal{E}_h^{\mathrm{int}}$ which share at least a vertex with a $K \in \mathcal{T}_h$, $\mathcal{T}_V$ for all triangles sharing the vertex $V \in \mathcal{V}_h$, and $\mathcal{T}_K$ for all triangles sharing at least a vertex with $K \in \mathcal{T}_h$. In addition to $\mathcal{T}_h$, in one of our approaches, we shall also consider dual partitions $\mathcal{D}_h$ of Ω; see Figure 1 for an example. We use $\mathcal{D}_h^{\mathrm{int,\,N}}, \mathcal{D}_h^{\mathrm{ext,\,D}}$ to denote respectively the interior dual volumes and exterior dual volumes with the Neumann boundary condition (imposed over the whole $\partial D \cap \partial\Omega$) and similarly exterior dual volumes with the Dirichlet boundary condition; $\mathcal{F}_h, \mathcal{F}_h^{\mathrm{int}}, \mathcal{F}_h^{\mathrm{ext}}$ then denotes the sides of $\mathcal{D}_h$. Finally, we will need a second simplicial mesh $\mathcal{S}_h$ of Ω, constructed by dividing each $D \in \mathcal{D}_h$ into a mesh $\mathcal{S}_D$ as indicated in Figure 1 and then taking $\mathcal{S}_h = \bigcup_{D \in \mathcal{D}_h} \mathcal{S}_D$. We will use the notation $\mathcal{G}_h$ ($\mathcal{G}_h^{\mathrm{int}}$, $\mathcal{G}_h^{\mathrm{ext}}$) for its sides and $\mathcal{G}_h^{\mathrm{N}}$ for the sides contained in $\overline{\Gamma_{\mathrm{N}}}$ and $\mathcal{G}_D$ for the sides of $\mathcal{G}_h$ contained in ∂D for $D \in \mathcal{D}_h$. Finally, for $\sigma = \sigma_{K,L} \in \mathcal{G}_h^{\mathrm{int}}$, we define the weighted average operator $\{\!\{\cdot\}\!\}_\omega$ by $\{\!\{\varphi\}\!\}_\omega := \omega_{K,\sigma}(\varphi|_K)|_\sigma + \omega_{L,\sigma}(\varphi|_L)|_\sigma$, whereas for $\sigma \in \mathcal{G}_h^{\mathrm{ext}}$, $\{\!\{\varphi\}\!\}_\omega := \varphi|_\sigma$. Here $\omega_{K,\sigma}$ are weights associated with each $K \in \mathcal{S}_h$ and $\sigma \in \mathcal{E}_K$ such that $0 \le \omega_{K,\sigma} \le 1$ and $\omega_{K,\sigma} + \omega_{L,\sigma} = 1$ for all $\sigma = \sigma_{K,L} \in \mathcal{G}_h^{\mathrm{int}}$.

We next denote by $(\cdot,\cdot)_S$ the L^2-scalar product on S, by $\|\cdot\|_S$ the associated norm (when $S = \Omega$, the index is dropped off), by $|S|$ the Lebesgue measure of S, and by h_S its diameter. Next, $H^1_{0,\mathrm{D}}(\Omega)$ and $H^1_{g,\mathrm{D}}(\Omega)$ are respectively the subspaces of $H^1(\Omega)$ of functions with traces vanishing and equal to g on Γ_{D}. We will also need the "broken Sobolev space", $H^1(\mathcal{T}_h) := \{\varphi \in L^2(\Omega); \varphi|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$. In sections 2.3 and 3 (2.4 and 4), we will suppose that $\mathbf{S}$ is a piecewise constant symmetric matrix on $\mathcal{T}_h$ ($\mathcal{D}_h$) and we will denote by $c_{\mathbf{S},K}, C_{\mathbf{S},K}$ ($c_{\mathbf{S},D}, C_{\mathbf{S},D}$) its smallest and biggest eigenvalues, respectively. For simplicity, let f, g, and u be piecewise polynomials.

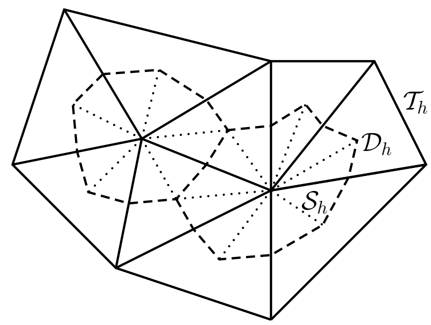

Figure 1: Original simplicial mesh $\mathcal{T}_h$, the associated dual mesh $\mathcal{D}_h$, and the fine simplicial mesh $\mathcal{S}_h$

2.2. *Continuous problem*

We define a bilinear form $\mathcal{B}$ by $\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h}(\mathbf{S}\nabla p, \nabla\varphi)_K$, $p,\varphi \in H^1(\mathcal{T}_h)$ and the corresponding energy semi-norm by $|||\varphi|||^2 := \mathcal{B}(\varphi,\varphi)$. In this way $\mathcal{B}(\cdot,\cdot)$ and $|||\cdot|||$ are well-defined for p,φ that are only piecewise regular. The weak formulation of problem [1a]–[1c] is then to find $p \in H^1_{g,\mathrm{D}}(\Omega)$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi) - \langle u,\varphi\rangle_{\Gamma_\mathrm{N}} \qquad \forall \varphi \in H^1_{0,\mathrm{D}}(\Omega). \tag{2}$$

2.3. *Cell-centered finite volume schemes*

A general cell-centered finite volume scheme for problem [1a]–[1c] on the mesh $\mathcal{T}_h$ can be written as: find p_K, $K \in \mathcal{T}_h$, the approximations to p such that

$$\sum_{\sigma\in\mathcal{E}_K} S_{K,\sigma} = f_K|K| \qquad \forall K \in \mathcal{T}_h, \tag{3}$$

where $f_K := (f,1)/|K|$ and $S_{K,\sigma}$ (functions of p_K) are the diffusive fluxes through the sides σ of an element K. We do not need the specific form of the fluxes; their continuity, imposing $S_{K,\sigma_{K,L}} = -S_{L,\sigma_{K,L}}$ for all $\sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}$, is our sole assumption.

2.4. *Vertex-centered finite volume schemes*

In vertex-centered finite volume schemes, both $\mathcal{T}_h$ and $\mathcal{D}_h$ are used, as we seek $p_h \in X_h^{\mathrm{D}} \subset H^1_{g,\mathrm{D}}(\Omega)$, the space of piecewise linear polynomials on $\mathcal{T}_h$, such that

$$-\langle \{\!\{\mathbf{S}\}\!\}_\omega \nabla p_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (f,1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int,\,N}}. \tag{4}$$

We have two basic choices of weights on a side $\sigma = \sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}$ in the above formula: $\omega_{D,\sigma} = \omega_{E,\sigma} = \frac{1}{2}$, which corresponds to the arithmetic averaging, and $\omega_{D,\sigma} = \frac{c_{\mathbf{S},E}}{c_{\mathbf{S},D}+c_{\mathbf{S},E}}$, $\omega_{E,\sigma} = \frac{c_{\mathbf{S},D}}{c_{\mathbf{S},D}+c_{\mathbf{S},E}}$, which corresponds to the harmonic averaging.

2.5. *Poincaré, Friedrichs, and trace inequalities*

The three following inequalities play a crucial role in our *a posteriori* error estimates. Let D be a polygon or a polyhedron. The Poincaré inequality states that

$$\|\varphi - \varphi_D\|_D^2 \le C_{\mathrm{P},D} h_D^2 \|\nabla\varphi\|_D^2 \qquad \forall \varphi \in H^1(D), \tag{5}$$

where φ_D is the mean value of φ over D and where the constant $C_{\mathrm{P},D}$ can for each convex D be evaluated as $1/\pi^2$. Next, the Friedrichs inequality states that

$$\|\varphi\|_D^2 \le C_{\mathrm{F},D,\Gamma_{\mathrm{D}}} h_D^2 \|\nabla\varphi\|_D^2 \qquad \forall \varphi \in H^1(D) \text{ such that } \varphi = 0 \text{ on } \partial\Omega \cap \partial D \neq \emptyset; \tag{6}$$

in general, $C_{\mathrm{F},D,\Gamma_{\mathrm{D}}} = 1$. Finally, for a side σ of D, the trace inequality states that

$$\|\varphi - \varphi_\sigma\|_\sigma^2 \quad \le \quad C_{\mathrm{t},D,\sigma} h_D \|\nabla\varphi\|_D^2. \tag{7}$$

For more details, we refer to [VOH 06, VOH 08] and the references therein.

3. Flux-based postprocessing and estimates

We present in this section the first type of *a posteriori* error estimates, *a priori* (but not exclusively) designed for cell-centered finite volume schemes of section 2.3.

3.1. *Guaranteed estimates*

Theorem 3.1 (Guaranteed estimate for flux-based postprocessing)**.** *Let p be the weak solution of problem* [1a]–[1c] *given by* [2] *and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary but such that $-\mathbf{S}\nabla\tilde{p}_h \in \mathbf{H}(\mathrm{div},\Omega)$, $-\mathbf{S}\nabla\tilde{p}_h \cdot \mathbf{n} = u_\sigma$, $u_\sigma := \langle u, 1\rangle_\sigma/|\sigma|$, for all $\sigma \in \mathcal{E}_h^{\mathrm{N}}$, and $-(\nabla\cdot(\mathbf{S}\nabla\tilde{p}_h), 1)_K = (f, 1)_K$ for all $K \in \mathcal{T}_h$. Let next $s_h \in H^1_{g,\mathrm{D}}(\Omega)$ be arbitrary and let the* nonconformity estimator *be given by*

$$\eta_{\mathrm{NC},K} := |||\tilde{p}_h - s_h|||_K,$$

the residual estimator *by*

$$\eta_{\mathrm{R},K} := m_K \|f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h)\|_K,$$

where $m_K^2 := C_{\mathrm{P}} h_K^2 / c_{\mathbf{S},K}$, with $C_{\mathrm{P}} = 1/\pi^2$ the constant from the Poincaré inequality [5]*, and the* Neumann boundary estimator *by*

$$\eta_{\Gamma_{\mathrm{N}},K} := 0 + \frac{\sqrt{h_K}}{\sqrt{c_{\mathbf{S},K}}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{N}}} \sqrt{C_{\mathrm{t},K,\sigma}} \|u_\sigma - u\|_\sigma,$$

where $C_{\mathrm{t},K,\sigma}$ is the constant from the trace inequality [7]*. Then*

$$|||p - \tilde{p}_h||| \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\Gamma_{\mathrm{N}},K})^2 \right\}^{1/2}.$$

3.2. *Construction of $\tilde{p}_h$ and s_h for cell-centered finite volume schemes*

A choice for $\tilde{p}_h$ is a locally constructed second-order polynomial whose normal fluxes across the sides of each K are given by $-S_{K,\sigma}$ and whose mean over K or value in a point $\mathbf{x}_K$ is given by p_K. For $s_h \in H^1_{g,\mathrm{D}}(\Omega)$, the basis is the so-called Oswald interpolate $\mathcal{I}_{\mathrm{Os}}$ of $\tilde{p}_h$, given in Lagrangian nodes by the average of the values of $\tilde{p}_h$, and adjusted so that $\mathcal{I}_{\mathrm{Os}}^{\Gamma_\mathrm{D}}(\tilde{p}_h) = g$ on Γ_D. For more details, we refer to [VOH 06].

3.3. *Local efficiency*

Theorem 3.2 (Local efficiency for flux-based postprocessing)**.** *Let the assumptions of Theorem 3.1 be verified, let $\tilde{p}_h$ and s_h be constructed as described in section 3.2, and let $\mathcal{T}_h$ be shape-regular, i.e., $\min_{K\in\mathcal{T}_h} |K|/h_K^d \geq \kappa_\mathcal{T}$ for a positive constant $\kappa_\mathcal{T}$. Put $c_{\mathbf{S},\mathcal{T}_K} := \min_{L\in\mathcal{T}_K} c_{\mathbf{S},L}$. Then, there holds*

$$\eta_{\mathrm{R},K}+\eta_{\mathrm{NC},K} \leq C\sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},\mathcal{T}_K}}}\Big(|||p-\tilde{p}_h|||_{\mathcal{T}_K}+|||p-\tilde{p}_h|||_{\#,\mathcal{E}_K^{\mathrm{int}}}\Big)+|||\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)-\mathcal{I}_{\mathrm{Os}}^{\Gamma_\mathrm{D}}(\tilde{p}_h)|||_K,$$

where the constant C depends only on the space dimension d, on the shape regularity parameter $\kappa_\mathcal{T}$, and on the polynomial degree k of f and where

$$|||p-\tilde{p}_h|||^2_{\#,\mathcal{E}_K^{\mathrm{int}}} := c_{\mathbf{S},\mathcal{T}_K} \sum_{\sigma\in\mathcal{E}_K^{\mathrm{int}}} h_\sigma^{-1} \|\langle [\![p-\tilde{p}_h]\!], 1\rangle_\sigma |\sigma|^{-1}\|^2_\sigma.$$

4. Potential-based postprocessing and estimates

We present here the second type of *a posteriori* error estimates, *a priori* (but not exclusively) designed for vertex-centered finite volume schemes of section 2.4.

4.1. *Guaranteed estimates*

Theorem 4.1 (Guaranteed estimate for potential-based postprocessing)**.** *Let p be the weak solution of problem [1a]–[1c] given by [2] and let $p_h \in H^1_{g,\mathrm{D}}(\Omega)$ be arbitrary. Let next $\mathbf{t}_h \in \mathbf{H}(\mathrm{div},\Omega)$ be arbitrary but such that $\mathbf{t}_h\cdot\mathbf{n} = u_\sigma$ for all $\sigma\in\mathcal{G}_h^\mathrm{N}$ and $(\nabla\cdot\mathbf{t}_h, 1)_D = (f,1)_D$ for all $D\in\mathcal{D}_h^{\mathrm{int,\,N}}$. Define the* diffusive flux estimator *by*

$$\eta_{\mathrm{DF},D} := \|\mathbf{S}^{\frac12}\nabla p_h + \mathbf{S}^{-\frac12}\mathbf{t}_h\|_D \qquad D\in\mathcal{D}_h,$$

the residual estimator *by*

$$\eta_{\mathrm{R},D} := m_D\|f-\nabla\cdot\mathbf{t}_h\|_D \qquad D\in\mathcal{D}_h,$$

where $m_D^2 := C_{\mathrm{P},D}h_D^2/c_{\mathbf{S},D}$ when $D\in\mathcal{D}_h^{\mathrm{int,\,N}}$ and $m_D^2 := C_{\mathrm{F},D,\Gamma_\mathrm{D}}h_D^2/c_{\mathbf{S},D}$ when $D\in\mathcal{D}_h^{\mathrm{ext,\,D}}$, with $C_{\mathrm{P},D}$ the constant from the Poincaré inequality [5] and $C_{\mathrm{F},D,\Gamma_\mathrm{D}}$ that from the Friedrichs inequality [6], and the Neumann boundary estimator *by*

$$\eta_{\Gamma_\mathrm{N},D} := 0 + \sum_{\sigma\in\mathcal{G}_D\cap\mathcal{G}_h^\mathrm{N}} \frac{\sqrt{h_{K_\sigma}}}{\sqrt{c_{\mathbf{S},K_\sigma}}}\sqrt{C_{\mathrm{t},K_\sigma,\sigma}}\|u_\sigma-u\|_\sigma,$$

where $K_\sigma \in \mathcal{S}_h$ is such that $\sigma \in \mathcal{E}_K$ and where $C_{\mathrm{t},K_\sigma,\sigma}$ is the constant from the trace inequality [7] *on K_σ. Then*

$$|||p - p_h||| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} + \eta_{\Gamma_{\mathrm{N}},D})^2 \right\}^{1/2} .$$

4.2. *Construction of $\mathbf{t}_h$ for vertex-centered finite volume schemes*

We define $\mathbf{t}_h$ in the Raviart–Thomas space on $\mathcal{S}_h$ by $\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{\!\{\mathbf{S}\nabla p_h \cdot \mathbf{n}_\sigma\}\!\}_\omega$ for all $\sigma \in \mathcal{G}_h \setminus \mathcal{G}_h^{\mathrm{N}}$, where p_h is the solution of and $\{\!\{\mathbf{S}\}\!\}_\omega$ is the averaging used in [4].

4.3. *Local efficiency*

Theorem 4.2 (Local efficiency for potential-based postprocessing)**.** *Let the assumptions of Theorem 4.1 be verified, let $\{\!\{\mathbf{S}\}\!\}_\omega$ be the harmonic averaging, and let p_h be the solution of* [4]*. Let next $\mathbf{t}_h$ be constructed as described in section 4.2 and let $\mathcal{T}_h$ be shape-regular, i.e., $\min_{K \in \mathcal{T}_h} |K|/h_K^d \geq \kappa_\mathcal{T}$ for some positive constant $\kappa_\mathcal{T}$. Then*

$$\eta_{\mathrm{DF},D} + \eta_{\mathrm{R},D} \leq C \max_{E;E\cap D \neq \emptyset} \sqrt{\frac{C_{\mathbf{S},E}}{c_{\mathbf{S},E}}} |||p - p_h|||_{\mathcal{T}_{V_D}} ,$$

where the constant C depends only on the space dimension d, on the shape regularity parameter $\kappa_\mathcal{T}$, on the polynomial degree k of f, and on $C_{\mathrm{P},D}$ or $C_{\mathrm{F},D,\Gamma_\mathrm{D}}$.

REMARK. — Note that Theorem 4.2 states that when $\mathbf{S}$ is scalar and piecewise constant on $\mathcal{D}_h$ and using the harmonic averaging, the potential-based *a posteriori* error estimate is robust, i.e., the effectivity index is independent of $\mathbf{S}$.

5. Numerical experiments

We consider here problem [1a]–[1c] with $\Omega = (-1,1) \times (-1,1)$ and $f = 0$, we assume that $\mathbf{S}$ is constant and equal to $s_i\, Id$ in the four axis quadrants, and we consider two cases, $s_1 = s_3 = 5$, $s_2 = s_4 = 1$ and $s_1 = s_3 = 100$, $s_2 = s_4 = 1$, respectively. An analytical solution, exhibiting a singularity at the origin, can be found here.

We have tested schemes of section 2.3 along with estimates of section 3 and, for square but possibly nonmatching meshes $\mathcal{D}_h$ and $\mathcal{T}_h$ constructed consequently using the square centers, schemes of section 2.4 along with estimates of section 4. In the first case and for the adaptive mode, each triangle where the estimated error is greater than 50% of the maximum of the estimators is refined into 4 sub-triangles and then the mesh is completed by a particular procedure so that it stays conforming and uniformly

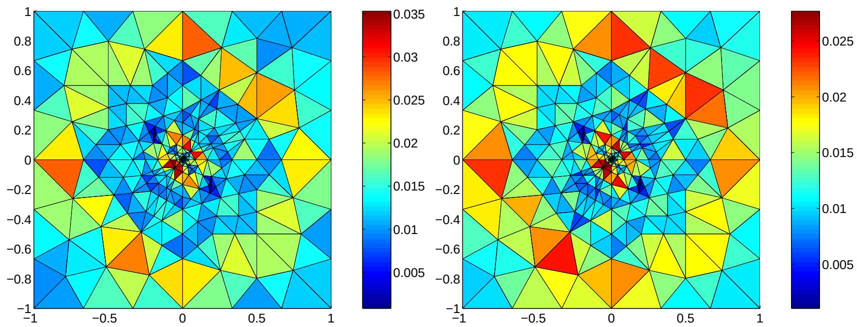

Figure 2: Estimated (left) and actual (right) error distribution, case 1, flux-based post-processing and estimates

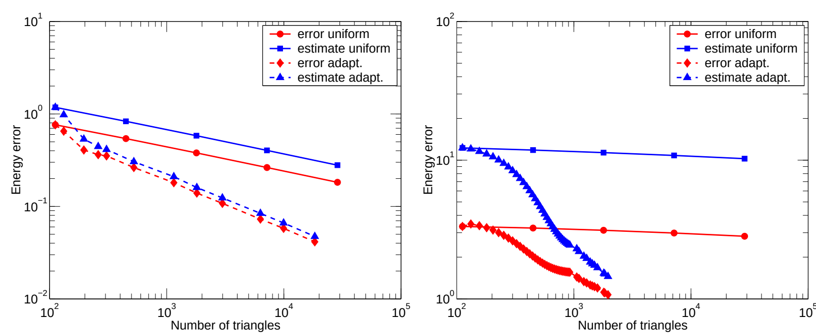

Figure 3: Estimated and actual energy error against the number of elements for case 1 (left) and case 2 (right), flux-based postprocessing and estimates

strictly Delaunay. In the second case, a square cell of the original dual mesh is refined into 9 identical subsquares if the estimated energy error is greater than 25% of the maximum of the estimators. We give in Figures 2 and 4 a comparison of the estimated and actual error distributions. We can see that for both types of estimates, the predicted distribution is excellent and in particular the singularity is well recognized, thanks to the local efficiency. Next in Figures 3 and 5 we report the estimated and actual energy error; these plots confirm in particular the guaranteed upper bound. In particular, for the estimates of section 3, the effectivity index for uniform mesh refinement is about 1.55 in the first case and 3.7 in the second. These estimates are thus not robust; refining the mesh adaptively, however, the effectivity index gets quite close to the optimal value of one. In contrast, full robustness is observed for estimates of section 4, where the effectivity index for uniform mesh refinement is constantly close to 2; its further improvement is possible using local minimization proposed and studied in [VOH 08]. Finally, it can clearly be seen from Figures 3 and 5 that the adaptive mesh refinement leads to much more efficient simulations.

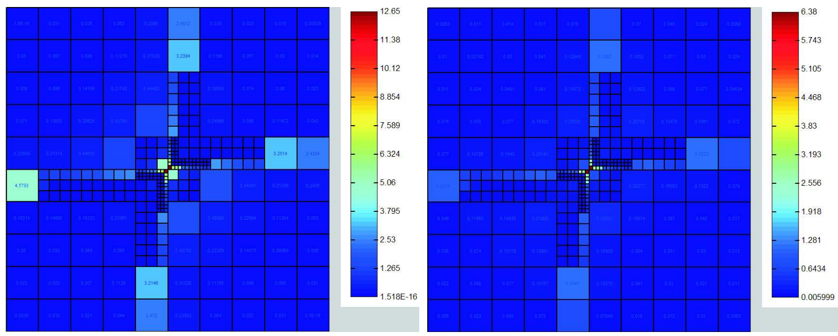

Figure 4: Estimated (left) and actual (right) error distribution, case 2, potential-based postprocessing and estimates

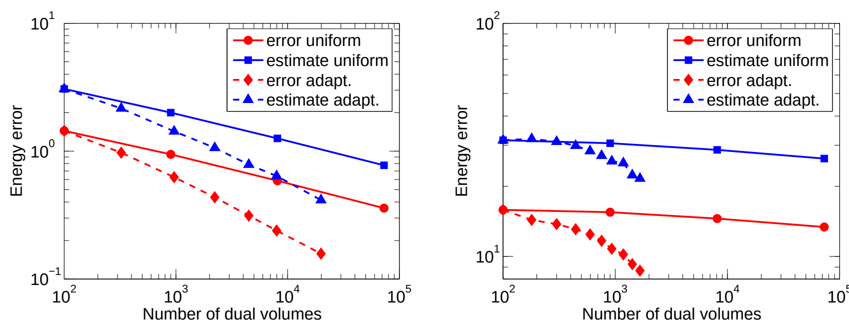

Figure 5: Estimated and actual energy error against the number of dual volumes for case 1 (left) and case 2 (right), potential-based postprocessing and estimates

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